Received: 24 July 2023 | Revised: 28 January 2024 | Accepted: 25 February 2024

IET Communications

DOI: 10.1049/cmu2.12762

The Institution of Engineering and Technology WILEY

ORIGINAL RESEARCH

On a class of column-weight 3 decomposable LDPC codes with the analysis of elementary trapping sets

G. Raeisi | **M. Gholami**

Department of Mathematical Sciences, Shahrekord University, Shahrekord, Iran

Correspondence
G. Raeisi, Department of Mathematical Sciences, Shahrekord University, Shahrekord, P.O. Box 115, Iran.
Email: g.raeisi@sku.ac.ir

Abstract

A column-weight k LDPC code with the parity-check matrix H is called decomposable if there exists a permutation π on the rows of H, such that $\pi(H)$ can be decomposed into k column-weight one matrix. In this paper, some variations of edge coloring of graphs are used to construct some column-weight three decomposable LDPC codes with girths at least six and eight. Applying the presented method on several known classes of bipartite graphs, some classes of column-weight three decomposable LDPC codes are derived having flexibility in length and rate. Interestingly, the constructed parity-check matrices based on the proper edge coloring of graphs can be considered as the base matrix of some high rate column-weight three quasi-cyclic (QC) LDPC codes with maximum-achievable girth 20. The paper also leads to a simple characterization of elementary trapping sets of the decomposable codes based on the chromatic index of the corresponding normal graphs. This characterization corresponds to a simple search algorithm finds all possible existing elementary trapping sets in a girth-6 or girth-8 column-weight 3 LDPC code which are layered super set of a short cycle in the Tanner graph of the code. Simulation results indicate that the QC-LDPC codes with large girths lifted from the constructed base matrices have good performances over AWGN channel.

1 | INTRODUCTION

Low-density parity-check (LDPC) codes were first introduced by Gallager [1] in his thesis in 1961 and have emerged as one of the top contenders for near-channel capacity error correction. LDPC codes are being considered in numerous applications including digital communication systems and magnetic recording channels [2]. Ever since their rediscovery, a great deal of research effort has been expended in the design and construction of these codes.

The design of LDPC codes can be categorized into two types of methods called random-like methods and structured methods. Although randomly constructed LDPC codes of large length give excellent bit-error rate (BER) performance [3], the memory required to specify the nonzero elements of a random matrix can be a major challenge for hardware implementation. Structured LDPC codes can lead to much simpler implementations, particularly for encoding.

An LDPC code is described by its (sparse) parity-check matrix [4]. Such a matrix can be efficiently represented by a bipartite graph, called Tanner graph [5]. To each parity-check matrix H of an LDPC code, the *Tanner graph* $\mathrm{TG}(H)$ is assigned which collects variable nodes and check nodes associated with the rows and the columns of H, respectively, and each edge connects a variable node to a check node if the intersection of the corresponding row and column of H has a nonzero entry.

LDPC codes perform well with iterative decoding based on belief propagation, such as the sum-product algorithm (SPA) or the min-sum algorithm (MSA) [6]. However, with iterative decoding, most LDPC codes have a common severe weakness, known as the error-floor. The error-floor of an LDPC code is characterized by the phenomenon that as the SNR continues to increase, the error probability suddenly drops at a rate much slower than that in the region of low to moderate SNR. The error-floor may preclude LDPC codes from applications where very low error rates are required, such as high-speed satellite communications, optical communications, hard-disk drives, and flash memories. Ever since the phenomenon of the error-floors of LDPC codes with iterative decoding became known [7], a great deal of research efforts has been expended in finding its causes and methods to resolve or mitigate the error-floor problem [7–13]. For the AWGN channel, the error-floor of an LDPC code is mostly caused by an undesirable structure, known as trapping sets, in the Tanner graph of the code. Extensive studies and simulation results show that most trapping sets that cause high error-floors of LDPC codes are trapping sets of small sizes. While the knowledge of trapping sets is most helpful in the design and analysis of LDPC codes, attaining such knowledge is generally a hard problem. Much research has been devoted to devising efficient search algorithms for finding small trapping sets, (see [7–20]).

One of the important parameters affecting the performance and determining the efficiency of iterative decoding algorithms for LDPC codes is the girth which determines the number of independent iterations [5]. It is well-known that the iterative sum-product decoding algorithm converges to the optimal solution provided that the Tanner graph of the code is free of short cycles. The effect of cycles on the practical performance of LDPC codes was demonstrated by simulation experiments when LDPC codes were rediscovered by MacKay and Neal [21] in the mid-1990's and the beneficial effects of using graphs free of short cycles were shown in [3]. Therefore, large girth Tanner graphs lead an increase in the number of correlation-free iterations and improve the convergence of the decoder. In addition, the performance in the error-floor region is predetermined by the girth, because trapping sets contain cycles in the Tanner graph [22], and so trapping sets containing short cycles are eliminated when the girth is increased. It is worth noting that the lower bound on the sizes of the minimum trapping sets grows exponentially with the girth for codes with column-weight at least three (see [23]).

Tanner [5] showed that the code's girth can be used as the lower bound of the minimum distance d_{min} of the code. In fact, Tanner determined a lower bound on the minimum distance that grows exponentially with the girth of the code. Specifically, for any regular LDPC code with girth g and variable node degree d_v,

$$d_{min} \geq \begin{cases} 1 + d_v\left(\sum_{i=1}^{\lfloor (g-2)/4 \rfloor} (d_v - 1)^{i-1}\right) & \text{if } g/2 \text{ is odd,} \\ 1 + d_v\left(\sum_{i=1}^{\lfloor (g-2)/4 \rfloor} (d_v - 1)^{i-1}\right) & \text{if } g/2 \text{ is even.} \\ \quad + (d_v - 1)^{\lfloor g-2/4 \rfloor} & \end{cases}$$

Accordingly, the design of large-girth LDPC codes is of great interest. Random and algebraic methods are two famous approaches for the constructions of LDPC codes with large girth. Among the random-like approaches, the progressive edge growth (PEG) algorithm [24] builds a Tanner graph by connecting the graph nodes edge-by-edge provided the added edge has minimal impact on the girth of the graph. Except for the PEG algorithm and its evolved construction algorithms, algebraic structured constructions of large-girth LDPC codes have been considered. Among the well-known structured LDPC codes, finite geometry LDPC codes and LDPC codes constructed from combinatorial designs [25–30] are adequate for high-rate LDPC codes. The error-correcting performance of these LDPC codes is verified under proper decoding algorithms but they have severe restrictions on flexibly choosing the code rate and length. Also, since finite geometry LDPC codes usually have much redundancy and large weights in their parity-check matrices, they are not suitable for a strictly power-constrained system with iterative message-passing decoding.

In an effort toward the algebraic constructions of LDPC codes, a quasi-cyclic (QC) LDPC code (see [4]) is getting more attention due to its linear-time encodability and small size of required memory. A QC-LDPC code can be viewed as a protograph code [31] whose parity-check matrix contains blocks of circulant matrices. Constructing families of QC-LDPC codes with large girth has been investigated by several authors (see [25–28, 32, 33]). For example, Steiner triple systems (STS) [29] and voltage-graphs [34] were used to construct some QC-LDPC codes with maximum girths 18 and 20, respectively.

The focus of the paper is on a class of LDPC codes called decomposable codes. A column-weight k LDPC code with the parity-check matrix H is called decomposable if there exists a permutation on the rows of H, such that under this permutation, H can be decomposed into k column-weight one matrix. In this paper, some well-structured block designs are presented whose incidence matrices can be considered as the parity-check matrix of some column-weight three decomposable LDPC codes with girths 6 or 8. The approach is based on the variations of edge coloring of graphs and the class of constructed LDPC codes has flexibility in code length and rate, as shown by several examples (see sections 3 and 4). Interestingly, in some cases, the constructed girth-6 LDPC codes have better lengths and minimum distances compared to the constructed girth-6 LDPC codes by Bocharava et al. [35]. In addition, it is shown that the constructed parity-check matrices can be considered as the mother matrices of some QC-LDPC codes having a maximum achievable girth 20 or 24. Moreover, trapping sets of decomposable codes will be analyzed.

The outline of the paper is organized as follows. In Section 2, we give the preliminaries and constructions and in Sections 3 and 4, we give some examples of the constructed decomposable codes. In Section 5, the trapping sets of the decomposable codes will be analyzed, and finally, in Sections 6 and 7, the QC-LDPC codes based on the constructed codes are considered and some performance comparisons are provided between the proposed QC-LDPC codes with different girths.

2 | PRELIMINARIES AND CONSTRUCTIONS

Let $v \geq k \geq 2$ and $\lambda \geq 1$ be given. In combinatorial mathematics [36], a (v, k, λ)-packing is a pair $(X, \mathcal{B})$, where X is a v-set of elements (points) and $\mathcal{B}$ is a collection of k-subsets of X (blocks), such that every 2-subset of points

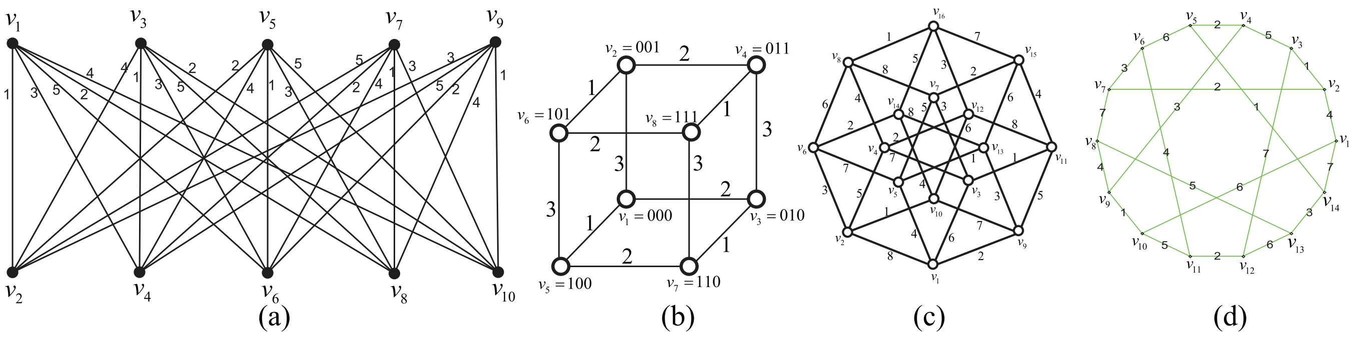

FIGURE 1 (a), (b) Proper edge colorings of $K_{5,5}$ and Q_3, (c), (d) Strong edge colorings of Q_4 and $G_{14}(1, 5, 13)$, respectively.

