Automatica 116 (2020) 108898

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Consensus of multi-agent systems via fully distributed event-triggered control☆

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ARTICLE INFO

Article history:
Received 12 October 2018
Received in revised form 23 January 2020
Accepted 30 January 2020
Available online 14 March 2020

Keywords:
Consensus
Multi-agent systems
Fully distributed protocol
Event-triggering control
Adaptive control

ABSTRACT

This article studies consensus of linear multi-agent systems (MASs) on undirected graphs. An adaptive event-triggering protocol is constructed for consensus control by using relative information between agents. Sufficient conditions are established for consensus of linear MASs without and with external disturbances, respectively. Zeno behavior is proved to be excluded. Moreover, a self-triggered realization based on sampled information is formulated for the protocol. Since the proposed protocol incorporates both adaptive control and event-triggered control, it can be implemented in a *fully distributed* way and only makes use of the *sampled relative* information between neighboring agents. Two numerical examples are finally provided for demonstrating the effectiveness and advantages of the theoretical results.

1. Introduction

Coordination of multi-agent systems (MASs) is definitely one of the research hotspots in systems and control over the last decade (Knorn, Chen, & Middleton, 2016; Qin, Ma, Yu, & Wang, 2018; Tang, Gao, Zhang, & Kurths, 2015; Wen, Yang, Luo, Wang, & Pan, 2019; Zhang, Xu, Karimi, Wang, & Yu, 2018). Consensus, as a fundamental control problem of MASs, is to design a distributed control protocol such that a group of autonomous agents reach an agreement in some sense, which can find many potential applications in, for instance, distributed optimization, spacecraft coordination, opinion dynamics and robot formation (Meng, Dimarogonas, & Johansson, 2017; Xia, Cao, & Johansson, 2016; Yang et al., 2019). In recent years, a great number of results have been reported on consensus control of various MASs (Knorn et al., 2016; Wen et al., 2019). One of the basic paradigms for designing distributed protocols for MASs is first to find some gain matrices related to local agent dynamics and then to determine some parameter(s) related to the communication graph. Such a design process can be applied not only to state-feedback protocols (Tuna, 2008), but also to kinds of output-feedback protocols (Li, Duan, Chen, & Huang, 2010; Li, Soh, & Xie, 2017, 2019a; Li, Soh, Xie, & Lewis, 2019b).

To design a protocol, the most important graph-related parameter might be the smallest nonzero eigenvalue of the Laplacian of the communication graph (Li et al., 2010, 2017, 2019a; Tuna, 2008). However, when the network scale is large, it may be quite tough to obtain such global information. In other words, the designed protocols are not scalable. For this reason, one of the recent trends in studying MASs is to devise adaptive mechanisms for tuning the coupling gains with no need of knowing and dealing with the precise Laplacian (DeLellis, diBernardo, & Garofalo, 2009; Li, Ren, Liu, & Xie, 2013; Mei, Ren, & Chen, 2016). In Li et al. (2013), adaptive consensus protocols are proposed for linear MASs on undirected graphs; in Mei et al. (2016), adaptive consensus of second-order MASs with unknown inertias is studied and several adaptive protocols with absolute or relative velocity feedback are proposed.

In many applications of MASs, each agent is usually equipped with limited energy storage and communication resources. Thus, how to reduce the occupation of network resources is of practical significance. Note that the aforementioned results are based on continuous connection between neighboring agents. Thus, a natural idea to reduce the consumption of communication resources

☆ This work is supported in part by the Natural Science Foundation of China under Grants 61903250, 61673176 and 61751305, and in part by the 111 Project under Grant B17017. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Dimos V. Dimarogonas under the direction of Editor Christos G. Cassandras.

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https://doi.org/10.1016/j.automatica.2020.108898
0005-1098/

is to design protocols which make use of the sampled, rather than continuous-time, knowledge between neighboring agents. In particular, many efforts recently have been made to apply the advanced event-triggered sampling techniques to consensus control of MASs (Cheng & Ugrinovskii, 2016; Dimarogonas, Frazzoli, & Johansson, 2012; Fan, Feng, Wang, & Song, 2013; Hu, Liu, & Feng, 2016; Li, Liao, Huang, & Zhu, 2015; Liu, Cao, Persis, & Hendrick, 2017; Yang, Ren, Liu, & Chen, 2016; Zhang, Tang, Liu, & Kurths, 2017; Zhu & Jiang, 2015). In event-triggered control, the violation of some state- and/or time-dependent conditions will trigger a sampling event and then the controller will update the feedback signal with the newly sampled information (Heemels, Johansson, & Tabuada, 2012). According to the sampled information, two types of event-triggering conditions can be recognized for the results on MASs. The first type depends on the absolute information about neighboring agents (see Dimarogonas et al., 2012; Liu et al., 2017; Yang et al., 2016), while the other one on the relative information between neighboring agents (see Cheng & Ugrinovskii, 2016; Fan et al., 2013; Hu et al., 2016; Li et al., 2015; Zhu & Jiang, 2015). However, note that, in some applications (e.g., deep-space exploration Smith & Hadaegh, 2005), the absolute information about agents cannot be precisely measured, while the relative information between agents can. In such a case, the results of the former type cannot be applied, while those of the latter type still can.

It is worth pointing out that the design conditions in the above results on event-triggered consensus still need to know the precise Laplacian. This drawback makes it difficult to apply these results on event-triggered consensus to MASs composed of a large number of agents. Thus, designing fully distributed event-triggering protocols with no need to deal with the Laplacian is important for keeping the protocol scalability. Accordingly, an emerging topic about MASs is to integrate both adaptive control and event-triggered control to achieve scalable and sampled-data feedback control (Cheng & Li, 2019; Qian, Liu, & Feng, 2019; Wang, Zhou, & Zhu, 2018; Yang, Zhang, Feng, & Yan, 2018; Zhu, Zhou, & Wang, 2018). In Qian et al. (2019) and Yang et al. (2018), adaptive event-triggered protocols are constructed for cooperative output regulation of heterogeneous linear MASs on undirected and directed graphs, respectively. Consensus of homogeneous linear MASs is studied in Cheng and Li (2019) and Wang et al. (2018) under state-feedback adaptive event-triggered protocols. However, it should be emphasized that all the results in Cheng and Li (2019), Qian et al. (2019), Wang et al. (2018) and Yang et al. (2018) are based on absolute information sensing, which, as aforementioned, are inapplicable when absolute information is difficult to measure in some applications. An exception is Zhu et al. (2018), which investigates consensus of linear MASs under an adaptive event-triggered protocol with relative information sensing. Nevertheless, it should be pointed out that the results in Zhu et al. (2018) are still rather restrictive. Roughly speaking, the feasibility of the additional inequality in Zhu et al. (2018, Theorem 1) has no general guarantee. Consequently, consensus of linear MASs has *not* been well investigated under adaptive event-triggered protocols with relative information sensing, which motivates this work. It should be further pointed out that the existence of event-triggered protocols based on absolute information sensing generally do not imply that based on relative information sensing, and vise versa. Thus, one usually cannot directly establish the relative information based counterpart from the existing results based on absolute information (e.g., Cheng & Li, 2019). Please refer to Remark 2 for more detailed explanations.

In this paper, we study the consensus problem of homogeneous linear MASs on undirected graphs. An adaptive event-triggered state-feedback protocol is proposed for consensus control without need to know and deal with the Laplacian of the communication graph. Sufficient conditions are established for consensus of linear MASs in the absence and presence of external disturbances, respectively. We also prove that Zeno behavior is excluded in the triggering process, and further provide a formulation of the relative states based on sampled information such that the proposed protocol can be realized in the self-triggered manner. Finally we present two numerical simulations for illustrating the effectiveness of the proposed theoretic results.

