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Highlights

- A model to study free flap-wise, chord-wise and axial vibrations of rotating nano-beams is presented for the first time.
- The model is developed based on Euler-Bernoulli beam theory and Eringen's nonlocal theory.
- The size-dependence of Coriolis and spin-softening effects are considered and discussed.
- The nonlocal parameter is found to affect differently to the first and the higher flexural modes.
- A virtual displacement based formulation with Ritz method for nano-beams is reported for the first time.

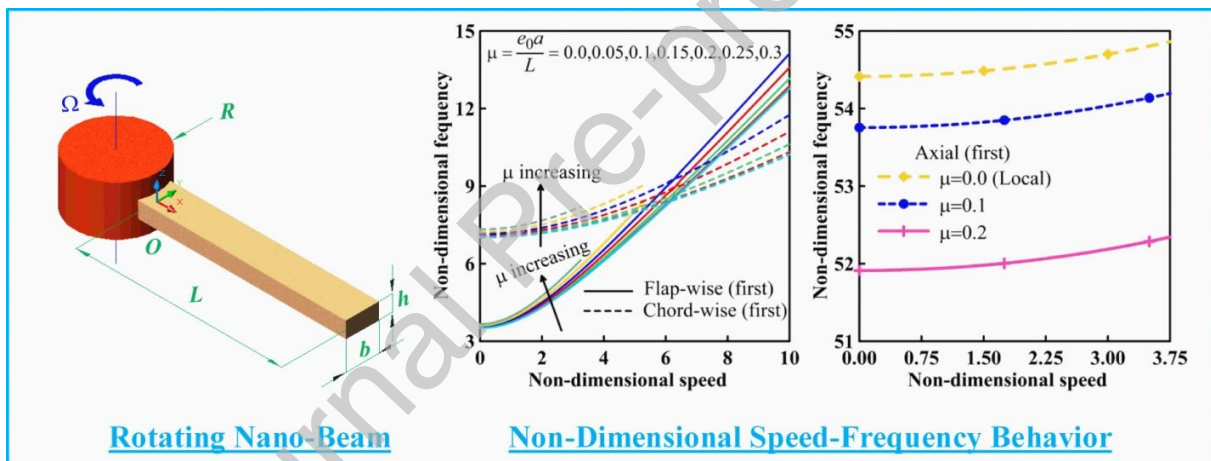
Free vibration analysis of rotating nano-beams for flap-wise, chord-wise and axial modes based on Eringen's nonlocal theory

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GRAPHICAL ABSTRACT



Abstract

A mathematical model based on Eringen's nonlocal elasticity theory is presented to analyze free vibration behavior of rotating nano-beams. The model is capable of studying the flap-wise and chord-wise vibrations as well as the axial vibration of rotating nano-beams. The model is based on Euler-Bernoulli beam theory, and incorporates spin-softening and Coriolis effects. Hamilton's principle is employed to derived the governing equations involving virtual displacements, and Ritz method is followed to discretize the governing equations. The governing equations are

transformed to an eigenvalue problem in state-space. The stable solutions in frequency domain are identified by appropriately examining the nature of the complex eigenvalues. The model is validated through some reduced problems available in the literature. The non-dimensional speed-frequency behaviors are presented and discussed for different normalized nonlocal parameters, hub parameters and section aspect ratios. The spin-softening and Coriolis effects are individually illustrated and discussed. The present advanced model for rotating nano-beams is reported for the first time. The present study would help in understanding the dynamics of rotating nano-beams in a comprehensive manner.

Keywords: Nano-beam; Eringen's nonlocal theory; Rotating beam; Free vibration; Spin-softening; Coriolis effect.