occurs in at most λ blocks of $\mathcal{B}$. A (v, k, λ)-BIBD (Balanced Incomplete Block Design) is a (v, k, λ)-packing in which every 2-subset of points occurs in exactly λ blocks of $\mathcal{B}$. As in this paper, we just consider $(v, 3, 1)$-packings, for simplicity, by a (v, l)-packing, we mean a $(v, 3, 1)$-packing with l blocks. The *incidence matrix* of a (v, l)-packing is a $v \times l$ binary matrix $\mathcal{H} = (h_{ij})$ in which the rows and columns correspond to the points and blocks, respectively, such that $h_{ij} = 1$ if the i-th point belongs to the j-th block and $h_{ij} = 0$, otherwise. For example, if $b_1 = \{1, 2, 4\}$, $b_2 = \{2, 3, 5\}$, $b_3 = \{3, 4, 6\}$, $b_4 = \{4, 5, 7\}$, $b_5 = \{1, 5, 6\}$, $b_6 = \{2, 6, 7\}$ and $b_7 = \{1, 3, 7\}$, then $\mathcal{B} = \{b_1, b_2, \ldots, b_7\}$ is a (7,7)-packing with the following incidence matrix.

$$\mathcal{H} = \begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

To go through the details of the construction, we need some basic definitions from graph theory [37]. Let $G = (V, E)$ be a graph with vertex and edge sets V and E, respectively. The *degree* of a vertex $v \in V$ is the number of edges incident with v and G is called *k-regular* if all vertices have degree k. The maximum degree of vertices of a graph G is denoted by $\Delta(G)$, or simply by Δ. Two vertices u and v are adjacent if there is an edge between u and v. For a subgraph H of G and $v \in V(G) \setminus V(H)$, we use $\deg_H(v)$ to denote the number of vertices in H which are adjacent to v.

We say that vertex v and edge e are *incident* if v is an endpoint of e. Also, two edges of G are incident if they have a vertex in common. Two edges are of *distance at most two* if they either share an endpoint or an endpoint of one is joined to an endpoint of the other by an edge. A *cycle* in a graph $G = (V, E)$ is a sequence of connected vertices and edges in the graph which starts and ends at the same vertex and contains each vertex no more than once. The *length of a cycle* is the number of edges it contains and the *girth* of a graph $G = (V, E)$ is the length of the shortest cycle in G.

A graph G is *bipartite* if the set of its vertices can be partitioned into two disjoint sets V_1 and V_2 such that no two vertices within either V_1 or V_2 are connected by an edge. A bipartite graph G with partite sets V_1 and V_2 is denoted by $G = (V_1, V_2)$. A bipartite graph $G = (V_1, V_2)$ is called *complete bipartite graph* if there is an edge between each vertex of V_1 and each vertex of V_2. A complete bipartite graph $G = (V_1, V_2)$ with $|V_1| = n$ and $|V_2| = m$ is denoted by $K_{n,m}$. For example, $K_{5,5}$ is depicted in Figure 1a.

In graph theory, a *proper edge coloring* or briefly *edge coloring* [38] of a graph is an assignment of colors to the edges of the graph so that no two adjacent edges have the same color. For example, a proper edge coloring of the Peterson graph with 4 colors is depicted in Figure 2a. The minimum required number of colors for the edge coloring of a given graph G is called the *chromatic index* of G and denoted by $\chi'(G)$. By the Vizing's theorem [39], the number of colors needed to color the edges of a simple graph is either its maximum degree Δ or $\Delta + 1$. For bipartite graphs, the number of colors is always Δ [38]. If G is a graph with maximum degree Δ and $\chi'(G) = \Delta$, then G is called type I graph and if $\chi'(G) = \Delta + 1$, then G is called type II graph.

Many variations of the edge coloring problem, in which an assignment of colors to edges must satisfy other conditions than non-adjacency, have been studied. One of these colorings is the strong edge coloring of graphs [40]. A *strong edge coloring* [38] assigns colors to edges such that every two edges of distance at most two have different colors. For example, a strong edge coloring of the Peterson graph with 5 colors is depicted in Figure 2b. The *strong chromatic index*, $\chi'_s(G)$, is the minimum number of colors in a strong edge coloring of G.

A low-density parity-check (LDPC) code is a linear block code for which the parity-check matrix H contains only a few 1's in comparison to the amount of 0's. An LDPC code with the parity-check matrix H is called r-row-regular (r-column-regular) if each row (column) of H has weight r. An LDPC code is known as a irregular code if its parity-check matrix is not row-regular or column-regular. Also, by a (j, k)-*regular* LDPC code we mean a code whose parity-check matrix is k-row-regular and j-column-regular. For an LDPC code with a $h \times n$ parity-check matrix H, the *code rate* R is defined as $1 - \frac{h}{n}$ (this assumes H is

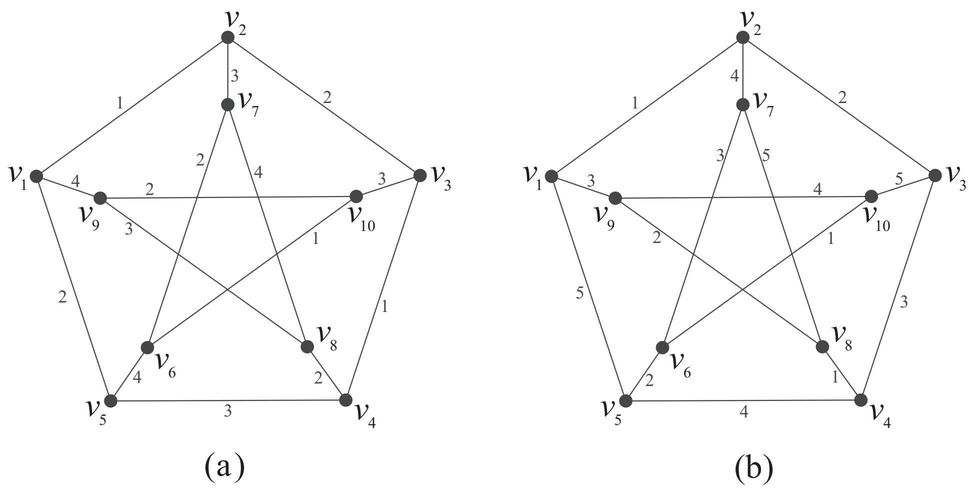

FIGURE 2 (a) Proper edge coloring, (b) strong edge coloring of the Peterson graph.

full rank) which is equal to $1 - \frac{j}{k}$, if H is (j, k)-regular. The *girth* of an LDPC code C with the parity-check matrix H, denoted by $g(H)$ or $g(C)$, is defined as the length of the shortest cycle in TG(H).

Let $H = (h_{i,j})_{m\times n}$ be an arbitrary matrix and S_m denote the set of all permutations on m points. For a permutation $\pi \in S_m$, we use $\pi(H)$ to denote the matrix obtained from H by applying the permutation π on the rows of H, that is, $\pi(H) = (h_{\pi(i),j})$. Now, let H be the parity-check matrix of a column-weight c LDPC code. We say that H is *decomposable* if and only if there exists a permutation π on the rows of H, such that $\pi(H)$ can be decomposed into c column-weight 1 matrix H_1, ..., H_c, that is, $H = \begin{pmatrix} H_1 \\ \vdots \\ H_c \end{pmatrix}$. A column-weight c LDPC code with the parity-check matrix H is called decomposable if H is decomposable.

Example 2.1. Consider a column-weight three LDPC code with the following parity-check matrix.

$$H = \begin{pmatrix}
1&0&0&1&0&0&1&0&0&0&0&0\\
1&0&0&0&0&1&0&1&0&1&0&0\\
0&1&0&0&1&0&0&0&0&1&0&0\\
0&1&0&1&0&0&0&0&1&0&1&0\\
0&0&1&0&0&0&0&1&0&0&1&0\\
0&0&1&0&1&0&1&0&0&0&0&1\\
0&0&0&0&0&1&0&0&1&0&0&1\\
1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1
\end{pmatrix}$$

Applying the permutation $\pi = (2\ 5\ 3)(4\ 6\ 7) \in S_{11}$ on the rows of H, we have

$$\pi(H) = \left(\begin{array}{cccccccccccc}
1&0&0&1&0&0&1&0&0&0&0&0\\
0&1&0&0&1&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&1&0&0&1&0&0&1\\
\hline
1&0&0&0&0&1&0&1&0&1&0&0\\
0&1&0&1&0&0&0&0&1&0&1&0\\
0&0&1&0&1&0&1&0&0&0&0&1\\
\hline
1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1
\end{array}\right),$$

which shows that H is the parity-check matrix of a decomposable code.