Compared with the relevant results in the literature, the main contributions of this article are threefold:

(1) A new adaptive event-triggered protocol is proposed for consensus of linear MASs. Compared with the existing event-triggered results in Cheng and Ugrinovskii (2016), Dimarogonas et al. (2012), Fan et al. (2013), Hu et al. (2016), Li et al. (2015), Liu et al. (2017), Yang et al. (2016), Zhang et al. (2017) and Zhu and Jiang (2015), global information in terms of the Laplacian is not needed in the design conditions. Thus, the protocol can be implemented in a *fully distributed* way.

(2) Different from the results in Cheng and Li (2019), Qian et al. (2019), Wang et al. (2018) and Yang et al. (2018) that need to know absolute information about agents, the proposed protocol is completely based on *relative information* between neighboring agents. Thus, it is applicable for the case where relative information is available while absolute information is not.

(3) The consensus conditions in the article are *feasible* under the common stabilizability assumption for agents, while that in Zhu et al. (2018, Theorem 1) does not have this guarantee. Thus, our results are applicable for more general linear MASs.

Notation. $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. $\mathbf{1}_n$ represents the n-dimensional column vector with all the entries being one and $\mathbf{I}_n$ is the $n \times n$ identity matrix (the subscript is omitted if the context is clear) and $\mathbb{N}$ is the set of all nonnegative integers. The symbol $\otimes$ stands for Kronecker product of two matrices. For a symmetric matrix A, $\lambda_{\mathrm{M}}(A)$ and $\lambda_{\mathrm{m}}(A)$ denote its maximum and minimum eigenvalues, respectively. $\|(\cdot)\|$ means the 2-norm of a vector $(\cdot)$. diag$\{\cdots\}$ means a (block) diagonal matrix with "$\cdots$" on the diagonal.

Denote by $\mathscr{G}(\mathscr{V},\mathscr{E})$ an undirected graph of N nodes, where $\mathscr{V}=\{1,\ldots,N\}$ is the node set and $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}$ is the edge set. Node j is said to be a neighbor of node i, if $(j,i)\in\mathscr{E}$ (thus, i is also a neighbor of j, since the graph is undirected). Denote by $\mathscr{N}_i$ the set of neighboring nodes of node i. Each edge of a graph is assigned with a weight a_{ij} such that $a_{ij}>0$ if $(j,i)\in\mathscr{E}$ and $a_{ij}=0$ otherwise. Then the adjacency matrix and Laplacian associated with the graph are denoted, respectively, by $\mathscr{A}=[a_{ij}]_{N\times N}$ and $\mathscr{L}=\left[l_{ij}\right]_{N\times N}$ with $l_{ii}=\sum_{k\in\mathscr{N}_i}a_{ik}$ and $l_{ij}=-a_{ij}$ for $(j,i)\in\mathscr{E}$. A path of a graph is an ordered sequence of edges connecting two nodes. An undirected graph is said to be connected if every node can be reached from every other node through any path.

Lemma 1 (*Fiedler, 1975*)**.** *For an undirected, connected graph* $\mathscr{G}$, $\lambda_1=0$ *is the simple zero eigenvalue of the Laplacian and* $\lambda_2=\min_{\mathbf{1}^{\mathrm{T}}x=0,\,x\neq0}\frac{x^{\mathrm{T}}\mathscr{L}x}{x^{\mathrm{T}}x}>0$ *is the smallest nonzero eigenvalue of the Laplacian.*

2. Problem statement

Consider N $(N\geq2)$ linear time-invariant agents as

$$\dot{x}_i(t)=Ax_i(t)+B_uu_i(t),\quad i=1,\ldots,N, \tag{1}$$

where $x_i\in\mathbb{R}^{n_x}$ and $u_i\in\mathbb{R}^{n_u}$ are the state and control input of agent i, respectively, and $A\in\mathbb{R}^{n_x\times n_x}$ and $B_u\in\mathbb{R}^{n_x\times n_u}$ with (A,B_u) stabilizable are known system matrices. The consensus problem is to design a feedback protocol such that the resulting closed-loop system satisfies $\lim_{t\to\infty}\left\|x_i(t)-x_j(t)\right\| = 0$, $i,j = 1,\ldots,N$.

Given an undirected communication graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, the relative state that can be accessed by agent i is given by

$$\tilde{x}_i(t) \triangleq \sum_{j\in\mathcal{N}_i} a_{ij}\left(x_i(t)-x_j(t)\right),\ \ i\in\mathcal{V}.$$

To avoid continuously consuming resources for obtaining $\tilde{x}_i(t)$, in this paper, we are interested in event-triggered protocols of the following form:

$$u_i(t) = c_i(t)Kz_i(t), \tag{2}$$

where $z_i(t) \triangleq \tilde{x}_i(t_k^i)$ for $t \in [t_k^i, t_{k+1}^i)$. In the above equation, $K \in \mathbb{R}^{n_u\times n_x}$ is a constant gain matrix; t_k^i with $t_0^i = 0$, $i\in\mathcal{V}$, $k\in\mathbb{N}$, is the kth sampling instant of $\tilde{x}_i(t)$; and $c_i(t) > 0$, $i\in\mathcal{V}$, are adaptive scalar gains. All K, t_k^i and c_i are protocol parameters to be designed. Particularly, the sampling instants t_k^i are determined by some event-triggering conditions that will be clear later. For convenience, define

$$k_t^i \triangleq \arg\min_{l\ \ l\in\mathbb{N};t\geq t_l^i}\left\{t-t_l^i\right\},$$

which denotes the last event of agent i at the time t.

Remark 1. The event-triggering protocols in Cheng and Li (2019), Dimarogonas et al. (2012), Liu et al. (2017), Qian et al. (2019), Wang et al. (2018), Yang et al. (2016, 2018) and Zhang et al. (2017) need absolute information (for instance, $x_i(t_k^i)$ and $x_j(t^j_{k_t^j})$, $j\in\mathcal{N}_i$) for agent i, which are infeasible in some applications. Contrarily the protocol in (2) only needs relative state information and thus does not have this restriction. Although event-triggering protocols with relative information sensing have been widely studied in Cheng and Ugrinovskii (2016), Fan et al. (2013), Hu et al. (2016), Li et al. (2015) and Zhu and Jiang (2015), all these results need to precisely know the Laplacian of the graph for determining the protocol gains and/or triggering conditions. In this paper, we will propose a method for adaptively tuning the gains $c_i(t)$ based on sampled relative information, circumventing the use of global information in term of the Laplacian.

In the sequel, if no confusion is caused, we will omit the explicit dependence of symbols on t. Define

$$e_i \triangleq z_i - \tilde{x}_i,\ \ \mathscr{C}\triangleq \mathrm{diag}\{c_1,\ldots,c_N\},$$
$$\epsilon = \mathrm{col}\{\epsilon_1,\ldots,\epsilon_N\} \triangleq (\mathscr{L}_\epsilon\otimes\mathbf{I})\,x,$$
$$\tilde{x}\triangleq\mathrm{col}\left\{\tilde{x}_1,\ldots,\tilde{x}_N\right\},\ \ z\triangleq\mathrm{col}\left\{z_1,\ldots,z_N\right\},\ \ e\triangleq\mathrm{col}\left\{e_1,\ldots,e_N\right\},$$

where $\mathscr{L}_\epsilon \triangleq \mathbf{I}-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}$. The signal ϵ_i is the error between the state of each agent and the average state of all agents. When the graph is undirected, the average state of all agents is usually the consensus state to be reached (Tuna, 2008). Thus, if ϵ is treated as the consensus error, that consensus is reached physically means that the state of all agents will approach to the average of them.