List of Symbols

b, h, L	width, thickness and length of the beam respectively
i	$\sqrt{-1}$
nu, nv, nw	number of functions for ϕ_i^u , ϕ_i^v and ϕ_i^w respectively
t	time
t_1, t_2	two different time instants
(u_0, v_0, w_0)	displacement components at the mid-plane
(u, v, w)	displacement components of any point on the beam
x, y, z	coordinates along the length, width and thickness directions respectively
A	cross sectional area
$\{\mathbf{D}\}$	vector of generalized coordinates having components d_j
E, ρ	Young's modulus and mass density respectively
F_c	centrifugal force field
$[\mathbf{G}]$	gyroscopic matrix of dimension $(nu + nv + nw)$ and having components $[g_{ij}]$
I_y, I_z	area moment of inertia about the y and z axis respectively

$[\mathbf{I}]$	identity matrix of dimension $(nu + nv + nw)$
$[\mathbf{K}]$	stiffness matrix of dimension $(nu + nv + nw)$ and having components $[k_{ij}]$
$[\mathbf{M}]$	mass matrix of dimension $(nu + nv + nw)$ and having components $[m_{ij}]$
N_x, M_y, M_z	axial force, and bending moment about the y and z axis respectively
R	hub radius
$[\mathbf{S}]$	a transformation matrix in state space
T, U, V	kinetic energy, strain energy and potential energy respectively
$\{\mathbf{X}\}$	state vector of dimension $2(nu + nv + nw)$ having form $\{\mathbf{D}^T, \dot{\mathbf{D}}^T\}^T$
$\{\mathbf{X}_c\}$	vector of dimension $2(nu + nv + nw)$ representing time-independent amplitude of motion in state space
α	eigenvalue
$\beta = R/L$	hub parameter
δ	operator denoting virtual change
ε_x, σ_x	local axial strain and nonlocal axial stress respectively
$\varepsilon_{ij}, \sigma_{ij}, \sigma_{ij}^l$	local strain tensor, nonlocal stress tensor and local stress tensor respectively
η, e_0, a	nonlocal parameter, material constant and an internal characteristic length respectively
$\kappa = b/h$	section aspect ratio
$\mu = e_0 a/L$	normalized nonlocal parameter
$\xi = L/h$	slenderness parameter
$(\phi_i^u, \phi_i^v, \phi_i^w)$	set of functions to approximate the displacement fields (u_0, v_0, w_0)
ω, ω^*	dimensional and non-dimensional frequency of vibration respectively
$\bar{\omega}, \bar{\bar{\omega}}$	normalized frequency of vibration to represent spin-softening and Coriolis effects respectively
$\omega_{\text{without Coriolis}}, \omega_{\text{with Coriolis}}$	frequency of vibration without and with Coriolis effect respectively

$\omega_{\text{without spin-soft.}}$	frequency of vibration without and with spin-softening effect respectively
$\omega_{\text{with spin-soft.}}$	
Ω, Ω^*	dimensional and non-dimensional angular speed respectively
∇^2	Laplace operator
$(\dot{\cdot}), (\ddot{\cdot})$	single dot and double dots represent single and double time derivatives respectively

1. Introduction

Due to tremendous advancements in nano-fabrication technology in recent times, the use of nano-electro-mechanical systems (NEMS) is expected to rise in various engineering applications. So it is worthwhile to study mechanical behavior of various NEMS components. Considering its applications in various rotating NEMS such as nano-turbines, nano-motors, rotating nano-actuators etc. [1-4], study of dynamic behavior of rotating nano-beams is an emerging research area. Over the past few years, Eringen's nonlocal elasticity theory (ENET) [5-7] is being considered for theoretically studying mechanical behavior of various nano-structures. It is to be mentioned that the classical continuum theory is incapable to address size-effects in the absence of material length scale parameters [8]. The present work aims to study the free vibration behavior of a rotating nano-beam based on ENET.

Study on free vibration behavior of rotating beams based on classical theory has been an interesting and challenging research problem for many years due to huge practical applications of rotating beam-like structures. Some of those important studies involving homogeneous beam materials are found in Refs. [9-18] for straight beam configuration, and in Refs. [19-22] for pre-twisted beam configuration. With the advent of functionally graded materials (FGMs), which constitute a class of advanced composites with continuous gradation of material properties, research studies on rotating functionally graded (FG) beams were taken up by various researchers. Some of those studies are found in Refs. [23-28] for straight beam configuration and in Refs. [29-31] for pre-twisted beam configuration.