Now, we go through the details of our method which constructs a girth-6 (girth-8) column-weight three LDPC code from an arbitrary proper (strong) edge colored graph. For this purpose, let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and let ϱ be an arbitrary edge coloring of G with t colors $1, 2, \dots, t$. For each k, $1 \le k \le t$, let B_k denote the family of all triples $\{i, j, n + k\}$, where $e = v_i v_j$ is an edge of G with color k. For instance, for the Peterson graph in Figure 2a, we have:

$$B_1 = \{\{1,2,11\},\{3,4,11\},\{6,10,11\}\},$$
$$B_2 = \{\{1,5,12\},\{2,3,12\},\{6,7,12\},\{9,10,12\},\{4,8,12\}\},$$
$$B_3 = \{\{2,7,13\},\{3,10,13\},\{8,9,13\},\{4,5,13\}\},$$
$$B_4 = \{\{7,8,14\},\{5,6,14\},\{1,9,14\}\}.$$

Set $\mathcal{B}_\varrho(G) = \bigcup_{i=1}^{t} B_i$. It is clear to see that $\mathcal{B}_\varrho(G)$ contains $n + t$ points $\{1, 2, \dots, n + t\}$, where $\{1, 2, \dots n\}$ is the set of indices of vertices of G and $\{n + 1, n + 2, \dots n + t\}$ is the set of colors in the edge coloring of G. In addition, each edge of G is contained in exactly one block of $\mathcal{B}_\varrho(G)$ and $\mathcal{B}_\varrho(G)$ has $m = |E(G)|$ blocks and so $\mathcal{B}_\varrho(G)$ is a $(n + t, m)$-packing. Let $\mathcal{H}_\varrho(G)$ denote the $(n + t) \times m$ incident matrix of $\mathcal{B}_\varrho(G)$. It is easy to see that $\mathcal{H}_\varrho(G)$ is the parity-check matrix of a column-weight 3 LDPC code, with length m and rate $\mathcal{R} = 1 - \frac{n+t}{m}$. We denote by $\mathcal{C}_\varrho(G)$ the constructed column-weight 3 LDPC code with the parity-check matrix $\mathcal{H}_\varrho(G)$ and we call it as the (G, ϱ)-code. In the rest (see Theorems 1 and 2), we prove that the girth of the constructed column-weight 3 LDPC code with parity-check matrix $\mathcal{H}_\varrho(G)$ is fully dependent on the type of the edge coloring ϱ. It is easy to see that $\mathcal{H}_\varrho(G)$ is the parity-check matrix of a decomposable code if and only if G is a bipartite graph. Hereafter, if ϱ is a proper (resp. strong) edge coloring of G, we use $\mathcal{H}_p(G)$ (resp. $\mathcal{H}_s(G)$) to denote the parity-check matrix $\mathcal{H}_\varrho(G)$.

3 | LDPC CODES FROM PROPER EDGE COLORED GRAPHS

In this section, we just concentrate on the proper edge coloring of simple graphs (graphs without loops and parallel edges) and we examine the constructed codes by several examples. First, for a given simple graph G, we prove that $\mathcal{H}_p(G)$ is the parity-check matrix of a 4-cycle free column-weight three LDPC code.

Theorem 1. *Let G be a simple graph with an arbitrary edge coloring ϱ. Then $\mathcal{H}_\varrho(G)$ is free of 4-cycles if and only if ϱ is proper.*

Proof. Let $|V(G)| = n$, $|E(G)| = m$ and t be the number of colors used in ϱ. Since G is a simple graph, $g(\mathcal{H}_\varrho(G)) = 4$ if and only if there are blocks (triples) $B, B' \in \mathcal{B}_\varrho(G)$ such that $B \cap B' = \{i, n + k\}$, for some $1 \leq i \leq n$ and $1 \leq k \leq t$. But this is equivalent to the existence of two edges e and e' having vertex v_i in common and both edges e and e' have the same color k, means that ϱ is not proper. Therefore, $g(\mathcal{H}_\varrho(G)) \geq 6$ if and only if ϱ is a proper edge coloring. □

In the rest of this section, we examine the constructed codes by several examples. As decomposable codes are considered in this paper, we just consider bipartite graphs. In the following examples, a graph G with a proper edge coloring p is presented and then, this coloring is used to construct a column-weight three LDPC code having parity-check matrix $\mathcal{H}_p(G)$.

Example 3.1 (Complete bipartite graphs). Let $l \geq 4$ and $K_{l,l}$ be the complete bipartite graph with partite sets $V_1 = \{1, 3, \dots, 2l - 1\}$ and $V_2 = \{2, 4, \dots, 2l\}$. Coloring each edge $\{2i - 1, 2j\}$, $1 \leq i, j \leq l$, by color $2(j - i) + 1 \pmod{l}$ yields a proper edge coloring of $K_{l,l}$ with l colors. For example, for $l = 5$ such a coloring is given in Figure 1a. Therefore, if p is the mentioned edge coloring of $K_{l,l}$, then $\mathcal{H}_p(K_{l,l})$ is the parity-check matrix of a girth-6 column-weight three LDPC code with rate $\mathcal{R} = 1 - \frac{3}{l} + \frac{1}{l^2}$. For example, the code constructed based on the proper edge coloring of $K_{5,5}$ has the parity-check matrix $\mathcal{H}_p(K_{5,5})$, rate 0.44 and length 25.

$$\mathcal{H}_p(K_{5,5}) = \begin{pmatrix}
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\
1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&0&0&0
\end{pmatrix}$$

Clearly $\mathcal{R}$ rapidly tends to 1 when n enlarges, as shown in Table 1. Since the degree of each vertex in $K_{l,l}$ is l, $\mathcal{H}_p(K_{l,l})$ is the parity-check matrix of a $(3, l)$-regular code. Compared to the constructed girth-6 $(3, l)$-regular codes in [35], the constructed codes based on $K_{l,l}$ have smaller lengths for each

TABLE 1 Codes based on the proper edge coloring of complete bipartite graphs and hypercubes with rate $\mathcal{R}$ and length n.

Complete bipartite graph $K_{l,l}$	l	3	4	5	6	7	8	9	10	11	12	13	14	16
	n	9	16	25	36	49	64	81	100	121	144	169	196	256
	$\mathcal{R}$	0.11	0.31	0.44	0.53	0.59	0.64	0.68	0.71	0.74	0.76	0.78	0.80	0.81
Hypercube graph Q_l	l	3	4	5	6	7	8	9	10	11	12	13	14	15
	n	12	32	80	192	448	1024	2304	5120	11264	24576	53248	114688	245760
	$\mathcal{R}$	0.17	0.41	0.55	0.64	0.70	0.74	0.77	0.80	0.82	0.83	0.85	0.86	0.87

TABLE 2 Constructed codes based on edge coloring of G having mother matrix $\mathcal{H}_{\varrho}(G)$.

Parity-check matrix $\mathcal{H}_{\varrho}(G)$	Regularity	Girth	Length	Minimum distance	QC
$\mathcal{H}_p(K_{n,n})$(*n odd*)	n	6	n^2	6(**4**, [35])	-
$\mathcal{H}_p(K_{n,n})$(*n even*)	n	6	n^2($\mathbf{n^2}+n$, [35])	6(**4**, [**35**])	-
$\mathcal{H}_p(G_{14}(1,5,13))$	3	6	21	6	-
$\mathcal{H}_p(G_{26}(1,5,17,25))$	4	6	52	10	-
$\mathcal{H}_s(Q_4)$	4	8	32(**36**, [**35**])	6	-
$\mathcal{H}_s(G_{26}(1,5,17,25))$	4	8	52	12	-
$\mathcal{H}_s(G_{42}(1,11,15,35,41))$	6	8	186	12	-
$\mathcal{H}_p(K_{4,4})$	4	20	164128(**1296000**, [**35**])	-	*
$\mathcal{H}_s(G_{26}(1,5,17,25))$	4	20	312416(**1296000**, [**35**])	-	*

even l and have better minimum distances for each l (see Table 2).

Example 3.2 (Hypercubes). The hypercube Q_l is a graph whose vertex set is all l-sequences $x = x_1x_2 \dots x_l$ with entries from $\{0,1\}$, and two vertices are adjacent if they differ in exactly one coordinate. Clearly, Q_l is a l-regular graph on 2^l vertices. For example, Q_3 is denoted in Figure 1c. Now, color each edge $xy \in Q_l$ by color i if x and y differs in the i-th coordinate, $1 \le i \le l$. It is easy to see that this coloring is a proper coloring of Q_l with l colors. For example, for $l = 3$ such a coloring is given in Figure 1b. Therefore, if p is the above edge coloring of Q_l, then the constructed code with parity-check matrix $\mathcal{H}_p(Q_l)$ is a column-weight three LDPC code with girth 6 and rate $\mathcal{R} = 1 - \frac{2^l + l}{l2^{l-1}} = 1 - \frac{2}{l} - \frac{1}{2^{l-1}}$. For $l = 3$, $\mathcal{H}_p(Q_3)$ is the parity-check matrix of a code with rate 0.17 and length 12 with the following parity-check matrix.

$$\mathcal{H}_p(Q_3) = \begin{pmatrix} 1&1&1&0&0&0&0&0&0&0&0&0\\ 1&0&0&1&1&0&0&0&0&0&0&0\\ 0&1&0&0&0&1&1&0&0&0&0&0\\ 0&0&0&1&0&1&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&1&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&1&0&0&1&0&1\\ 0&0&0&0&0&0&0&1&0&0&1&1\\ 0&0&1&0&1&0&0&1&0&0&0&0\\ 0&1&0&1&0&0&1&0&0&1&1&0\\ 1&0&0&0&0&1&0&0&1&0&0&1 \end{pmatrix}$$

Since the degree of each vertex in Q_l is l, $\mathcal{H}_p(Q_l)$ has 2^l rows of weight l and l rows of weight 2^{l-1}. Therefore, the constructed code with the parity-check matrix $\mathcal{H}_p(Q_l)$ is irregular.

4 | LDPC CODES FROM STRONG EDGE COLORED GRAPHS

In this section, the strong edge coloring of some known bipartite graphs is used to construct some classes of column-weight three decomposable LDPC codes with girth 8. First, we prove that $\mathcal{H}_s(G)$ is the parity-check matrix of a girth-8 column-weight three LDPC code, provided that G is free of triangles.