Note that

$$\begin{aligned}\mathrm{col}\left\{u_1,\ldots,u_N\right\} &= \mathrm{col}\left\{c_1Kz_1,\ldots,c_NKz_N\right\} = (\mathscr{C}\otimes K)\,z\\ &= (\mathscr{C}\otimes K)\left(\tilde{x}+e\right).\end{aligned}$$

Thus the corresponding closed-loop system is given by

$$\dot{x} = (\mathbf{I}\otimes A)\,x + (\mathscr{C}\otimes B_uK)\left(\tilde{x}+e\right). \tag{3}$$

Furthermore, the signal ϵ evolves according to

$$\dot{\epsilon} = (\mathbf{I}\otimes A)\,\epsilon + (\mathscr{L}_\epsilon\mathscr{C}\otimes B_uK)\left(\tilde{x}+e\right). \tag{4}$$

Since 0 is a simple eigenvalue of $\mathscr{L}_\epsilon$ with $\mathbf{1}$ as the eigenvector, according to the definition of ϵ, it can be proved that $x_i(t) - x_j(t) \to 0$ as $t\to\infty$ if and only if $\epsilon(t)\to 0$ as $t\to\infty$. Thus, as an intermediate result, it is seen that consensus is reached if and only if $\epsilon(t)$ in (4) is convergent. In view of this fact, ϵ is called the consensus error in this article (one may use other definitions, but the objective is the same).

3. Consensus without external disturbances

In this section, in the absence of external disturbances, we study the consensus problem under the fully distributed event-triggered protocol (2). Moreover, a self-triggered realization of the protocol will be presented.

3.1. Consensus condition

Let $P\in\mathbb{R}^{n_x\times n_x}$ be the positive definite matrix satisfying

$$A^{\mathrm{T}}P + PA - PB_uB_u^{\mathrm{T}}P + Q = 0 \tag{5}$$

for a given positive definite matrix $Q\in\mathbb{R}^{n_x\times n_x}$. Furthermore, consider the following law of adaption,[1]

$$\dot{c}_i(t) = \alpha\left\|K\tilde{x}_i(t_k^i)\right\|^2,\ \ t\in[t_k^i,t_{k+1}^i),\ \ i\in\mathcal{V}, \tag{6}$$

where the initial conditions $c_i(0)$ and α are any positive constants. We further propose the following triggering condition for determining the sampling instants:

$$t_{k+1}^i = \inf_{t>t_k^i}\{t|f_i(e_i(t),\tilde{x}_i(t),t)=0\},\ \ i\in\mathcal{V},\ \ k\in\mathbb{N}, \tag{7}$$

where

$$f_i = \|Ke_i(t)\|^2 - \omega\left\|K\tilde{x}_i(t)\right\|^2 - \frac{\theta}{c_i(t)}\mathrm{e}^{-\delta t} \tag{8}$$

with ω, θ and δ being positive constants to be specified.

Before proceeding further, it should be pointed out that (8) requires the knowledge of the continuous-time relative state $\tilde{x}_i(t)$ for obtaining $e_i(t)$. Thus, if agent i obtains $\tilde{x}_i(t)$ by direct measurement, then it will be required to *continuously* monitor the state of neighboring agents, which does not obey our objective that only the sampled information about neighboring agents is used. In Section 3.2, we will provide sampled-data based formulations of $x_i(t)$, so that continuously monitoring neighboring agents is circumvented.

Now we present the following theorem. It not only provides a sufficient condition for the existence of the protocol (2) under (6)–(8), but also shows that Zeno behavior is excluded.

Theorem 1. *Consider the MAS (1) and the protocol (2), (6)–(8) on an undirected, connected graph $\mathcal{G}$. Let the gain K be given by $K = -B_u^{\mathrm{T}}P$ with P satisfying (5). Then the closed-loop system (3) reaches consensus and $c_i(t)$, $i\in\mathcal{V}$, converge to some positive constants, if ω, θ and δ are any constants such that*

$$\omega\in(0,1)\,,\ \ \theta>0,\ \ \delta>0. \tag{9}$$

Moreover, the system (3) does not exhibit Zeno behavior.

[1] The term "adaptive" in this paper is to indicate that the scalars c_i are real-time adjusted without using global information. In this sense, the proposed protocol can handle unknown connection weights. However, c_i are not of the meaning of parameter estimation, and K is designed on the basis of precisely known agent dynamics. Thus, the meaning of adaption for the proposed protocol does not completely comply with traditional adaptive control laws that aim to handle unknown system parameters through parameter estimation/identification. In this paper, we still call the proposed protocol as an adaptive one in order to emphasize the real-time tuning feature of c_i, and also to follow the convention of the existing results on fully distributed control of MASs (see, e.g., DeLellis et al., 2009; Li et al., 2013; Mei et al., 2016).

Proof. We first prove the achievement of consensus. Consider a candidate Lyapunov function as

$$V(t)=\epsilon^{\mathrm{T}}(t)\left(\mathscr{L}\otimes P\right)\epsilon(t)+\sum_{i\in\mathscr{V}}\frac{(1-\omega)}{4\alpha(1+\omega)}\left(c_i(t)-\bar{c}\right)^2, \tag{10}$$

where $\bar{c}$ is any positive constant satisfying $\bar{c}\geq\frac{2(1+\omega)(1+\gamma)}{\lambda_2\gamma(1-\omega)(1-\omega-\omega\gamma)}$ (γ is any constant belonging to $\left(0,\frac{1}{\omega}-1\right)$). The derivative of V along the solution of ϵ in (4) with $K=-B_u^{\mathrm{T}}P$ is given by

$$\begin{aligned}\dot{V}&=2\epsilon^{\mathrm{T}}\left(\mathscr{L}\otimes P\right)\dot{\epsilon}+\sum_{i\in\mathscr{V}}\frac{1-\omega}{2\alpha(1+\omega)}\left(c_i-\bar{c}\right)\dot{c}_i\\&=2\epsilon^{\mathrm{T}}\left(\mathscr{L}\otimes PA\right)\epsilon-2\tilde{x}^{\mathrm{T}}\left(\mathscr{C}\otimes K^{\mathrm{T}}K\right)\tilde{x}-2\tilde{x}^{\mathrm{T}}\left(\mathscr{C}\otimes K^{\mathrm{T}}K\right)e\\&\quad+\sum_{i\in\mathscr{V}}\frac{1-\omega}{2(1+\omega)}\left(c_i-\bar{c}\right)\left\|K\tilde{x}_i(t^i_{k^i_t})\right\|^2,\end{aligned}$$

where we have used the fact $\mathscr{L}\mathscr{L}_\epsilon=\mathscr{L}$. It is easy to verify that

$$-2\tilde{x}^{\mathrm{T}}\left(\mathscr{C}\otimes K^{\mathrm{T}}K\right)e\leq\tilde{x}^{\mathrm{T}}\left(\mathscr{C}\otimes K^{\mathrm{T}}K\right)\tilde{x}+e^{\mathrm{T}}\left(\mathscr{C}\otimes K^{\mathrm{T}}K\right)e.$$