The present literature survey reveals that the research studies involving nano-beams based on ENET have been on the rise. Some of the research investigations on non-rotating nano-beams

based on ENET can be briefly categorized as follows: Bending behaviors of homogeneous nano-beams were studied in Refs. [32-35]; Buckling behaviors of homogeneous nano-beams were studied in Refs. [33-34]; Vibration behaviors of homogeneous nano-beams were studied in Refs. [33-37]; Bending behaviors of FG nano-beams were studied in Ref. [38]; Buckling behaviors of FG nano-beams were studied in Refs. [38-40]; Vibration behaviors of FG nano-beams were studied in Refs. [38,39,41,42]. Research studies on carbon nanotubes (CNTs) which are frequently used in NEMS, are also being performed in recent times. Vibration behaviors of CNTs have been studied in Refs. [43-46] based on ENET, and in Ref. [47] based on ENET coupled with strain gradient theory. Some of the recent studies reveal that the researchers are considering ENET in conjunction with various other continuum based non-classical theories to study the mechanical behavior of nano-beams. Some of them are briefly mentioned here. Malekzadeh and Shojaee [48] studied non-linear free vibration behavior of nano-beams employing ENET and Gurtin-Murdoch surface elasticity theory (SET) [49]. Attia and Mahmoud [50] studied bending and buckling behaviors of nano-beams based on ENET, coupled with modified couple stress theory (MCST) [51] and Gurtin-Murdoch SET. Barretta et al. [52] presented a first strain gradient model for studying static bending response of FG nano-beams based on ENET. Lv and Liu [53] studied non-linear static bending response of FG nano-beams based on nonlocal strain gradient theory. Şimşek [54] investigated static bending, buckling, and free and forced vibration behaviors of FG nano-beams based on nonlocal strain gradient theory. Li and Hu [55] investigated the post-buckling behaviors of FG nano-beams based on nonlocal strain gradient theory. Esfahani et al. [56] carried out non-linear vibration analysis of an electrostatic FG nano-beam resonator employing nonlocal strain gradient theory and Gurtin-Murdoch SET. Apart from these, Shanab et al. [57] studied non-linear bending behavior of FG nano-beams based on Gurtin-Murdoch SET.

Theoretical investigation of free vibration behaviors of rotating micro-beams is also an emerging research area due its present and potential applications in various micro-electromechanical systems (MEMS). The review of the recent literature reveals that micro-size structural elements are generally modeled using either MCST or modified strain gradient theory (MSGT), both being continuum based non-classical theories. MCST based free vibration behaviors of rotating micro-beams were reported in Refs. [58-60] for homogeneous materials and in Refs. [61-63] for FGMs, whereas MSGT based free vibration behaviors of rotating micro-beams were reported in Ref. [64]. The works of Refs. [58, 60-62, 64] dealt with flap-wise

vibration modes, whereas Refs. [59,63] dealt with both chord-wise and flap-wise modes. Free vibration behaviors of pre-twisted rotating FG micro-beams were reported in Ref. [65] based on MSGT and in Ref. [66] based on MCST.

Based on ENET, various theoretical studies on free flap-wise vibration of rotating nano-beams have been reported considering Euler-Bernoulli Beam theory (EBT) [67-76] and Timoshenko beam theory (TBT) [77-81] as follows: Pradhan and Murmu [67] employed differential quadrature method (DQM) for rotating nano-beams; Aranda-Ruiz et al. [68] used pseudo-spectral collocation method for non-uniform section rotating nano-beams; Ghadiri et al. [69] employed a power series method to obtain the exact solution of the natural frequencies in high-temperature environment for cantilever and propped cantilever rotating nano-beams; Ghafarian and Ariaei [70] employed differential transform method (DTM) for multiple elastically-connected rotating nano-beam systems; Considering surface effects, Mohammadi et al. [71] employed DQM for viscoelastic rotating nano-beams embedded in visco-Pasternak elastic medium and operating in thermal environment; Pourasghar et al. [72] used DQM for rotating nano-beams subjected to axial follower force; Employing DQM, non-linear free flap-wise vibrations of cantilever and propped cantilever rotating nano-beams were studied by Ghadiri and Shafiei [73] and by Shafiei et al. [74] considering high-temperature environment; Khaniki [75] used Eringen's two-phase local/nonlocal model and employed modified generalized DQM (GDQM) for rotating nano-beams; Atanasov and Stojanović [76] employed Galerkin method to study forced vibration behavior of rotating nano-beams; Ghadiri et al. [77] used GDQM for cantilever and propped cantilever rotating nano-beams, incorporating surface elasticity and temperature effects; Preethi et al. [78] used finite element method (FEM) for laminated rotating nano-cantilever beams and employed surface effects in the model; Azimi et al. [79,80] used DQM for cantilever and propped cantilever rotating FG nano-beams in high-temperature environment; Kiani and Soltani [81] determined natural vibrational frequencies for longitudinal, chord-wise and flap-wise modes of double-nano-beam systems employing Galerkin method. **Recently, Malik and Das [82] presented a primitive model to study the free vibration behavior of rotating nano-beams based on ENET, involving both chord-wise and flap-wise modes.** Apart from these, Baghani et al. [83] investigated dynamics and stability behaviors of rotating nano-beams based on ENET coupled with Gurtin-Murdoch SET. Employing ENET, wave dispersion characteristics and flap-wise free vibration behaviors of rotating CNTs were reported in Refs. [84,85].