Theorem 2. *Let G be a triangle-free graph with an arbitrary edge coloring ϱ. Then $g(\mathcal{H}_{\varrho}(G)) \ge 8$ if and only if ϱ is a strong edge coloring.*

Proof. Let t be the number of colors in the edge coloring ϱ, $|V(G)| = n$, $|E(G)| = m$ and $B_{\varrho}(G)$ be the $(n+t, m)$-packing associated with $\mathcal{H}_{\varrho}(G)$. By Theorem 1, $g(\mathcal{H}_{\varrho}(G)) \ge 4$ if and only if ϱ is proper. Since any strong edge coloring of G is a proper edge coloring, it is sufficient to prove that $\mathcal{H}_{\varrho}(G)$ is free of 6-cycles. Since G is triangle-free, $\mathcal{H}_{\varrho}(G)$ contains a 6-cycle if and only if there are some distinct blocks $B, B', B'' \in B_{\varrho}(G)$ such that for some vertices v_{i_1} and v_{i_2}, $1 \le i_1 \ne i_2 \le n$, and color $c = n + i_3$, $1 \le i_3 \le t$, we have $B \cap B'' = \{i_1\}$, $B' \cap B'' = \{i_2\}$ and $B \cap B' = \{n + i_3\}$. But, this means that vertices v_{i_1} and v_{i_2} are adjacent in G (via block B'') and also edges e and e' (different from the edge $v_{i_1}v_{i_2}$) corresponding to the blocks B and B' are incident with vertices v_{i_1} and v_{i_2}, respectively, such that both e and e' have the same color i_3. Therefore, $\mathcal{H}_{\varrho}(G)$ is free of

TABLE 3 Some odd transformation modular Golomb Ruler d' with k marks modulo $2n$ [42, 43].

$2n$	k	$d' = (d'_1, \ldots, d'_k)$
26	4	(1,5,17,25)
42	5	(1,11,15,35,41)
62	6	(1,15,21,25,33,61)
96	7	(1,29,51,71,85,89,95)
114	8	(1,25,29,41,47,61,105,113)
146	9	(1,13,21,69,95,101,105,129,145)
182	10	(1,3,13,21,47,53,69,83,107,111)
240	11	(1,93,105,125,155,159,181,195,223,233,239)
266	12	(1,5,13,49,59,81,87,111,137,151,153,171)
336	13	(1,39,61,69,75,93,127,171,175,191,217,325,335)
366	14	(1,31,99,103,109,143,157,169,185,193,231,249,345,365)
510	15	(1,23,27,71,79,109,167,183,233,243,297,391,491,497,509)
510	16	(1,21,23,63,67,117,141,147,155,173,245,255,303,315,331,367)
546	17	(1,11,31,69,71,85,147,151,173,179,197,269,303,311,355,367,403)
614	18	(1,5,21,45,107,113,165,167,179,197,261,297,307,335,377,385,411,433)
720	19	(1,7,63,65,83,135,173,189,221,233,257,267,369,397,411,419,485,511,515)
762	20	(1,49,61,87,111,143,151,179,209,251,255,325,335,379,413,431,545,551,565,567)

6-cycles if and only if each two edges e and e' with distance two receive distinct colors, means that ϱ is a strong edge coloring. This observation completes the proof of the theorem. □

Example 4.1 (Hypercubes). Let Q_l be the hypercube graph introduced in Example 3.2. It is proved [41] that a strong edge coloring of Q_l needs at most $2l$ colors. To see this, represent a vertex x of Q_l by a 0-1 vector $v(x)$ of length l and let E_i be the set of all edges xy in which $v(x)$ and $v(y)$ differ in the i-th coordinate, $1 \le i \le l$. A refinement $E_i = E_i^1 \cup E_i^2$ of this edge partition is obtained in the following way: an edge $xy \in E_i$ belongs to E_i^j, $(1 \le i \le l, 1 \le j \le 2)$, if and only if the sum of all coordinates of $v(x)$ (or $v(y)$) except for the i-th one, is congruent to $j \pmod 2$. Obviously, assigning color $i \times j$ to each edge in E_i^j yields an strong edge coloring of Q_l by $2l$ colors. For example, for $l = 4$ such a coloring is given in Figure 1c. In this figure, the 0-1 vector associated with each vertex v_i is the representation of $i-1$ in base 2. Therefore, if s is the above strong edge coloring of Q_l, then $\mathcal{H}_s(Q_l)$ can be considered as the parity-check matrix of a column-weight three LDPC code with girth 8 and rate $\mathcal{R} = 1 - \frac{2^l+2l-1}{l2^{l-1}}$. For example, the code with the parity-check matrix $\mathcal{H}_s(Q_4)$ has rate 0.28 and length 32.

Clearly $\mathcal{R}$ rapidly tends to 1 when l enlarges. Since the degree of each vertex in Q_l is l, $\mathcal{H}_s(Q_l)$ have 2^l rows of weight l and $2l$ rows of weight 2^{l-2}. In particular, $\mathcal{H}_s(Q_4)$ is a (3,4)-regular code with length 32. Interestingly, the constructed code based on $\mathcal{H}_s(Q_4)$ has smaller length than the (3,4)-QC-LDPC codes with girth 8 constructed in [35].

Example 4.2 (Graphs based on Golomb rulers). By a length-k Modular Golomb Ruler modulo n [42], we mean a set of k residues $a_1, a_2, \ldots, a_k$ such that the differences $a_i - a_j$, $i \neq j$, are all distinct modulo n. Note that each pair generates two differences: $a_i - a_j$ and $a_j - a_i$, which are both considered in modulo n. For example $\{0, 2, 6\}$ is a Modular Golomb ruler modulo 7. In [42], the author proved that the set $a = \{a_1, a_2, \ldots, a_k\}$ is a modular Golomb Ruler modulo n if and only if the odd transformation $d' = \{2a_1+1, 2a_2+1, \ldots, 2a_k+1\}$ is a modular Golomb Ruler modulo $2n$. In [42, 43], the authors present some odd transformation modular Golomb rulers modulo $2n$, shown in Table 3, and they used these odd transformation modular Golomb rulers to construct some regular graphs with girth 6, in the following sense. Let $a = \{a_1, a_2, \ldots, a_k\}$ be a modular Golomb Ruler modulo n and let d' be the odd transformation of a. Let $G_{2n}(d')$ be the graph with vertex set $V = \{1, 2, \ldots, 2n\}$ and edge set $E = \cup_{i=1}^{k} B_i$, where $B_i = \{\{2j, 2j + d'_i \pmod{2n}\} : 1 \le j \le n\}$, $1 \le i \le k$. It is proved [43] that if $a = \{a_1, a_2, \ldots, a_k\}$ is a modular Golomb Ruler modulo n with odd transformation d', then $G_{2n}(d')$ is a k-regular bipartite graph with girth 6. As an example, for modular Golomb Ruler $a = \{0, 2, 6\}$ with odd transformation $d' = \{1, 5, 13\}$, $G_{14}(d')$ is a 3-regular graph with girth 6 on 14 vertices, as shown in Figure 1d.

Now, one can easily check that coloring each edge $\{i, j\}$ of $G_{2n}(d')$ by color $i + j \pmod n$ gives a strong edge coloring of $G_{2n}(d')$ with n colors. As an example, a strong edge coloring of $G_{14}(1, 5, 13)$ with 7 colors is shown in Figure 1d. Therefore, if s is such a strong edge coloring of $G_{14}(1, 5, 13)$, then $H_s(G_{14}(1, 5, 13))$ is the parity-check matrix of a column-weight three code with girth 8.

Therefore, if s is such a strong edge coloring of $G_{2n}(d')$, then $\mathcal{H}_s(G_{2n}(d'))$ is the parity-check matrix of a girth-8 column-weight three LDPC code having rate $\mathcal{R} = 1 - \frac{3}{k}$, which tends

TABLE 4 Codes based on the strong edge coloring of hypercube and Golomb ruler graphs with rate $\mathcal{R}$ and length n.

Hypercube graph Q_l	l	4	5	6	7	8	9	10	11	12	13	14	15
	n	32	80	192	448	1024	2304	5120	11264	24576	53248	114688	245760
	$\mathcal{R}$	0.28	0.49	0.61	0.69	0.74	0.77	0.80	0.82	0.83	0.85	0.86	0.87
l-Regular Golomb ruler graph	l	3	4	5	6	7	8	9	10	11	12	13	15
	n	21	52	105	186	336	456	657	910	1320	1596	2184	3825
	$\mathcal{R}$	0.0	0.25	0.40	0.50	0.57	0.62	0.67	0.70	0.73	0.75	0.77	0.80

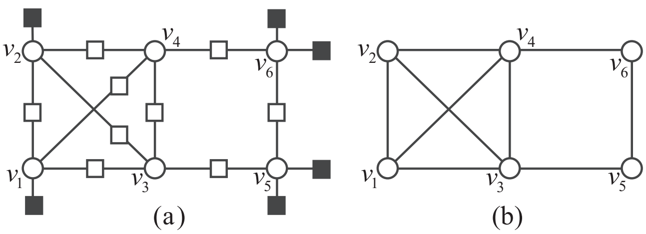

FIGURE 3 (a) A (6,6) trapping set and (b) its normal graph.

to 1 when k enlarges, as shown in Table 4. Note that $\mathcal{H}_s(G_{2n}(d'))$ can be considered as the parity-check matrix of a $(3, k)$-regular code.

5 | TRAPPING SETS OF DECOMPOSABLE LDPC CODES

The aim of this section is to characterize and analyze the trapping sets of decomposable LDPC codes. Let H be the parity-check of an LDPC code $\mathcal{C}$ and $G = TG(H)$ be the Tanner graph of $\mathcal{C}$ with vertex set $V = V_1 \cup V_2$, where V_1 and V_2 are the sets of variable nodes and check nodes of $\mathcal{C}$, respectively. For a subset S of V_1, let $N(S)$ be the check nodes in V_2 adjacent with some vertices in S and let $G[S]$ be the subgraph of G induced by the vertices of S, that is, the graph containing nodes $S \cup N(S)$ and edges $\{e = uv \in E(G) : u \in S, v \in N(S)\}$. Moreover, let $N_o(S)$ and $N_e(S)$ be the sets of check nodes in $N(S)$ with odd and even degrees in $G[S]$, respectively. By a (a, b)-*trapping set* in G, we mean a size-a subset $S \subseteq V$ such that $|N_o(S)| = b$. A (a, b)-trapping set S is called *elementary* if all nodes in $N(S)$ have degrees one or two. Elementary trapping sets (ETSs) are known to be the main cause of error floor in LDPC coding schemes [18, 19].