Moreover,

$$\begin{aligned}&\sum_{i\in\mathscr{V}}\frac{1-\omega}{2(1+\omega)}c_i\left\|K\tilde{x}_i(t^i_{k^i_t})\right\|^2\\&=\sum_{i\in\mathscr{V}}\frac{1-\omega}{2(1+\omega)}c_i\left\|K\tilde{x}_i+Ke_i\right\|^2\\&\leq\sum_{i\in\mathscr{V}}\frac{1-\omega}{2(1+\omega)}c_i\left(2\left\|K\tilde{x}_i\right\|^2+2\left\|Ke_i\right\|^2\right)\\&=\frac{1-\omega}{1+\omega}\tilde{x}^{\mathrm{T}}\left(\mathscr{C}\otimes K^{\mathrm{T}}K\right)\tilde{x}+\frac{1-\omega}{1+\omega}e^{\mathrm{T}}\left(\mathscr{C}\otimes K^{\mathrm{T}}K\right)e.\end{aligned}$$

Note that (8) implies $\|Ke_i\|^2\leq\omega\left\|K\tilde{x}_i\right\|^2+\frac{\theta}{c_i}\mathrm{e}^{-\delta t}$, which further implies $e^{\mathrm{T}}\left(\mathscr{C}\otimes K^{\mathrm{T}}K\right)e\leq\omega\tilde{x}^{\mathrm{T}}\left(\mathscr{C}\otimes K^{\mathrm{T}}K\right)\tilde{x}+N\theta e^{-\delta t}$. Thus, it follows that

$$\begin{aligned}\dot{V}&\leq2\epsilon^{\mathrm{T}}\left(\mathscr{L}\otimes PA\right)\epsilon+\frac{2}{1+\omega}e^{\mathrm{T}}\left(\mathscr{C}\otimes K^{\mathrm{T}}K\right)e-\frac{2\omega}{1+\omega}\tilde{x}^{\mathrm{T}}\left(\mathscr{C}\otimes K^{\mathrm{T}}K\right)\tilde{x}\\&\quad-\sum_{i\in\mathscr{V}}\frac{\bar{c}\left(1-\omega\right)}{2(1+\omega)}\left\|K\tilde{x}_i(t^i_{k^i_t})\right\|^2\\&\leq2\epsilon^{\mathrm{T}}\left(\mathscr{L}\otimes PA\right)\epsilon-\sum_{i\in\mathscr{V}}\frac{\bar{c}\left(1-\omega\right)}{2(1+\omega)}\left\|K\tilde{x}_i(t^i_{k^i_t})\right\|^2+\frac{2N\theta}{1+\omega}\mathrm{e}^{-\delta t}.\end{aligned}$$

It can be verified that

$$\begin{aligned}\left\|K\tilde{x}_i\right\|^2&=\left\|K\tilde{x}_i(t^i_k)-Ke_i\right\|^2\\&=\left\|K\tilde{x}_i(t^i_k)\right\|^2+\left\|Ke_i\right\|^2-2\tilde{x}_i^{\mathrm{T}}(t^i_k)K^{\mathrm{T}}Ke_i\\&\leq\left\|K\tilde{x}_i(t^i_k)\right\|^2+\left\|Ke_i\right\|^2+\frac{1}{\gamma}\left\|K\tilde{x}_i(t^i_k)\right\|^2+\gamma\left\|Ke_i\right\|^2,\end{aligned}$$

which, combined with $\|Ke_i\|^2\leq\omega\left\|K\tilde{x}_i\right\|^2+\frac{\theta}{c_i}\mathrm{e}^{-\delta t}$, implies

$$\begin{aligned}-\left\|K\tilde{x}_i(t^i_k)\right\|^2&\leq\gamma\left\|Ke_i\right\|^2-\frac{\gamma}{1+\gamma}\left\|K\tilde{x}_i\right\|^2\\&=\gamma\left(\left\|Ke_i\right\|^2-\omega\left\|K\tilde{x}_i\right\|^2\right)-\left(\frac{\gamma}{1+\gamma}-\gamma\omega\right)\left\|K\tilde{x}_i\right\|^2\\&\leq\frac{\gamma\theta}{c_i}\mathrm{e}^{-\delta t}-\frac{\gamma\left(1-\omega-\omega\gamma\right)}{1+\gamma}\left\|K\tilde{x}_i\right\|^2.\end{aligned}$$

Thus, it further follows that

$$\begin{aligned}\dot{V}&\leq2\epsilon^{\mathrm{T}}\left(\mathscr{L}\otimes PA\right)\epsilon-\frac{\bar{c}\left(1-\omega\right)\gamma\left(1-\omega-\omega\gamma\right)}{2(1+\omega)\left(1+\gamma\right)}\sum_{i\in\mathscr{V}}\left\|K\tilde{x}_i\right\|^2\\&\quad+\sum_{i\in\mathscr{V}}\frac{\bar{c}\left(1-\omega\right)\gamma\theta}{2(1+\omega)c_i}\mathrm{e}^{-\delta t}+\frac{2N\theta}{1+\omega}\mathrm{e}^{-\delta t}\\&\leq2\epsilon^{\mathrm{T}}\left(\mathscr{L}\otimes PA\right)\epsilon-\bar{c}\rho\epsilon^{\mathrm{T}}\left(\mathscr{L}^2\otimes K^{\mathrm{T}}K\right)\epsilon+\upsilon\mathrm{e}^{-\delta t},\end{aligned} \tag{11}$$

where $\rho=\frac{(1-\omega)\gamma(1-\omega-\omega\gamma)}{2(1+\omega)(1+\gamma)}$, $\upsilon=\frac{N\bar{c}(1-\omega)\gamma\theta}{2(1+\omega)\min_{i\in\mathscr{V}}c_i(0)}+\frac{2N\theta}{1+\omega}$ and the fact $c_i(t)\geq c_i(0)$ was used. Since $0<\gamma<\frac{1}{\omega}-1$ and $\omega<1$, we have $\rho>0$.

Let $\varepsilon=\mathrm{col}\{\varepsilon_1,\ldots,\varepsilon_N\}\triangleq\left[\ N^{-1/2}\mathbf{1}\ \ \mathscr{U}\ \right]^{\mathrm{T}}\epsilon$, where $\left[\ N^{-1/2}\mathbf{1}\ \ \mathscr{U}\ \right]$ is a unitary matrix such that $\Lambda\triangleq\mathrm{diag}\{0,\lambda_2,\ldots,\lambda_N\}=\left[\ N^{-1/2}\mathbf{1}\ \ \mathscr{U}\ \right]^{\mathrm{T}}\mathscr{L}\left[\ N^{-1/2}\mathbf{1}\ \ \mathscr{U}\ \right]$. It follows from (5), (11) and the fact $\bar{c}\geq\frac{1}{(1-\omega)\lambda_2}$ that

$$\begin{aligned}\dot{V}&\leq2\varepsilon^{\mathrm{T}}\left(\Lambda\otimes PA\right)\varepsilon-\bar{c}\rho\varepsilon^{\mathrm{T}}\left(\Lambda^2\otimes K^{\mathrm{T}}K\right)\varepsilon+\upsilon\mathrm{e}^{-\delta t}\\&=\sum_{i=2}^{N}\lambda_i\varepsilon_i^{\mathrm{T}}\left[A^{\mathrm{T}}P+PA-\bar{c}\rho\lambda_iPBB^{\mathrm{T}}P\right]\varepsilon_i+\upsilon\mathrm{e}^{-\delta t}\\&\leq\sum_{i=2}^{N}\lambda_i\varepsilon_i^{\mathrm{T}}\left(A^{\mathrm{T}}P+PA-PBB^{\mathrm{T}}P\right)\varepsilon_i+\upsilon\mathrm{e}^{-\delta t}\\&=-\sum_{i=2}^{N}\lambda_i\varepsilon_i^{\mathrm{T}}Q\varepsilon_i+\upsilon\mathrm{e}^{-\delta t}.\end{aligned} \tag{12}$$