The literature review presented in the previous paragraph reveals that, except Ref. [82], the chord-wise and axial free vibration behaviors of rotating nano-beams have not been studied yet though it is an important aspect for understanding dynamics of rotating nano-beams. The present work develops an advanced mathematical model that is capable to study both flap-wise and chord-wise free vibrations as well as axial free vibration behavior of rotating nano-beams. Additionally, this makes it possible to illustrate the mode-switching phenomena between the flap-wise and chord-wise motions for various modes. Also the present model incorporates both spin-softening and Coriolis effects for a rotating nano-beam, which is also reported for the first time. The size-dependence of spin-softening and Coriolis effects are illustrated and discussed separately which is also not available in the literature. Unlike the previous studies, the present results are shown for relatively higher non-dimensional rotational speeds, wherever possible. The present method of virtual displacement based governing equations and Ritz approximations for solving rotating nano-beam-type problems is reported for the first time through this work.

In the present work, the governing equations are derived based on EBT and employing Hamilton's principle whereas ENET is considered to address size-effect in nano-scale. Ritz method is followed to discretize the governing equations in algebraic form, which are then transformed to the state-space to develop an eigenvalue problem. The stable solutions of the problem are obtained by considering only the pure imaginary roots of the eigenvalue problem. The results are presented in non-dimensional speed-frequency plane for the first six flexural vibration modes involving both flap-wise and chord-wise motions for different non-local parameters, hub parameters and section aspect ratios. The speed-frequency behavior for the first axial vibration mode is also presented. The size-dependent spin-softening effect is reported for the chord-wise and axial modes, and the influence of Coriolis effect on the axial mode is presented. The relevant mode shapes are illustrated. The present study would help in understanding the dynamics of rotating nano-beams in a comprehensive manner.

2. Mathematical Formulation

A beam with length L , width b , thickness h , and attached to a rigid hub of radius R is considered to be rotating with constant angular speed Ω (Fig. 1) about the hub axis. A rectangular Cartesian coordinate system $x-y-z$, originated at the hub-beam joint O is considered to be rotating with the same speed Ω where x , y and z axes are aligned along the length, width and

thickness directions of the beam respectively, and z axis is parallel to the hub axis. The Young's modulus and density of the beam are denoted by E and ρ respectively. Also, $A(=bh)$ represents the cross sectional area, and $I_y(=bh^3/12)$ and $I_z(=hb^3/12)$ represent the area moments of inertia about the y and z axes respectively. It is to be mentioned that the flexural vibratory motion taking place in the plane of rotation ($x-y$) is called chord-wise vibration and that occurring perpendicular to the plane of rotation ($x-z$) is called flap-wise vibration.