An ETS S is called regular if the degree of all nodes of S in $G[S]$ is the same. The *normal graph* of a regular (a, b)-ETS S is a graph on a vertices and obtained by removing all degree-1 check nodes and their edges from $G[S]$, and by replacing each degree-2 check node with an edge. In Figure 3, a (6,6)-trapping set and its associated normal graph is presented. Here, we use the notation τ to denote the set of all trapping sets S in a Tanner graph G whose induced subgraph $G[S]$ is connected and for which every node $v \in S$ is connected to at least two nodes in $N_e(S)$, that is, the degree of each vertex in the normal graph of G is at least two.

Definition 5.1. Let S and S' be two ETS in τ of size α and α', respectively, such that $S \subseteq S'$ and $\alpha < \alpha'$. We say that S' is a layered superset (LSS) of S if there exists a nested sequence of ETSs $S = S^{(0)} \subset S^{(1)} \subset \ldots \subset S^{(\alpha'-\alpha)} = S'$, such that $S^{(i)} \in \tau$ has size $\alpha + i$, for $i = 0, \ldots, \alpha' - \alpha$.

Example 5.2. The (6,6)-trapping set $S' = \{v_1, \ldots, v_6\}$ in Figure 3 is an LSS of $S = \{v_3, v_4, v_5, v_6\}$, because of the existence of the following nested LSS in τ:

$$S = S^{(0)} \subset S^{(1)} = \{v_1, v_3, v_4, v_5, v_6\} \subset S^{(2)} = S'.$$

However, S' can't be considered as an LSS of $S'' = \{v_1, v_2, v_3\}$, because for $i = 5, 6$, $S'' \cup \{v_i\} \notin \tau$. Moreover, $S^{(1)} = S'' \cup \{v_4\} \in \tau$, but $S^{(1)} \cup \{v_i\} \notin \tau$, for $i = 5, 6$.

Definition 5.3. Let G be the given graph and H be a proper subgraph of G with $|V(H)| \geq 2$. We say G is H-*generate* of order t if the vertices of $V(G) \setminus V(H)$ can be ordered as $u_1, u_2, \ldots, u_t$, such that if $H_1 = H$ and H_i $(i \geq 2)$ is the subgraph of G induced by $V(H) \cup \{u_1, \ldots, u_{i-1}\}$, then $\deg_{H_i}(u_i) \geq 2$, for each i, $1 \leq i \leq t$. In fact, G can be constructed from H by adding vertices $u_1, u_2, \ldots, u_t$ sequentially such that each vertex u_i has at least two endpoints belong to the earlier subgraph spanned by $V(H) \cup \{u_1, \ldots, u_{i-1}\}$.

Example 5.4. Let G be the graph shown in Figure 3b and let $H = G[v_3, v_4, v_5, v_6]$ be the subgraph of G induced by the vertices v_3, v_4, v_5, v_6. Clearly G is a H-generate graph, because G can be constructed from H by adding vertex v_1 to $H_1 = H$ and then v_2 to $H_2 = H_1[v_1]$ such that $\deg_{H_1}(v_1) = 2$ and $\deg_{H_2}(v_2) = 3$. However, G is not a H-generate for $H = G[v_1, v_2, v_3]$, because in any ordering of the remaining vertices v_4, v_5, v_6, some conditions of Definition 5.3 are not satisfied. For example, $\deg_H(v_4) = 3$ but $\deg_{H[v_4]}(v_i) = 1 < 2$, for $i = 5, 6$.

The layered superset relationship between an ETS and generated subgraphs is essential to the rest of the paper. Therefore, in the following proposition, we give a characterization for the elementary trapping sets and generate normal graphs.

Proposition 5.5. *Let S' and S, $S \subseteq S'$, be two ETS in τ with normal graphs H' and H, respectively. Then S' is an LSS of S if and only if H' is a H-generate graph.*

Proof. Let S' be an LSS of S with the corresponding chain $S = S^{(0)} \subset S^{(1)} \subset \dots \subset S^{(\alpha'-\alpha)} = S'$ and $S^{(i)}/S^{(i-1)} = \{u_i\}$. Since $S^{(i)} \in \tau$, there are at least two elements $x, y \in N_e(S^{(i)})$ adjacent to u_i. Then $\deg_{S^{(i)}}(x) = \deg_{S^{(i)}}(y) = 2$ and so there exist two elements $x', y' \in S^{(i-1)}$ adjacent to x, y, respectively. Hence, x', y' are the neighbors of u_i in the normal graph of $H'[S^{(i)}]$, which plays the same role as H_i in the definition of the H-generate graph. □

Now, in the following theorem, we give a characterization for the Elementary trapping sets of a decomposable code based on the chromatic index of the corresponding normal graph.

Theorem 3. *Let G be a simple graph with maximum degree k, $k \geq 3$. If G is a normal graph of an ETS of a 4-cycle free decomposable column-weight k LDPC code, then $\chi'(G) = k$.*

Proof. Since the parity-check matrix is decomposed by k binary matrix $H_1, H_2, \dots, H_k$ with column-weight 1, we can label all check nodes in H_i by color i. Thus, all check nodes are labeled by k colors. Now, we consider the same labeling in an ETS which means that all check nodes in an ETS are labeled by k colors. According to the definition of a decomposable LDPC code, it is easy to prove that there are no two check nodes with one variable node connected to those check nodes that have the same label. Through constructing a normal graph from an ETS, all 1-degree check nodes are removed and each 2-degree check node is replaced by an edge. Now, to color the edges of the normal graph, it is sufficient to consider the same labeling we have for check nodes. The label of each 2-degree check node is given to its corresponding edge in the normal graph. Therefore, the normal graph has a k-edge coloring which is proper because the girth of the Tanner graph is at least 6. Thus, the chromatic number is k. □

Gallager Codes [1] and LDPC Codes constructed in Sections 3 and 4 are some classes of 4-cycle free decomposable codes and the Elementary trapping sets of such codes can be completely determined by Theorem 1.

Consider the case where an ETS S of size a is an LSS of an ETS S' of size a'. Clearly, starting from S', the successive application of Proposition 5.5 will result in finding all ETSs which are layered supersets of S'. In the rest of the paper, we investigate the structures of all ETSs in column-weight three LDPC codes with girth values 6 and 8 and then, we study all non-isomorphic structures of different classes of (a, b)-ETSs. In Algorithm 1, for each class of ETSs with given values of a and b, we first find all non-isomorphic structures and then examine each of these structures to find out whether the structure is an LSS of any of its cycles. In this algorithm, for given even integer g, and positive integer $a_{\max}$ with $g < a_{\max}$, all l-cycles, that is, cycles of length l with their chords, will be generated in $\mathcal{C}_l$ and then $\mathcal{C}_l$ is used to construct all $\mathcal{C}_l$-generate graphs with at most $a_{\max}$ vertices having girth at least g. Then, the non-isomorph relation between the constructed graphs is checked and finally, the algorithm returns the number of all non-isomorphic (a, b)-ETS

ALGORITHM 1 Non-isomorphic Class I and Class II ETSs which are LSS of a Cycle.

Require: Let g be even and $a_{\max}$ be an integer with $g < a_{\max}$;
$\mathcal{G}_1 \leftarrow \emptyset$.
for l from g **to** $a_{\max}$ **do**
Let $\mathcal{C}_l$ be the set of all l-cycles with girth at least g and $T_l \leftarrow \emptyset$.
for i from 0 **to** $a_{\max} - l$ **do**
$T_l \leftarrow \{\mathcal{C}_l\text{-generate graphs of order } i\} \cup T_l$;
end for
if $l = g$ **then**
$\mathcal{G} \leftarrow T_l$;
else
$\mathcal{G}_1 \leftarrow \mathcal{G}_1 \cup T_{l-1}$ and $\mathcal{G} \leftarrow T_l \setminus \mathcal{G}_1$;
end if
for l from g **to** $a_{\max}$ **do**
for t from 0 **to** l **do**
$f_1(l, t) \leftarrow 0$ and $f_2(l, t) \leftarrow 0$;
end for
end for
for all $G \in \mathcal{G}$
$a \leftarrow |V(G)|$ and $b \leftarrow 3a - 2|E(G)|$;
if $\chi'(G) = \Delta(G)$ **then**
$f_1(a, b) \leftarrow f_1(a, b) + 1$;
else
$f_2(a, b) \leftarrow f_2(a, b) + 1$;
end if
for l from g **to** $a_{\max}$ **do**
for t from 0 **to** l **do**
if $f_1(l, t) > 0$ **then**
The number of $[l, t]$ ETS of Class I is $f_1(l, t)$;
end if
if $f_2(l, t) > 0$ **then**
The number of $[l, t]$ ETS of Class II is $f_2(l, t)$;
end if
end for
end for
end for

which are LSS of an l-cycle. The algorithm also return the number of all type I and type II (a, b)-ETSs. For each value of a, we mostly consider the values of b which satisfy $b/a \leq 1$. Having applied Algorithm 1, the results for column-weight three LDPC codes with girth values 6 and 8 are reported in Tables 5 and 6. For column-weight three LDPC codes with girths 6 and 8, the multiplicity of non-isomorphic structures are also listed in these tables, for different classes of ETSs. In these tables, $[a, b]_t^s$ is used to denote a (a, b)-ETS such that t is the number of all (a, b)-ETSs and s is the number of (a, b)-ETSs whose normal graphs having chromatic index 4. For a given (a, b), if all (a, b)-ETSs

TABLE 5 All non-isomorphic $[a, b]$ ETS for girth-6 column-weight 3 LDPC codes which are LSS of an L-cycle.