Thus,

$$\begin{aligned}0\leq V(t)&=\int_0^t\dot{V}(\tau)\mathrm{d}\tau+V(0)\leq\int_0^t\upsilon\mathrm{e}^{-\delta\tau}\mathrm{d}\tau+V(0)\\&\leq\frac{\upsilon}{\delta}\left(1-\mathrm{e}^{-\delta t}\right)+V(0)\leq\frac{\upsilon}{\delta}+V(0),\ \ t\geq0,\end{aligned} \tag{13}$$

which implies that $V(t)$ is bounded. It is seen from the definition of $V(t)$ that the boundedness of $V(t)$ implies that of $c_i(t)$. Since $c_i(t)$ are monotonically increasing, $c_i(t)$ converge to some positive constants. The boundedness of $V(t)$ implies that of $\epsilon(t)$, which further implies the boundedness of $\tilde{x}(t)$ and $e(t)$ (since $\tilde{x}=(\mathscr{L}\otimes\mathbf{I})x=(\mathscr{L}\mathscr{L}_\epsilon\otimes\mathbf{I})x=(\mathscr{L}\otimes\mathbf{I})\epsilon$). Note that $\epsilon^{\mathrm{T}}\epsilon$ is differentiable with respect to t; moreover, its derivative can be written as $2\epsilon^{\mathrm{T}}[(\mathbf{I}\otimes A)\epsilon+(\mathscr{L}_\epsilon\mathscr{C}\otimes B_uK)(\tilde{x}+e)]$. Thus, the derivative of $\epsilon^{\mathrm{T}}\epsilon$ is bounded, which implies that $\epsilon^{\mathrm{T}}\epsilon$ is uniformly continuous (Slotine & Li, 1991, Page 123). Moreover, it follows from (12) and (13) that

$$\begin{aligned}\lim_{t\to\infty}\int_0^t\epsilon^{\mathrm{T}}(\tau)\epsilon(\tau)\mathrm{d}\tau&=\lim_{t\to\infty}\int_0^t\varepsilon^{\mathrm{T}}(\tau)\varepsilon(\tau)\mathrm{d}\tau\\&\leq\frac{1}{\lambda_{\mathrm{m}}(Q)\lambda_2}\lim_{t\to\infty}\int_0^t\sum_{i=2}^{N}\lambda_i\varepsilon_i^{\mathrm{T}}(\tau)Q\varepsilon_i(\tau)\mathrm{d}\tau\\&\leq\frac{1}{\lambda_{\mathrm{m}}(Q)\lambda_2}\left(\frac{\upsilon}{\delta}+V(0)-V(\infty)\right)<\infty,\end{aligned}$$

where the fact $\varepsilon_1\equiv0$ was used. By Khalil (2002, Lemma 8.2), it follows that $\epsilon^{\mathrm{T}}(t)\epsilon(t)\to0$ as $t\to\infty$, or equivalently $\epsilon(t)\to0$ as $t\to\infty$, that is, consensus is reached.

Next, we show that Zeno behavior is excluded. Note that $e_i(t)=\tilde{x}_i(t^i_k)-\tilde{x}_i(t)$ for $t\in[t^i_k,t^i_{k+1})$. Thus, $\frac{\mathrm{d}\|Ke_i\|^2}{\mathrm{d}t}=2e_i^{\mathrm{T}}K^{\mathrm{T}}K\dot{e}_i=-2e_i^{\mathrm{T}}K^{\mathrm{T}}K\dot{\tilde{x}}_i=-2e_i^{\mathrm{T}}K^{\mathrm{T}}K\sum_{j\in\mathscr{N}_i}a_{ij}(Ax_i+B_uu_i-Ax_j-B_uu_j)\leq\left|2e_i^{\mathrm{T}}K^{\mathrm{T}}K(A\tilde{x}_i+\sum_{j=1}^{N}l_{ij}B_uu_j)\right|$ for $t\in[t^i_k,t^i_{k+1})$. Since $\tilde{x}(t)$ and $c_i(t)$, $i\in\mathscr{V}$, have been proved to be bounded, $u_i(t)$, $i\in\mathscr{V}$, are bounded. Moreover, $e_i(t)$, $i\in\mathscr{V}$, are bounded. Thus, with $\chi_i\triangleq\sup_{t\geq0}\left|2e_i^{\mathrm{T}}(t)K^{\mathrm{T}}K(A\tilde{x}_i(t)+\sum_{j=1}^{N}l_{ij}B_uu_j(t))\right|$, we have $\chi_i<\infty$.

Using $\left\|Ke_i(t_k^i)\right\|^2 = 0$ and $\frac{\mathrm{d}\|Ke_i(t)\|^2}{\mathrm{d}t} \leq \chi_i$, we have

$$\|Ke_i(t)\|^2 \leq \chi_i(t - t_k^i), \quad t \in [t_k^i, t_{k+1}^i). \tag{14}$$

The triggering condition (8) implies that, at $t = t_{k+1}^i$, there holds $\left\|Ke_i(t_{k+1}^i)\right\|^2 = \omega\left\|K\tilde{x}_i(t_{k+1}^i)\right\|^2 + \theta c_i^{-1}\mathrm{e}^{-\delta t_{k+1}^i}$, which together with (14) shows

$$\theta c_i^{-1}\mathrm{e}^{-\delta t_{k+1}^i} \leq \left\|Ke_i(t_{k+1}^i)\right\|^2 \leq \chi_i T_k^i, \text{ with } T_k^i \triangleq t_{k+1}^i - t_k^i. \tag{15}$$

The above inequality implies that $T_k^i > 0$ for any finite horizon. Moreover, we can show $t_k^i \to \infty$ as $k \to \infty$. If this is not true, by denoting $t_\infty^i = \lim_{k\to\infty} t_k^i$, then $t_\infty^i < \infty$, and moreover $\lim_{k\to\infty} T_k^i = 0$. Thus, it is known from (15) that, for $k \to \infty$, $0 < \theta c_i^{-1}\mathrm{e}^{-\delta t_\infty^i} \leq \chi_i 0 = 0$, which is a contradiction. Consequently, no Zeno behavior is exhibited. ■

Remark 2. Some comments about Theorem 1 and its comparisons with the existing results are provided as follows.