Considering u_0 , v_0 and w_0 as the mid-plane displacements along the respective axes, the displacement fields (u, v, w) and the axial strain field (ε_x) for any point (x, y, z) at any time instant t are expressed following EBT as given below:

$$\begin{aligned} u(x, y, z, t) &= u_0(x, t) - y \frac{\partial v_0(x, t)}{\partial x} + z \frac{\partial w_0(x, t)}{\partial x}, \\ v(x, y, z, t) &= v_0(x, t), \\ w(x, y, z, t) &= w_0(x, t), \end{aligned} \quad (1)$$

$$\varepsilon_x = \frac{\partial u_0}{\partial x} - y \frac{\partial^2 v_0}{\partial x^2} + z \frac{\partial^2 w_0}{\partial x^2}. \quad (2)$$

While writing Eq. (1), the right hand rule of rotation is followed for the Cartesian coordinate system $x-y-z$, e.g., for any positive rotation (as per the right hand rule) of beam section about z -axis, any point (x, y, z) on the beam having positive y -coordinate will be axially displaced along the negative x -direction, and vice-versa for any point having negative y -coordinate. The governing equations are derived employing Hamilton's principle which is given as follows:

$$\int_{t_1}^{t_2} (\delta T - \delta U - \delta V) dt = 0, \quad (3)$$

where the operator δ is used to denote the virtual form of kinetic energy (T), strain energy (U) and work potential (V).

The expression of kinetic energy in rotating coordinates ($x-y-z$) is expressed as follows [18,59],

$$T = \frac{\rho A}{2} \int_0^L \left\{ \left(\frac{\partial u}{\partial t} - \Omega v \right)^2 + \left(\frac{\partial v}{\partial t} + \Omega u \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right\} dx. \quad (4)$$

The axial stress (σ_x) resultants in the form of axial force (N_x) and bending moments (M_z, M_y) are given by,

$$N_x = \int_A \sigma_x dA, M_z = -\int_A y \sigma_x dA, M_y = \int_A z \sigma_x dA. \quad (5)$$

The rotating beam is subjected to centrifugal force (F_c) field, which is derived to the following form,

$$F_c(x) = \rho A \Omega^2 \int_x^L (R+x) dx = \frac{\rho A \Omega^2}{2} [(R+L)^2 - (R+x)^2]. \quad (6)$$

Substituting in Eq. (3), the expressions of δT using (4), δU considering (5) and δV considering (6), the following equation is obtained:

$$\begin{aligned} & \int_{t_1}^{t_2} \int_0^L \left[\rho A \left\{ \frac{\partial u_0}{\partial t} \delta \left(\frac{\partial u_0}{\partial t} \right) + \frac{\partial v_0}{\partial t} \delta \left(\frac{\partial v_0}{\partial t} \right) + \frac{\partial w_0}{\partial t} \delta \left(\frac{\partial w_0}{\partial t} \right) \right\} + \rho I_z \frac{\partial^2 v_0}{\partial t \partial x} \delta \left(\frac{\partial^2 v_0}{\partial t \partial x} \right) + \rho I_y \frac{\partial^2 w_0}{\partial t \partial x} \delta \left(\frac{\partial^2 w_0}{\partial t \partial x} \right) \right. \\ & + \Omega \rho A \left\{ -v_0 \delta \left(\frac{\partial u_0}{\partial t} \right) - \frac{\partial u_0}{\partial t} \delta v_0 + \frac{\partial v_0}{\partial t} \delta u_0 + u_0 \delta \left(\frac{\partial v_0}{\partial t} \right) \right\} \\ & + \Omega^2 \left\{ \rho A (u_0 \delta u_0 + v_0 \delta v_0) + \rho I_z \frac{\partial v_0}{\partial x} \delta \left(\frac{\partial v_0}{\partial x} \right) + \rho I_y \frac{\partial w_0}{\partial x} \delta \left(\frac{\partial w_0}{\partial x} \right) \right\} \Big] dx \\ & - \int_0^L \left[N_x \delta \left(\frac{\partial u_0}{\partial x} \right) + M_z \delta \left(\frac{\partial^2 v_0}{\partial x^2} \right) + M_y \delta \left(\frac{\partial^2 w_0}{\partial x^2} \right) \right] dx \\ & - \int_0^L F_c \left[\frac{\partial v_0}{\partial x} \delta \left(\frac{\partial v_0}{\partial x} \right) + \frac{\partial w_0}{\partial x} \delta \left(\frac{\partial w_0}{\partial x} \right) \right] dx \Big] dt = 0. \end{aligned} \quad (7)$$

It is to be mentioned that while deriving δV due to centrifugal stiffening effect, the von Kármán non-linear strain-displacement relations i.e., $\frac{1}{2} \left(\frac{\partial v_0}{\partial x} \right)^2$ and $\frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2$ are employed [18, 58]. It is further noted that the following simplifications are considered due to bi-symmetric nature of the beam section to derive $\delta T : \int_A \{y, z, yz\} dA = 0$.