L	$[a, b]$	Total	Type II
6	$[3,3]_1, [4,0]_1, [4,2]_1, [5,1]_1^1$	4	1
8	$[4,4]_1, [5,3]_2, [6,0]_2, [6,2]_3, [7,1]_3^3$	11	3
10	$[5,5]_1, [6,2]_1, [6,4]_2, [7,1]_1^1, [7,3]_6, [8,0]_3, [8,2]_9, [9,1]_9^9$	32	10
12	$[6,4]_1, [6,6]_1, [7,3]_3, [7,5]_3, [8,0]_2, [8,2]_7, [8,4]_{12}, [9,1]_7^7, [9,3]_{31}^1, [10,0]_{12}^1, [10,2]_{50}^1, [11,1]_{50}^{50}$	188	60
14	$[7,5]_1, [7,7]_1, [8,2]_1, [8,4]_6, [8,6]_3, [9,1]_2^2, [9,3]_{18}, [9,5]_{19}, [10,0]_5, [10,2]_{34}, [10,4]_{75}, [11,1]_{35}^{35},$ $[11,3]_{210}^3, [12,0]_{43}^1, [12,2]_{347}^4, [13,1]_{346}^{346}$	1145	391
16	$[8,8]_1, [9,3]_4, [9,5]_{13}, [9,7]_4, [10,0]_1, [10,2]_{15}, [10,4]_{57}, [10,6]_{31}, [11,1]_{16}^{16}, [11,3]_{136}, [11,5]_{172},$ $[12,0]_{27}, [12,2]_{249}^1, [12,4]_{710}^1, [13,1]_{256}^{256}, [13,3]_{1957}^{18}, [14,0]_{271}^4, [14,2]_{3297}^{26}, [15,1]_{3273}^{3273}$	10489	3595

TABLE 6 All non-isomorphic $[a, b]$ ETS for girth-8 column-weight 3 LDPC codes which are LSS of an L-cycle.

L	$[a, b]$	Total	Type II
8	$[5,3]_1, [6,0]_1, [6,2]_1, [7,1]_1^1$	5	1
10	$[6,4]_1, [7,3]_2, [8,0]_1, [8,2]_3, [9,1]_3^3$	11	3
12	$[7,3]_1, [7,5]_2, [8,0]_1, [8,2]_2, [8,4]_6, [9,1]_1^1, [9,3]_{13}^1, [10,0]_5^1, [10,2]_{19}^1, [11,1]_{19}^{19}$	71	23
14	$[8,4]_2, [8,6]_2, [9,3]_4, [9,5]_{10}, [10,0]_1, [10,2]_7, [10,4]_{34}, [11,1]_4^4, [11,3]_{85}^1, [12,0]_{16}, [12,2]_{127}^1,$ $[13,1]_{127}^{127}$	420	133
16	$[9,5]_7, [9,7]_3, [10,2]_2, [10,4]_{21}, [10,6]_{19}, [11,3]_{29}, [11,5]_{88}, [12,0]_6, [12,2]_{47}^1, [12,4]_{324}^1, [13,1]_{30}^{30},$ $[13,3]_{792}^5, [14,0]_{92}^1, [14,2]_{1204}^6, [15,1]_{1204}^{1204}$	3869	1248
18	$[10,4]_4, [10,6]_{14}, [0,8]_3, [11,3]_6, [11,5]_9, [11,7]_{27}, [12,2]_{11}, [12,4]_{145}, [12,6]_{182}, [13,1]_4^4, [13,3]_{230}^2,$ $[13,5]_{981}^1, [14,0]_{14}, [14,2]_{323}^2, [14,4]_{3611}^4, [15,1]_{212}^{212}, [15,3]_{8948}^{21}, [16,0]_{668}^2, [16,2]_{13663}^{31}, [17,1]_{13663}^{3663}$	42719	3942

have chromatic index 3, we just write $[a, b]_t$. For example, in the first row of Table 5, $[4, 2]_1$ means that there is just one (4,2)-ETS whose normal graph has a chromatic index 3 and $[5, 1]_1^1$ means that there is just one (5,1)-ETS whose normal graph has a chromatic index 4. In these tables, by type II we mean an ETS whose normal graph has chromatic index 4. Note that by Theorem 3, a (a, b)-ETS of Type II is avoidable in a decomposable LDPC code.

6 | QC-LDPC CODES WITH GIRTHS 20 AND 24

In this section, we prove that the parity-check matrices of the constructed column-weight 3 LDPC codes can be considered as the mother matrix of some column-weight three QC-LDPC code with girth at most 20 or 24, depending on G and edge coloring ϱ of G.

For given positive integers s and N, $0 \leq s \leq N-1$, we use $\mathcal{I}_N^s$ to denote the $N \times N$ matrix obtained from $N \times N$ identity matrix $\mathcal{I}_N$ by shifting each column s positions to the left. In the other words, $\mathcal{I}_N^0 = \mathcal{I}$ and for $1 \leq s \leq N-1$,

$$\mathcal{I}_N^s = \begin{pmatrix} 0 & \mathcal{I}_s \\ \mathcal{I}_{N-s} & 0 \end{pmatrix}.$$

A (m, N)-*slope vector* is a length-$2m$ vector S such that each component of S belongs to $\{0, 1, \ldots, N-1\}$. Now, consider an n-vertex graph G on m edges and let ϱ denote proper or strong edge coloring of G with t colors.

For the (m, N)-slope vector $S = (s_1, s_2, \ldots, s_{2m})$, let $\mathcal{H}_\varrho(G, (m, N), S)$ be the binary $(n+t)N \times mN$ matrix obtained from $\mathcal{H}_\varrho(G)$ by replacing the non-zero elements in the j-th column, $1 \leq j \leq m$, from top to bottom in order of placement, by permutation matrices $\mathcal{I}$, $\mathcal{I}^{s_{2j-1}}$ and $\mathcal{I}^{s_{2j}}$, respectively. Clearly, $\mathcal{H}_\varrho(G, (m, N), S)$ is the parity-check matrix of a column-weight 3 QC-LDPC code, denoted by $\mathcal{C}_\varrho(G, (m, N), S)$, with design rate $\mathcal{R} = 1 - \frac{n+t}{m}$.

In [33], Kim et al. proved that the maximum achievable girth of the QC-LDPC codes based on the mother matrix H with $g(H) \geq g$, is at least $3g$. It is worth notice that the maximum achievable girth means the maximum of the girth that can be achieved from the QC-LDPC codes with mother matrix H. Therefore, $\mathcal{C}_\varrho(G, (m, N), S)$ can achieve girth 18 if ϱ is a proper edge coloring of G and $\mathcal{C}_\varrho(G, (m, N), S)$ can achieve girth 24 if ϱ is a strong edge coloring and $g(\mathcal{H}_\varrho(G)) \geq 8$. In the sequel, we prove that if G is a triangle-free graph with a proper edge coloring p, then the constructed QC-LDPC code $\mathcal{C}_p(G, (m, N), S)$ has maximum achievable girth 20. Interestingly, the constructed codes with girth 20 have smaller lengths rather than the QC-LDPC codes with the same girths used by Bocharova [35]. To determine the maximum achievable girth of $\mathcal{C}_p(G, (m, N), S)$, we need the following theorem.

Theorem 4 [33]. *Let* $\mathbf{H}$ *be a binary matrix and let* $\mathfrak{F}$ *be the class of QC-LDPC codes having* $\mathbf{H}$ *as the mother matrix. If*

$g(\mathfrak{F}) := \max_{C\in\mathfrak{F}}\, g(C)$, then $g(\mathfrak{F}) \ge 20$, if $\mathbf{H}$ *does not contain any row-column permutation of the following matrices and also their transposes ($g(\mathfrak{F})$ is the maximum achievable girth of the QC-LDPC codes with mother matrix* $\mathbf{H}$*):*

$$P_{12} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, P_{14} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, P_{16,1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$P_{16,2} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, P_{18,1} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$P_{18,2} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Theorem 5. *Let G be an n-vertex graph and let p be a proper edge coloring of G with t colors. If G is triangle-free, then the constructed QC-LDPC codes with the mother matrix $\mathcal{H}_p(G)$ have the maximum achievable girth of at least 20.*

Proof. Let g denote the maximum achievable girth of the QC-LDPC codes with the base matrix $\mathcal{H}_p(G)$. To show $g \ge 20$, by Theorem 4, it is sufficient to prove that $\mathcal{H}_p(G)$ dose not contain $P_{12}, P_{14}, P_{16,1}, P_{16,2}, P_{18,1}, P_{18,2}$ nor their transposes. Since p is a proper edge coloring of G and $\mathcal{H}_p(G)$ is free of 4-cycles, $\mathcal{H}_p(G)$ cannot contain any row-column permutation of P_{12}, P_{14}, $P_{16,1}$, $P_{16,2}$, $P_{18,1}$ and also their transposes.

On the other hand, $\mathcal{H}_p(G)$ contains $P_{18,2}$, or $P^T_{18,2}$, if and only if there exist some blocks B_1, B_2, B_3 and B_4 of $\boldsymbol{B}_p(G)$ such that $B_1 \cap B_2 \cap B_3 = \{i\}$ for some $v_i \in V$, and if $B_4 = \{j, k, h\}$ then $\{i, j\} \subset B_1, \{i, k\} \subset B_2$ and $\{i, h\} \subset B_3$. Now, if $i > n$ and $i = n + i_1$ (i.e. i_1 is a color), then $j, k, h \le n$ (i.e. v_j, v_k, v_h correspond to some vertices of G), because each of blocks B_1, B_2 and B_3 contains exactly one element greater than n. Thus, all elements of B_4 are less than $n + 1$, a contradiction. So, let $i \le n$. As, B_4 have one element greater than n, let $j, k \le n$ and $h > n$, $h = n + h_1$ (i.e. h_1 is a color and v_j, v_k are vertices of G). Then, $v_j v_k$ is an edge of G with color h_1. This means that vertices v_i, v_j, v_k form a triangle in G, which is impossible. This contradiction shows that $g \ge 20$ and this completes the proof. □

In [30], the authors provided a deterministic algorithm generating all (m, N)-slope vectors S, such that $C_\varrho(G, (m, N), S)$ has girth at least $2g, g \ge 3$. Applying the proposed algorithm in [30], for graphs Q_3, K_6, $K_{4,4}$, $G_{14}(1, 5, 13)$, Q_4 and $G_{26}(1, 5, 17, 25)$ with $m = 12, 15, 16, 21, 32, 52$ edges, respectively, and N chosen by a computer search, Tables 7 and 8 presents some (m, N)-slope vectors S such that the corresponding codes $C_\varrho(G, (m, N), S)$ have girths $g = 8, 10, 12, 14, 16, 18, 20$. As shown in Table 2, in some cases, the constructed codes have better lengths and minimum distances compared to the codes

TABLE 7 QC-LDPC codes based on proper edge coloring of graph G having mother matrix $\mathcal{H}_p(G)$.