1. As aforementioned, results about event-triggered consensus of MASs can be categorized into two classes that are based on absolute information sensing and relative information sensing, respectively. To the best of the authors' knowledge, except Zhu et al. (2018), most of the existing results about fully distributed event-triggered consensus fall into the former class (Cheng & Li, 2019; Qian et al., 2019; Wang et al., 2018; Yang et al., 2018). The proposed protocol (2) is of the latter class.
2. Protocol (2) and Theorem 1 *cannot* be obtained by simply extending the existing results based on absolute information sensing to the consensus problem based on relative information as considered in this paper. Take Cheng and Li (2019) for instance. *Firstly*, because different information is employed in the two cases, the protocols in Cheng and Li (2019) do not imply the existence of distributed event-triggered protocols based on relative information, even if $t_k^i = t_k^j$, $j \in \mathcal{N}_i$, is further enforced for agents i and j.[2] This obstacle makes the protocols for the two cases essentially different from each other, which necessitates to design different event-triggered protocols for the two cases, respectively. Moreover, the adaptive gains $c_{ij}(t)$ in Cheng and Li (2019) are the weight of the relative information of each edge (e.g., $\tilde{x}_i - \tilde{x}_j$ in (3) therein), while $c_i(t)$ in this paper are that of the collective relative information at each node. To our understanding, $c_{ij}(t)$ therein cannot be simply merged into a single weight, like $c_i(t)$ in this paper, of the collective relative information at each node. *Secondly*, we provide an event-triggering condition which, in terms of the coefficient of the state-dependent term, are different from the one in Cheng and Li (2019). This coefficient (i.e., ω in (8) in this paper and $\frac{1}{4(1+\delta c_{ij})}$ in Cheng & Li, 2019) cannot be arbitrarily large but should be properly constrained to ensure consensus. As a matter of fact, properly characterizing the bound for this coefficient is one of the main technical challenges in designing event-triggered protocols (see, e.g., Cheng & Ugrinovskii, 2016; Fan et al., 2013; Hu et al., 2016; Li et al., 2015; Zhu & Jiang, 2015). Particularly, note that the Lyapunov function in this paper is different from the one in Cheng and Li (2019), and we need to technically construct the key term $\sum_{i\in\mathcal{V}} \frac{(1-\omega)}{4\alpha(1+\omega)}(c_i(t) - \bar{c})^2$ and properly choose the constant $\bar{c}$. From this aspect, the event-triggering condition proposed in this paper is technically different from those in Cheng and Li (2019). *Thirdly*, the adaptive gains $c_{ij}(t)$ in Cheng and Li (2019) are required to satisfy $c_{ij}(t) = c_{ji}(t)$ for all $t \geq 0$. However, this symmetry is difficult to be maintained all the time, for instance, when $c_{ij}(0)$ and $c_{ji}(0)$ are not precisely equal or the transmitted states are corrupted by communication noises. The adaptive gains $c_i(t)$ in this paper are not required to have this symmetry.
3. In Zhu et al. (2018), fully distributed event-triggered consensus of MASs based on relative information sensing was studied very recently, and a different adaption law and a different event-triggering condition have been presented therein. However, the consensus condition therein needs the feasibility of $\sigma\lambda_{\mathrm{m}}(P^{-1}) \geq \lambda_{\mathrm{M}}(B_u B_u^{\mathrm{T}} P)$, where P is a positive definite matrix solving (5) with $Q = \sigma\mathbf{I}$. Although this inequality is related to local agent dynamics only, there is no guarantee of its feasibility in general. Take a simple MAS for instance, where $A = 1$ and $B_u = 1$. Eq. (5) reduces to $2P - P^2 + \sigma = 0$, which gives $P = \frac{2+\sqrt{4+4\sigma}}{2} = 1 + \sqrt{1+\sigma}$. Then $\sigma\lambda_{\mathrm{m}}(P^{-1}) = \frac{\sigma}{1+\sqrt{1+\sigma}}$ and $\lambda_{\mathrm{M}}(B_u B_u^{\mathrm{T}} P) = P = 1 + \sqrt{1+\sigma}$. The regarded inequality thus reduces to $\frac{\sigma}{1+\sqrt{1+\sigma}} \geq 1 + \sqrt{1+\sigma}$, or equivalently $0 \geq 2 + 2\sqrt{1+\sigma}$, which cannot be true for any $\sigma \geq 0$. Thus, the consensus condition in Zhu et al. (2018) is quite *restrictive* (we find that it is also infeasible for Example 1). Theorem 1 in this article does not have this restriction.

3.2. Self-triggered realization

Note that the triggering condition (7) needs the continuous-time knowledge of $\tilde{x}_i(t)$ to compute $e_i(t)$. Directly measuring such relative information requires agents to continuously monitor neighbors. To circumvent this drawback, inspired by Cheng and Ugrinovskii (2016), we further provide a self-triggered realization for the protocol. Specifically, $\tilde{x}_i(t)$ for $t \in [t_k^i, t_{k+1}^i)$ can be directly computed as

$$\tilde{x}_i(t) = \underbrace{\left[\mathrm{e}^{A(t-t_k^i)} + l_{ii}\int_{t_k^i}^{t} c_i(\tau)\mathrm{e}^{A(t-\tau)}B_u K\,\mathrm{d}\tau\right]\tilde{x}_i(t_k^i)}_{\triangleq x_i^1(t_k^i, t)} \\ - \underbrace{\int_{t_k^i}^{t} \mathrm{e}^{A(t-\tau)}B_u K \sum_{j\in\mathcal{N}_i} c_j(\tau)a_{ij}\tilde{x}_j(t_{k_\tau^j}^j)\mathrm{d}\tau}_{\triangleq x_i^2(t_k^i, t)}. \tag{16}$$

The formulation of $\tilde{x}_i(t)$ as above only requires the sampled relative information between agents. Note that the two terms x_i^1 and x_i^2, from the perspective of agent i, rely on different types of relative information:

1. x_i^1 on the sampled relative information measured by agent i itself;
2. x_i^2 on the sampled relative information measured by the neighbors of agent i.

In addition, note that, although $c_j(t)$, $j \in \mathcal{N}_i$, are continuous-time signals not directly related to agent i (they are the adaptive gains of neighboring agents), it is unnecessary for agent i to obtain them from neighbors in a continuous-time way. In fact, as long as $\tilde{x}_j(t_k^j)$ and $c_j(0)$, $j \in \mathcal{N}_i$, have been transmitted to agent i, it can compute $c_j(t)$, $j \in \mathcal{N}_i$, in an iterative way as

$$c_j(t) = c_j(t_k^j) + \alpha\left\|K\tilde{x}_j(t_k^j)\right\|^2 (t - t_k^j), \ t \in [t_k^j, t_{k+1}^j).$$

[2] Intuitively, it appears that an event-triggered protocol based on absolute information can be written in the form of relative information by letting $t_k^i = t_k^j$. However, such an enforcement does *not* result in the desired result. Since the graph is undirected and connected, enforcing $t_k^i = t_k^j$ actually implies $t_k^1 = \cdots = t_k^N$, that is, the resulting event-triggering mechanism works in a centralized manner.

Thus, with $\tilde{x}_i(t)$ given as in (16), the proposed protocol (2) with (6)–(8) does not need to continuously monitor neighboring agents. In fact, based on (16), the protocol under the triggering condition (7) reduces to a *self-triggered* one.

Remark 3. To obtain x_i^2, it is necessary for agent i to implement certain communication medium which can transmit the events and the sampled information from the neighbors to agent i. This is a common case for most of the existing results on event-triggered protocols with absolute information sensing (e.g., Cheng & Li, 2019; Dimarogonas et al., 2012; Liu et al., 2017; Yang et al., 2016) or on self-triggered realizations of event-triggered protocols with relative information sensing (e.g., Cheng & Ugrinovskii, 2016; Fan et al., 2013; Hu et al., 2016; Li et al., 2015; Zhu & Jiang, 2015). It is worth pointing out that (16) is an equivalent formulation of $\tilde{x}_i$, thus the value of $x_i(t)$ via computation according to (16), theoretically, has no difference when compared with that via direct measurement.

4. Consensus with external bounded disturbances

In this section, we extend the results in the previous section to the consensus problem with external bounded disturbances. Specifically, consider the following N linear agents,

$$\dot{x}_i(t) = Ax_i(t) + B_u u_i(t) + w_i(t),\ \ i \in \mathscr{V}, \tag{17}$$

where $w_i(t) \in \mathbb{R}^n$ are external disturbances satisfying $\sup_{t\ge 0} \|w_i(t)\| \le \bar{w}_i$ and other symbols have the same meaning of those in (1). We are still interested in the adaptive event-triggered protocol in (2), for which it is easy to see that the consensus error ϵ satisfies

$$\dot{\epsilon} = (\mathbf{I} \otimes A)\, \epsilon + (\mathscr{L}_\epsilon \mathscr{C} \otimes B_u K)\, z + (\mathscr{L}_\epsilon \otimes \mathbf{I})\, w. \tag{18}$$

Because of the presence of the external disturbance w, it is in general impossible to reach precise consensus. Instead, our objective in this section is to find a protocol (2) such that the consensus error ϵ is uniformly ultimately bounded, which is called *bounded consensus* in this article.