It is worthwhile to explain here the use of von Kármán type non-linear terms. The present problem can be visualized as a two-step phenomenon. The first step involves generation of centrifugal force (F_c) which has been determined using relation (6), and the beam configuration remains essentially straight during this phenomenon. The second step involves small amplitude

oscillatory motion of the centrifugally deformed beam in its neighborhood, and thus necessitates determination of the virtual work done (δV) by the centrifugal force. It is known [86] that the von Kármán type non-linear terms ($\frac{1}{2}\left(\frac{\partial v_0}{\partial x}\right)^2$ and $\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2$) represent membrane or axial strains due to the respective transverse displacements (v_0 and w_0). Hence to determine the virtual work done by the centrifugal force in moving through the axial displacements as a result of membrane strains during vibratory motion, the von Kármán type non-linear terms are employed in Eq. (7).

Integrating Eq. (7) by parts suitably, it leads to the following Euler-Lagrange equations of motion along with the boundary conditions (B.C.) at $x = 0$ and $x = L$:

$$\frac{\partial N_x}{\partial x} + \Omega^2 \rho A u_0 = \rho A \frac{\partial^2 u_0}{\partial t^2} - 2\Omega \rho A \frac{\partial v_0}{\partial t}, \quad (8)$$

$$\text{B. C.: } N_x = 0 \text{ or, } \delta u_0 = 0. \quad (8a)$$

$$-\frac{\partial^2 M_z}{\partial x^2} + \frac{\partial}{\partial x} \left(F_c \frac{\partial v_0}{\partial x} \right) + \Omega^2 \rho A v_0 - \Omega^2 \rho I_z \frac{\partial^2 v_0}{\partial x^2} = \rho A \frac{\partial^2 v_0}{\partial t^2} - \rho I_z \frac{\partial^4 v_0}{\partial t^2 \partial x^2} + 2\Omega \rho A \frac{\partial u_0}{\partial t}, \quad (9)$$

$$\text{B. C.: } -\frac{\partial M_z}{\partial x} + F_c \frac{\partial v_0}{\partial x} - \Omega^2 \rho I_z \frac{\partial v_0}{\partial x} + \rho I_z \frac{\partial^3 v_0}{\partial t^2 \partial x} = 0 \text{ or, } \delta v_0 = 0, \quad (9a)$$

$$\text{B. C.: } M_z = 0 \text{ or, } \delta \left(\frac{\partial v_0}{\partial x} \right) = 0. \quad (9b)$$

$$-\frac{\partial^2 M_y}{\partial x^2} + \frac{\partial}{\partial x} \left(F_c \frac{\partial w_0}{\partial x} \right) - \Omega^2 \rho I_y \frac{\partial^2 w_0}{\partial x^2} = \rho A \frac{\partial^2 w_0}{\partial t^2} - \rho I_y \frac{\partial^4 w_0}{\partial t^2 \partial x^2}, \quad (10)$$

$$\text{B. C.: } -\frac{\partial M_y}{\partial x} + F_c \frac{\partial w_0}{\partial x} - \Omega^2 \rho I_y \frac{\partial w_0}{\partial x} + \rho I_y \frac{\partial^3 w_0}{\partial t^2 \partial x} = 0 \text{ or, } \delta w_0 = 0, \quad (10a)$$

$$\text{B. C.: } M_y = 0 \text{ or, } \delta \left(\frac{\partial w_0}{\partial x} \right) = 0. \quad (10b)$$

While deriving Eqs. (8-10), the virtual displacements ($\delta u_0, \delta v_0, \delta w_0$) and the virtual derivatives

$\left\{ \delta \left(\frac{\partial v_0}{\partial x} \right), \delta \left(\frac{\partial w_0}{\partial x} \right) \right\}$ are equated to zero at the initial (t_1) and final (t_2) time instants.