Graph	Block size	Girth	Length	Slope vector
$K_{4,4}$	10258	20	164128	[0,0,0,0,0,0,0,0,0,1,3,7,16,30,40,87,0,106,67,219,294,550,684,1209,0,1327,320,2033,910,3373,3853,7290]
	2990	18	47840	[0,0,0,0,0,0,0,0,0,1,3,7,12,20,30,44,0,65,35,125,167,242,335,496,0,590,82,777,279,1161,1213,2077]
	724	16	11584	[0,0,0,0,0,0,0,0,0,1,3,7,9,19,21,32,0,17,9,45,55,105,92,174,0,135,30,141,84,306,303,497]
	219	14	3504	[0,0,0,0,0,0,0,0,0,1,2,4,3,7,10,15,0,17,3,22,26,37,51,8,0,27,9,67,28,102,127,157]
	30	12	480	[0,0,0,0,0,0,0,0,0,1,2,3,4,5,6,7,0,8,2,6,6,11,21,27,0,22,19,18,7,16,19,7]
	14	10	224	[0,0,0,0,0,0,0,0,0,1,2,3,4,5,6,7,0,3,3,7,1,5,12,9,0,10,5,10,2,9,8,4]
	3	8	48	[0,0,0,0,0,0,0,0,0,1,0,1,0,2,0,2,0,1,2,2,2,1,1,2,0,1,1,1,2,2,2,1]
$G_{14}(1, 5, 13)$	9710	20	203910	[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,3,7,9,27,22,61,26,102,68,194,56,310,396,0,454,528,599,1250,636,2061,881,3319,1172,4981,1479,7192]
	2771	18	58191	[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,2,5,7,20,9,45,12,75,24,128,19,194,197,0,202,230,256,535,218,805,215,1140,262,1592,309,2203]
	674	16	14154	[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,2,5,7,15,11,29,17,50,20,82,22,113,23,0,115,19,55,100,70,152,94,282,78,298,60,460]
	175	14	3675	[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,2,3,3,5,4,8,5,16,6,15,7,19,18,0,20,25,27,61,17,66,11,95,22,129,26,147]
	31	12	651	[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,2,1,4,1,6,2,9,2,10,2,13,14,0,13,5,15,12,8,15,8,9,6,26,7,18]
	16	10	336	[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,2,1,4,1,6,1,8,1,10,1,12,2,1,2,0,3,7,2,12,3,8,2,13,3,6]
	3	8	63	[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,1,0,1,0,1,1,1,1,1,1,2,0,2,0,2,0,1,2,1,2,1,2,1,2]

TABLE 8 QC-LDPC codes based on strong edge coloring of graph G having mother matrix $\mathcal{H}_s(G)$.

Graph	Block size	Girth	Length	Slope vector
$G_{26}(1,5,17,25)$	6008	20	312416	[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,2,0,0,0,1,0,1,0,2,0, 0,0,1,0,1,0,2,0,2,3,8,7,16,11,23,15,32,26,51,35,63,54,74,68,108,40,85,91,143, 113,132,106,176,57,170,177,362,256,505,278,584,388,811,513,1010,578,1219, 751,1482,857,1626,873,1850,1089,1990,542,1941,1033,2618,1428,3064]
	1467	18	76284	[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,2,0,0,0,1,0,1,0,2,0, 0,0,1,0,1,0,2,0,2,3,8,7,13,4,16,8,23,3,21,25,36,32,52,39,46,43,57,13,28,48,54,35, 56,39,73,39,118,108,224,127,233,135,252,155,321,138,282,237,427,245,469, 286,537,193,482,315,593,345,518,445,820]
	325	16	16900	[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,2,0,1,0,0,0,2,0, 1,0,0,0,2,0,1,0,3,0,3,2,10,4,15,0,5,4,14,1,15,9,20,17,23,6,19,18,8,15,21,7,25,20, 35,24,59,28,69,25,64,41,90,45,78,60,126,47,95,72,140,93,166,98,154,31,140, 65,148,26,133]
	70	14	3640	[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,2,0,1,0,0,0,2,0, 1,0,0,0,2,0,1,0,3,0,3,0,6,1,4,1,5,0,9,1,5,1,6,9,12,4,8,3,10,5,11,4,11,9,6,8,20,13,27, 2,22,19,16,15,36,10,26,27,42,26,48,24,45,18,42,23,37,27,47,33,13]
	16	12	832	[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,0, 0,0,1,0,0,0,1,0,1,0,1,0,1,0,1,0,1,0,1,1,2,2,3,1,2,2,4,1,3,2,1,1,3,2,1,2,6,3,7,2,6,1,7, 6,12,1,9,2,7,1,8,2,5,3,8,15,7,12,6,2,12]
	5	10	260	[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,0, 0,0,1,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,2,0,2,0,2,1,3,0,1,1,2,0,4,0,4,0,2,3,2,1,2,3,4,2,3, 4,3,1,4,2,4,2,4,3,1,3,3,3,1,2,4,2,1]

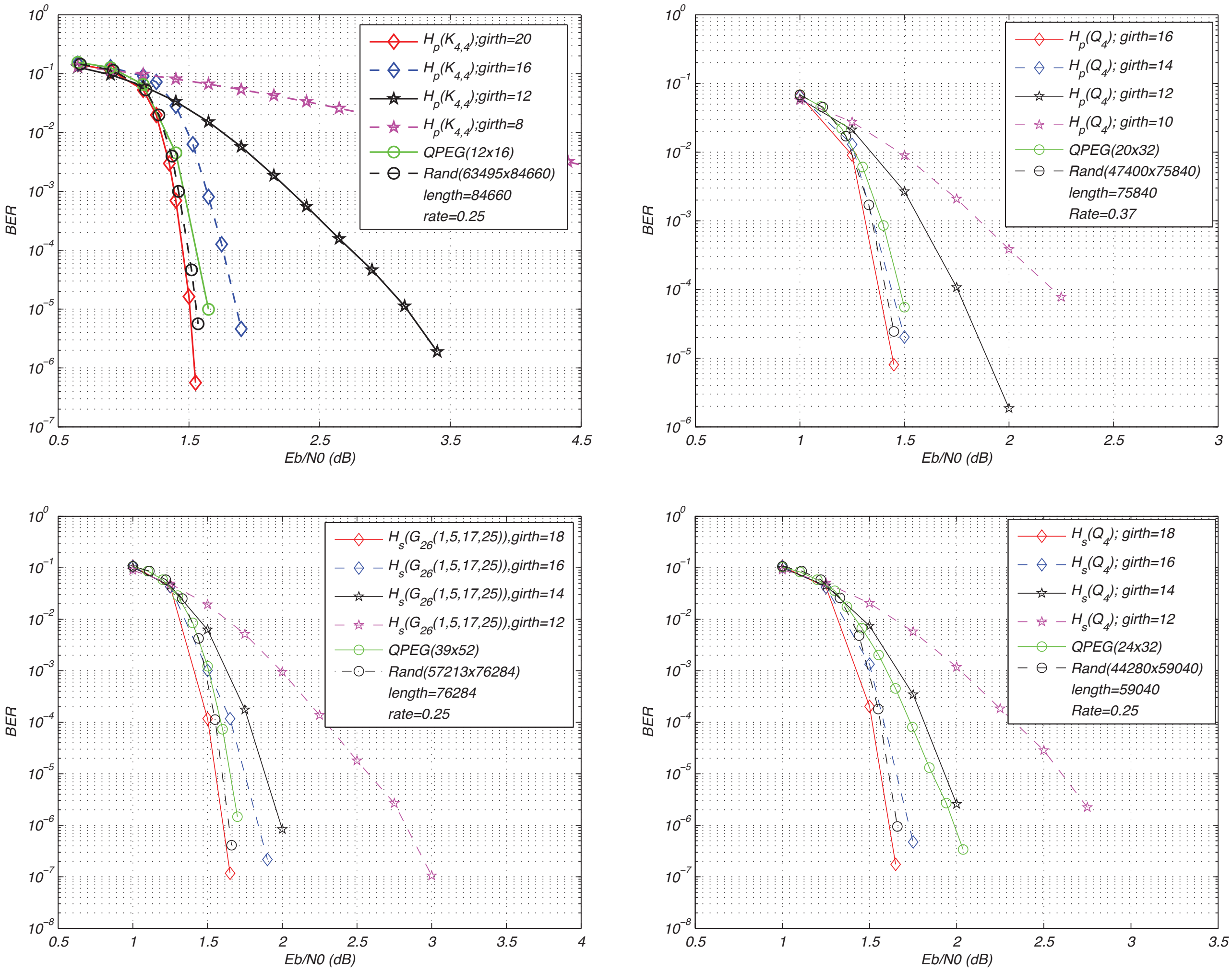

FIGURE 4 The BER performances of the constructed codes with different girths having mother matrices $\mathcal{H}_p(G)$ and $\mathcal{H}_s(G)$ with maximum iteration number 20.

constructed in [35]. For example, the length of the constructed girth-20 QC-LDPC code with parity-check matrix $\mathcal{H}_p(K_{4,4})$ is remarkably less than the length of girth-20 voltage graph based QC-LDPC codes [35] with the same regularity.

7 | SIMULATION RESULTS

Using software available online [44], we have obtained simulation results on additive white Gaussian noise (AWGN) channel with BPSK modulation. The decoding algorithm is the sum-product algorithm under the constraints of a maximum of 20 iterations. The random-like MacKay codes [3] constructed by this software, have no 4-cycles in their Tanner graphs.