Since the consensus error ϵ in (18) does not converge to zero in general, and so do $\tilde{x}_i$. If the law of adaption (6) is still applied, $c_i(t)$ could increase unboundedly. Thus, to ensure the boundedness of $c_i(t)$, the following modified law of adaption is applied:

$$\dot{c}_i(t) = \alpha \left\| K\tilde{x}_i(t_k^i) \right\|^2 - \phi c_i(t),\ \ (i) \in \mathscr{V},\ \ t \in [t_k^i, t_{k+1}^i), \tag{19}$$

where the initial conditions $c_i(0)$ are any nonnegative constants and the parameters α and ϕ are any positive constants. Note that $c_i(t) \ge 0$ for all $t \ge 0$. Moreover, we still employ the triggering condition (7) but functions f_i are re-defined as

$$f_i \triangleq \|Ke_i(t)\|^2 - \omega \left\| K\tilde{x}_i(t) \right\|^2 - \frac{\theta}{\beta + c_i(t)} \mathrm{e}^{-\delta t} \tag{20}$$

with ω, θ, δ and β being positive constants. The following theorem shows that the protocol (2) combined with (7), (19) and (20) can solve the bounded consensus problem.

Theorem 2. *Consider the MAS (17) and the protocol (2), (7), (19) and (20) on an undirected, connected graph $\mathscr{G}$. Then the closed-loop system (18) reaches bounded consensus and $c_i(t)$, $i \in \mathscr{V}$, are uniformly ultimately bounded, if ω, θ, δ and β are any constants such that*

$$\omega \in (0, 1)\,,\ \ \theta > 0,\ \ \delta > 0,\ \ \beta > 0. \tag{21}$$

Moreover, the system (18) does not exhibit Zeno behavior.

Proof. Still consider the candidate Lyapunov function in (10). Using similar arguments for obtaining (12), one can show that the derivative of V along the solution of ϵ in (18) satisfies

$$\begin{aligned}\dot{V} \le & -\sum_{i=2}^{N} \lambda_i \varepsilon_i^{\mathrm{T}} Q \varepsilon_i + \nu \mathrm{e}^{-\delta t} + 2\epsilon^{\mathrm{T}} (\mathscr{L} \otimes P)\, w \\ & - \sum_{i\in\mathscr{V}} \frac{(1-\omega)\,(c_i - \bar{c})}{2\alpha(1+\omega)} \phi c_i \\ = & -\epsilon^{\mathrm{T}} (\mathscr{L} \otimes Q)\, \epsilon + \nu \mathrm{e}^{-\delta t} + 2\epsilon^{\mathrm{T}} (\mathscr{L} \otimes P)\, w \\ & - \sum_{i\in\mathscr{V}} \frac{(1-\omega)\,(c_i - \bar{c})}{2\alpha(1+\omega)} \phi c_i,\end{aligned}$$

where $\nu = \sup_{t\ge 0} \left\{ \sum_{i\in\mathscr{V}} \left[\frac{\bar{c}(1-\omega)\gamma\theta}{2(1+\omega)(\beta+c_i(t))} + \frac{2\theta c_i(t)}{(1+\omega)(\beta+c_i(t))} \right] \right\}$. Since $c_i(t) \ge 0$, it follows that $0 < \nu < \infty$. Since $-(c_i - \bar{c})\, c_i \le -\frac{(c_i-\bar{c})^2}{2} + \frac{\bar{c}^2}{2}$ and $2\epsilon^{\mathrm{T}} (\mathscr{L} \otimes P)\, w \le \gamma \epsilon^{\mathrm{T}} (\mathscr{L} \otimes P)\, \epsilon + \gamma^{-1} w^{\mathrm{T}} (\mathscr{L} \otimes P)\, w$ for any constant $\gamma > 0$, we have

$$\begin{aligned}\dot{V} \le & -\epsilon^{\mathrm{T}} (\mathscr{L} \otimes Q)\, \epsilon + \gamma \epsilon^{\mathrm{T}} (\mathscr{L} \otimes P)\, \epsilon + \frac{1}{\gamma} w^{\mathrm{T}} (\mathscr{L} \otimes P)\, w \\ & + \nu \mathrm{e}^{-\delta t} + \sum_{i\in\mathscr{V}} \frac{\phi(1-\omega)\bar{c}^2}{4\alpha(1+\omega)} - \sum_{i\in\mathscr{V}} \frac{\phi(1-\omega)\,(c_i-\bar{c})^2}{4\alpha(1+\omega)}.\end{aligned}$$

Let η and γ be sufficiently small positive constants such that $\eta P + \gamma P - Q$ is negative semi-definite and $\phi \ge \eta$. Then

$$\begin{aligned}\dot{V} \le & -\eta V + \epsilon^{\mathrm{T}} [\mathscr{L} \otimes (\eta P + \gamma P - Q)] \epsilon + \frac{1}{\gamma} w^{\mathrm{T}} (\mathscr{L} \otimes P)\, w \\ & + \nu \mathrm{e}^{-\delta t} + \sum_{i\in\mathscr{V}} \frac{\phi(1-\omega)\bar{c}^2}{4\alpha(1+\omega)} - \sum_{i\in\mathscr{V}} \frac{(\phi-\eta)\,(1-\omega)\,(c_i-\bar{c})^2}{4\alpha(1+\omega)} \\ \le & -\eta V + \mu,\end{aligned}$$

where $\mu \triangleq \frac{1}{\gamma} \lambda_{\mathrm{M}}(\mathscr{L} \otimes P) \sum_{i\in\mathscr{V}} \bar{w}_i^2 + \nu + \sum_{i\in\mathscr{V}} \frac{\phi(1-\omega)\bar{c}^2}{4\alpha(1+\omega)}$ and the fact $\|w_i\|^2 \le \bar{w}_i^2$ has been used. From the Comparison Lemma (see Khalil, 2002, Lemma 3.4), we obtain

$$V(t) \le \mathrm{e}^{-\eta t} V(0) + \frac{\mu}{\eta}(1 - \mathrm{e}^{-\eta t}).$$

It is seen that $V(t)$ is bounded for all $t \ge 0$. Thus, from (10), the boundedness of the consensus error $\epsilon(t)$ and the adaptive gains $c_i(t)$ can be concluded as well.

The exclusion of Zeno behavior follows from similar arguments as those for the corresponding part of Theorem 1. So the remaining proof is omitted for brevity. ■

Remark 4. It is seen from the proof of Theorem 2 that $\epsilon(t)$ and $c_i(t)$ will asymptotically converge to a bounded set in terms of $V(t)$ as $\left\{ V(t) : 0 < V(t) \le \frac{\mu}{\eta} \right\}$, which can be viewed as an estimation of the attraction region of $\epsilon(t)$ and $c_i(t)$. From the definition of μ, one sees that the bound of this set increases with $\bar{w}_i$ increasing, which complies with the intuition that larger disturbances result in larger consensus errors in general. Note that, even if $\bar{w}_i = 0$, that is, the system is free of disturbances, following the proof of Theorem 2, we can only show the boundedness of $\epsilon(t)$ under the law of adaption (19). The law of adaption (19) cannot guarantee precise consensus in general. This is a difference between the two laws of adaption in (6) and (19).