In Figure 4, we used QPEG ($a \times b$) to denote the QC-LDPC code obtained by applying the algorithm presented in [30] to a $a \times b$ base matrix generated by PEG having girth at least 6. In addition, $Rand(a \times b)$ denotes a random-like code represented by a $a \times b$ parity-check matrix with girth of at least 6 and column-weight three. Applying the presented algorithm in [30] on the base matrices constructed by the proper edge colorings of K_6 and Q_4 and the strong edge colorings of Q_4 and $G_{26}(1, 5, 17, 25)$ some QC-LDPC codes with girths 12, 14, 16, 18 are constructed. In this figure for a given graph G, $\mathcal{H}_p(G)$ and $\mathcal{H}_s(G)$ are used to denote the parity-check matrices of the constructed codes based on proper edge coloring p and strong edge coloring s of G.

For a fixed graph G, a performance comparison among the constructed QC-LDPC codes with different girths and their random-like counterparts and QPEG are given in Figure 4. The figure confirms the superiority of the constructed codes having large girth with respect to the random like codes and QPEG and also that the girth has a direct impact on the code performance.

AUTHOR CONTRIBUTIONS

G. Raeisi: Writing—review & editing. **M. Gholami**: Writing—review & editing.

ACKNOWLEDGEMENTS

The authors would like to thank the anonymous referees for their valuable comments that improved the quality of the manuscript.

CONFLICT OF INTEREST STATEMENT

The authors have no conflicts of interest to declare. All authors have seen and agree with the contents of the manuscript and certify that the submission is original work and is not under review at any other publication.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

ORCID

G. Raeisi https://orcid.org/0000-0003-0313-8059

REFERENCES

1. Gallager, R.G.: Low-Density Parity-Check Codes. MIT Press, Cambridge, MA (1963)
2. Song, H., Liu, J., Kumar, B.V.: Large girth cycle codes for partial response channels. IEEE Trans. Magn. 40(4), 3084–3086 (2004)
3. MacKay, D.J.C.: Good error-correcting codes based on very sparse matrices. IEEE Trans. Inform. Theor. 45(2), 399–431 (1999)
4. Ryan, W., Lin, S.: Channel Codes: Classical and Modern. Cambridge University Press, Cambridge (2009)
5. Tanner, R. M.: A recursive approach to low complexity codes. IEEE. Trans. Inform. Theor. 27(5), 533–547 (1981)
6. Ryan, W.E., Lin, S.: Channel Codes: Classical and Modern. Cambridge University Press, New York (2009)
7. MacKay, D., Postol, M. S.: Weaknesses of Margulis and Ramanujan-Margulis low-density parity-check codes. Electron. Notes Theoretical Comput. Sci. 74, 97–104 (2003)
8. Laendner, S., Milenkovic, O.: LDPC codes based on Latin squares: Cycle structure, stopping set, and trapping set analysis. IEEE Trans. Commun. 55(2), 303–312 (2007)
9. McGregor, A., Milenkovic, O.: On the hardness of approximating stopping and trapping sets. IEEE Trans. Inf. Theor. 56(4), 1640–1650 (2010)
10. Ivkovic, M., Chilappagari, S.K., Vasic, B.: Designing LDPC codes without small trapping sets by using Tanner graph covers. In: IEEE International Symposium on Information Theory, pp. 2266–2270. IEEE, Piscataway (2007)
11. Ivkovic, M., Chilappagari, S.K., Vasic, B.: Eliminating trapping sets in low-density parity-check codes by using Tanner graph covers. IEEE Trans. Inf. Theor. 54(8), 3763–3768 (2008)
12. Han, Y., Ryan, W.E.: Low-floor detection/decoding of LDPC coded partial response channels. IEEE J. Sel. Areas Commun. 28(2), 252–260 (2010)
13. Cavus, E., Haymes, C.L., Daneshrad, B.: Low BER performance estimation of LDPC codes via application of importance sampling to trapping sets. IEEE Trans. Commun. 57(7), 1886–1888 (2009)
14. Amirzade, F., Sadeghi, M.R., Panario, D.: QC-LDPC construction free of small size elementary trapping sets based on multiplicative subgroups of a finite field. Adv. Math. Commun. 14(3), 397–411 (2020)
15. Sadeghi, M.R., Amirzade, F.: Edge coloring technique to analyze elementary trapping sets of spatially-coupled LDPC convolutional codes. IEEE Comm. Lett. 24(4), 711–715 (2020)
16. Amirzade, F., Sadeghi, M.R., Panario, D.: QC-LDPC codes with large column weight and free of small size ETSs. IEEE Comm. Lett. 26(3), 500–504 (2022)
17. Karimi, M., Banihashemi, A.H.: On characterization of elementary trapping sets of variable-regular LDPC codes. IEEE Trans. Inform. Theor. 60(9), 5188–5203 (2014)
18. Ivkovic, M., Chilappagari, S.K., Vasic, B.: Eliminating trapping sets in low-density parity-check codes by using Tanner graph covers. IEEE Trans. Inf. Theor. 54(8), 3763–3768 (2008)
19. Laendner, S., Hehn, T., Milenkovic, O., Huber, J.B.: The trapping redundancy of linear block codes. IEEE Trans. Inf. Theor. 55(1), 53–63 (2009)
20. Hashemi, Y., Banihashemi, A.H.: 2015. On characterization and efficient exhaustive search of elementary trapping sets of variable-regular LDPC codes. IEEE Comm. Lett. 19(3), 323–326 (2015)
21. MacKay, D.J.C., Neal, R.M.: Near Shannon limit performance of low density parity-check codes. Electron. Lett. 33(6), 1645–1646 (1996)
22. Tian, T., Jones, C.R., Villasenor, J.D., Wesel, R.D.: Selective avoidance of cycles in irregular LDPC code construction. IEEE Trans. Commun. 50(8), 1242–1247 (2004)
23. Di, C., Proietti, D., Telatar, I.E., Richardson, T.J., Urbanke, R.L.: Finite length analysis of low-density parity-check codes on the binary erasure channel. IEEE Trans. Inform. Theor. 48(6), 1570–1579 (2002)
24. Hu, X.Y., Eleftheriou, E., Arnold, D.M.: Regular and irregular progressive edge-growth Tanner graphs. IEEE Trans. Inf. Theor. 51(1), 386–398 (2005)

25. Kou, Y., Lin, S., Fossorier, M.P.: Low-density parity-check codes based on finite geometries: A rediscovery and new results. IEEE Trans. Inf. Theor. 47(7), 2711–2736 (2001)
26. Johnson, S.J., Weller, S.R.: Codes for iterative decoding from partial geometries. IEEE Trans. Commun. 52(2), 236–243 (2004)
27. Kamiya, N.: High-rate quasi-cyclic low-density parity-check codes derived from finite affine planes. IEEE Trans. Inform. Theor. 53(4), 1444–1459 (2007)
28. Vasic, B., Milenkovic, O.: Combinatorial constructions of low-density parity-check codes for iterative decoding. IEEE Trans. Inform. Theor. 50(6), 1156–1176 (2004)
29. Esmaeili, M., Gholami, M.: Structured QC-LDPC codes with girth 18 and column-weight $J \geq 3$. Int. J. Electron. Commun. (AEUE) 64, 202–217 (2010)
30. Gholami, M., Samadieh, M., Raeisi, G.: Column-weight three QC-LDPC codes with girth 20. IEEE Commun. Lett. 17(7), 1439–1442 (2013)
31. Thorpe, J.: Low-density parity-check (LDPC) codes constructed from protograph. IPN Progr. JPL. Rep. 42(154), 42–154 (2003)
32. Lin, S., Song, S., Tai, Y.Y., Lan, L., Zeng, L.: Algebraic constructions of nonbinary quasi-cyclic LDPC codes. In: Proceedings of IEEE International Symposium on Information Theory. IEEE, Piscataway (2006)
33. Kim, S., No, J.S., Chung, H., Shin, D.J.: Quasi-cyclic low-density parity-check codes with girth larger than 12. IEEE Trans. Inform. Theor. 53(8), 2885–2891 (2007)
34. Gross, J.L.: Voltage graphs. Discrete Math. 9(3), 239–246 (1974)
35. Bocharova, I.E., Hug, F., Johannesson, F., Kudryashov, B.D., Satyukov, R.V.: Searching for voltage graph-based LDPC tail-biting codes with large girth. IEEE Trans. Inform. Theor. 58, 2265–2279 (2012)
36. Colbourne, C., Jeffrey, D.: Handbook of Combinatorial Designs. CRC Press, Boca Raton, FL (2007)
37. West, D.B.: Introduction to Graph Theory. Prentice Hall, Upper Saddle River (2001)
38. Stiebitz, M., Scheide, D., Toft, B., Favrholdt, L.M.: Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture, Vol. 75. John Wiley & Sons, Hoboken, NJ (2012)
39. Vizing, V.G.: On an estimate of the chromatic class of a p-graph. Diskret. Analiz. 3, 25–30 (1964)
40. Fouquet, J.L., Jolivet, J.L.: Strong edge coloring of graphs and applications to multi-k-gons. Ars Combinatoria 16, 141–150 (1983)
41. Faudree, R.J., Schelp, R.H., Gyárfás, A., Tuza, Z., The strong chromatic index of graphs. Ars Combinatoria 29, 205–211 (1990)
42. Dimitromanolakis, A.: Analysis of the Golomb ruler and the Sidon setproblems, and determination of large, near-optimal golomb rulers. Master's Thesis, Technical University of Crete, Chania (2002)
43. Raeisi, G., Gholami, M.: A Simple method for constructing k6-graphs using modular Golomb rulers, preprint.
44. Software for low density parity check (LDPC) codes. http://www.cs.utoronto.ca/~radford/ldpc.software.html

How to cite this article: Raeisi, G., Gholami, M.: On a class of column-weight 3 decomposable LDPC codes with the analysis of elementary trapping sets. IET Commun. 18, 583–596 (2024). https://doi.org/10.1049/cmu2.12762