Remark 5. As long as ω, α, θ, δ, β and ϕ satisfy the specifications in the theorems, the basic consensus or bounded consensus requirement can be ensured. However, in practice, a proper selection of these constants in general should be done on a case-by-case basis, and moreover trade-offs often need to be explored

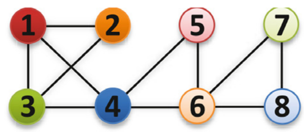

Fig. 1. The graph used in the examples. All edge weights are 1.

to accommodate more specifications. Intuitively speaking, large ω, θ and small δ, β can help enlarge event intervals, but in turn could reduce the convergence rate of the consensus error; a small α can help reduce the growing rate of c_i, but in turn could reduce the convergence rate of the consensus error; a large ϕ can help reduce the ultimate c_i in the presence of external disturbances, but in turn could result in the increasing of the consensus error. Finally, note that it is straightforward to further extend the proposed results to the case that the parameters ω, α, θ, δ, β and ϕ vary with agents.

5. Numerical examples

In this section, two numerical examples are provided to illustrate the effectiveness of the proposed results. We set the discretization time step for all simulations to 10^{-4}.

Example 1 (*Consensus Without Disturbances*). Consider the MAS (1) with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the network consists of 8 agents that communicate with each other according to the graph in Fig. 1. Let $Q = \mathbf{I}$, then the gain K obtained by Theorem 1 is $K = \begin{bmatrix} -1 & -1.7321 \end{bmatrix}$. Let $\alpha = 0.005$ and select $\omega = 0.5$, $\theta = 0.1$ and $\delta = 0.05$. For simulation, let the initial conditions $x_1(0) = \begin{bmatrix} -6 & 4 \end{bmatrix}^{\mathrm{T}}$, $x_2(0) = \begin{bmatrix} 2 & -4 \end{bmatrix}^{\mathrm{T}}$, $x_3(0) = \begin{bmatrix} -3 & -3 \end{bmatrix}^{\mathrm{T}}$, $x_4(0) = \begin{bmatrix} 0 & 3 \end{bmatrix}^{\mathrm{T}}$, $x_5(0) = \begin{bmatrix} 3 & -1 \end{bmatrix}^{\mathrm{T}}$, $x_6(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$, $x_7(0) = \begin{bmatrix} 4 & 3 \end{bmatrix}^{\mathrm{T}}$, $x_8(0) = \begin{bmatrix} 5 & -2 \end{bmatrix}^{\mathrm{T}}$ and $c_i(0) = 0.01$, $i \in \mathscr{V}$. Fig. 2 displays the simulation results under the designed protocol. The convergence of the consensus errors $\epsilon_i(t)$ is verified, that is, consensus is reached. The second sub-figure clearly illustrates that the protocol updates the relative information in a discrete-time manner, while the third one demonstrates the convergence of the adaptive gains $c_i(t)$ to some finite positive constants. The last sub-figure shows the minimum ($\min_k\{T_k^i\}$) and median ($\mathrm{med}_k\{T_k^i\}$) of the sampling intervals of the controllers. Particularly, it is shown that the values of $\min_k\{T_k^i\}$ are much larger than the discretization time step 10^{-4} for simulation, illustrating the exclusion of Zeno behavior in the closed-loop system.

With $Q = \sigma\mathbf{I}$, we obtain $P = [p_{ij}]_{2\times 2}$ with $p_{11} = \sqrt{\sigma^2 + 2\sigma^{\frac{3}{2}}}$, $p_{12} = p_{21} = \sqrt{\sigma}$ and $p_{22} = \sqrt{\sigma + 2\sqrt{\sigma}}$. The expressions of $\sigma\lambda_{\mathrm{m}}(P^{-1})$ and $\lambda_{\mathrm{M}}(B_u B_u^{\mathrm{T}} P)$ are omitted for brevity, but it can be verified that no $\sigma > 0$ can ensure $\sigma\lambda_{\mathrm{m}}(P^{-1}) \geq \lambda_{\mathrm{M}}(B_u B_u^{\mathrm{T}} P)$ (see Remark 2.3). Thus, the method in Zhu et al. (2018) is *infeasible* for this example.

Example 2 (*Consensus with Disturbances*). Let us consider the MAS (17) with the same A and B_u as those in Example 1 and moreover $w_i = B_u v_i$, where $v_i = \sin(3\pi t + \frac{i\pi}{8})$, $i \in \mathscr{V}$. We can employ the same design results as those in Example 1 for K, α, ω, θ and δ in the protocol (2), (7), (19) and (20). Additionally, take $\phi = 0.01$ and $\beta = 0.001$ for (19) and (20). Fig. 3 depicts the simulation results under the same initial conditions as those in Example 1. The effectiveness of the designed protocol is obvious. Particularly, it is seen that the consensus errors $\epsilon_i(t)$ and the adaptive gains $c_i(t)$ are all bounded, and the event-triggering processes work properly and do not exhibit Zeno behavior.

6. Conclusion

This article is devoted to event-triggered consensus of linear MASs on undirected graphs. An adaptive event-triggered protocol has been proposed for consensus control. The triggering conditions and adaptive gains depend on the relative information between neighboring agents. Sufficient conditions have been derived for the existence of the proposed protocol that ensures precise or bounded consensus, which do not need to know the Laplacian of the communication graph so that the protocol can be designed in a fully distributed way. It has been shown that Zeno behavior is excluded from the triggering process. Compared with the existing results, the proposed protocol is based on relative information sensing between neighboring agents; moreover, its existence is guaranteed for any linear MAS of stabilizable agents. The effectiveness and advantages have been clearly illustrated by numerical examples.

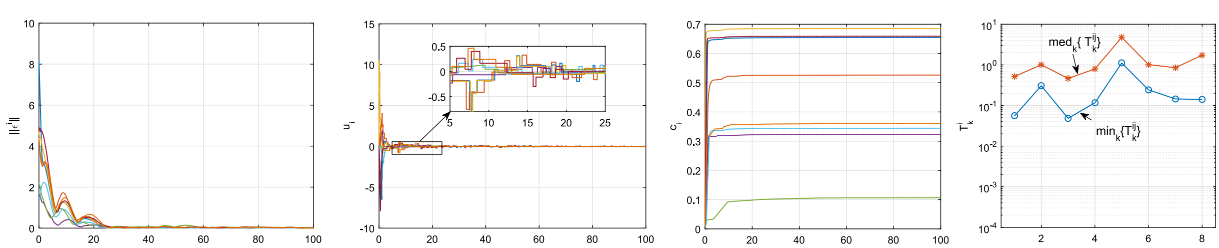

Fig. 2. Simulation results in Example 1.

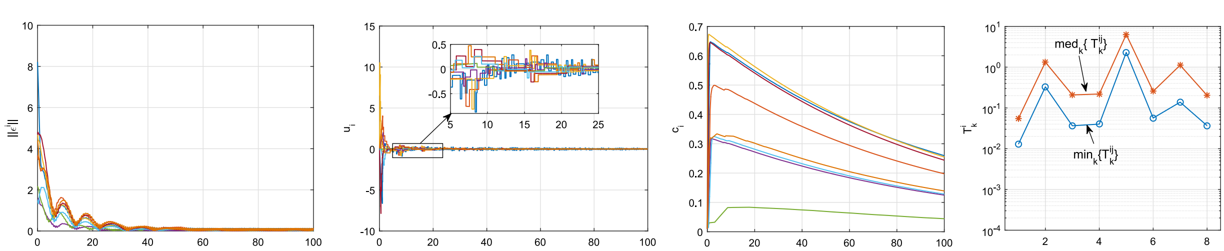

Fig. 3. Simulation results in Example 2.

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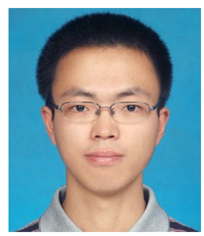

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