2.1. Nonlocal force and moments

According to Eringen's nonlocal elasticity theory [5-7,33,34], the stress field (σ_{ij}) (also called nonlocal stress field) at any representative point in an elastic continuum depends not only on strain field at that point (which leads to macroscopic or local stress (σ_{ij}^l)) but also on strains at all other points of the body. This is mathematically expressed in an equivalent differential form as follows:

$$(1 - \eta^2 \nabla^2) \sigma_{ij} = \sigma_{ij}^l, \quad (11)$$

where $\eta = e_0 a$ is the nonlocal parameter that considers size-effect in nano-scale, e_0 is a material constant, a is an internal characteristic length, ∇^2 is Laplace operator, and σ_{ij}^l is related to the local strain field (ε_{ij}) using Hooke's law. For the present problem, Eq. (11) is reduced to the following form:

$$\sigma_x - \eta^2 \frac{\partial^2 \sigma_x}{\partial x^2} = E \varepsilon_x. \quad (12)$$

Integrating Eq. (11) suitably over the cross sectional area, and using relations (2) and (5), the following nonlocal force-displacement and moment-displacement relations are obtained:

$$N_x - \eta^2 \frac{\partial^2 N_x}{\partial x^2} = EA \frac{\partial u_0}{\partial x}, \quad (13a)$$

$$M_z - \eta^2 \frac{\partial^2 M_z}{\partial x^2} = EI_z \frac{\partial^2 v_0}{\partial x^2}, \quad (13b)$$

$$M_y - \eta^2 \frac{\partial^2 M_y}{\partial x^2} = EI_y \frac{\partial^2 w_0}{\partial x^2}. \quad (13c)$$

Employing Eqs. (13a)-(13c) appropriately in Eqs. (8)-(10), the nonlocal force and moments are obtained in terms of displacements as follows:

$$N_x = EA \frac{\partial u_0}{\partial x} + \eta^2 \rho A \left(-\Omega^2 \frac{\partial u_0}{\partial x} - 2\Omega \frac{\partial^2 v_0}{\partial t \partial x} + \frac{\partial^3 u_0}{\partial t^2 \partial x} \right), \quad (14a)$$

$$M_z = EI_z \frac{\partial^2 v_0}{\partial x^2} + \eta^2 \left[\frac{\partial}{\partial x} \left(F_c \frac{\partial v_0}{\partial x} \right) + \rho \Omega^2 \left\{ A v_0 - I_z \frac{\partial^2 v_0}{\partial x^2} \right\} - 2\Omega \rho A \frac{\partial u_0}{\partial t} - \rho \left\{ A \frac{\partial^2 v_0}{\partial t^2} - I_z \frac{\partial^4 v_0}{\partial t^2 \partial x^2} \right\} \right], \quad (14b)$$

$$M_y = EI_y \frac{\partial^2 w_0}{\partial x^2} + \eta^2 \left[\frac{\partial}{\partial x} \left(F_c \frac{\partial w_0}{\partial x} \right) - \rho \Omega^2 I_y \frac{\partial^2 w_0}{\partial x^2} - \rho \left\{ A \frac{\partial^2 w_0}{\partial t^2} - I_y \frac{\partial^4 w_0}{\partial t^2 \partial x^2} \right\} \right]. \quad (14c)$$

It can be seen that putting the nonlocal parameter $\eta=0$ in (14a)-(14c) leads to the classical or local force and moments expressions.

2.2. Governing equations

Substituting the expressions of nonlocal force and moments given by (14a)-(14c) into Eq. (7), the equations of motion are obtained as follows:

$$\begin{aligned}
 & \int_0^L \int_{t_1}^{t_2} E \left\{ A \frac{\partial u_0}{\partial x} \delta \left(\frac{\partial u_0}{\partial x} \right) + I_z \frac{\partial^2 v_0}{\partial x^2} \delta \left(\frac{\partial^2 v_0}{\partial x^2} \right) + I_y \frac{\partial^2 w_0}{\partial x^2} \delta \left(\frac{\partial^2 w_0}{\partial x^2} \right) \right\} \\
 & + \left\{ F_c \left\{ \frac{\partial v_0}{\partial x} \delta \left(\frac{\partial v_0}{\partial x} \right) + \frac{\partial w_0}{\partial x} \delta \left(\frac{\partial w_0}{\partial x} \right) \right\} + \eta^2 \left\{ \frac{\partial}{\partial x} \left(F_c \frac{\partial v_0}{\partial x} \right) \delta \left(\frac{\partial^2 v_0}{\partial x^2} \right) + \frac{\partial}{\partial x} \left(F_c \frac{\partial w_0}{\partial x} \right) \delta \left(\frac{\partial^2 w_0}{\partial x^2} \right) \right\} \right\} \\
 & - \Omega^2 \rho \left\{ \left\{ A u_0 \delta u_0 + A v_0 \delta v_0 + I_z \frac{\partial v_0}{\partial x} \delta \left(\frac{\partial v_0}{\partial x} \right) + I_y \frac{\partial w_0}{\partial x} \delta \left(\frac{\partial w_0}{\partial x} \right) \right\} \right. \\
 & \left. + \eta^2 \left\{ A \frac{\partial u_0}{\partial x} \delta \left(\frac{\partial u_0}{\partial x} \right) - A v_0 \delta \left(\frac{\partial^2 v_0}{\partial x^2} \right) + I_z \frac{\partial^2 v_0}{\partial x^2} \delta \left(\frac{\partial^2 v_0}{\partial x^2} \right) + I_y \frac{\partial^2 w_0}{\partial x^2} \delta \left(\frac{\partial^2 w_0}{\partial x^2} \right) \right\} \right\} \\
 & + 2\Omega \rho A \left\{ \left\{ -\frac{\partial v_0}{\partial t} \delta u_0 + \frac{\partial u_0}{\partial t} \delta v_0 \right\} - \eta^2 \left\{ \frac{\partial^2 v_0}{\partial t \partial x} \delta \left(\frac{\partial u_0}{\partial x} \right) + \frac{\partial u_0}{\partial t} \delta \left(\frac{\partial^2 v_0}{\partial x^2} \right) \right\} \right\} \\
 & + \rho \left\{ \left\{ A \frac{\partial^2 u_0}{\partial t^2} \delta u_0 + A \frac{\partial^2 v_0}{\partial t^2} \delta v_0 + A \frac{\partial^2 w_0}{\partial t^2} \delta w_0 + I_z \frac{\partial^3 v_0}{\partial t^2 \partial x} \delta \left(\frac{\partial v_0}{\partial x} \right) + I_y \frac{\partial^3 w_0}{\partial t^2 \partial x} \delta \left(\frac{\partial w_0}{\partial x} \right) \right\} \right. \\
 & + \eta^2 \rho \left\{ A \frac{\partial^3 u_0}{\partial t^2 \partial x} \delta \left(\frac{\partial u_0}{\partial x} \right) - A \frac{\partial^2 v_0}{\partial t^2} \delta \left(\frac{\partial^2 v_0}{\partial x^2} \right) + I_z \frac{\partial^4 v_0}{\partial t^2 \partial x^2} \delta \left(\frac{\partial^2 v_0}{\partial x^2} \right) - A \frac{\partial^2 w_0}{\partial t^2} \delta \left(\frac{\partial^2 w_0}{\partial x^2} \right) \right. \\
 & \left. \left. + I_y \frac{\partial^4 w_0}{\partial t^2 \partial x^2} \delta \left(\frac{\partial^2 w_0}{\partial x^2} \right) \right\} \right\} dx dt = 0. \tag{15}
 \end{aligned}$$

In Eq. (15), the terms underscored by a single line arise due to the centrifugal force field, which is induced due to rotation of the beam. As the centrifugal force field (F_c) developed in a rotating beam is always tensile in nature for the present beam configuration (can be seen from Eq. (6)), it induces stiffening/hardening effect onto the beam. This effect is incorporated through stretching-bending coupling as explained previously in connection with the use of von Kármán type non-linear terms. This effect is the main cause of stiffening in a rotating beam (which leads to increase in vibration frequency, which has been discussed in section 3). It can be seen that, out of

these terms, those associated with the square of the nonlocal parameter η are due to the size-dependent effect of the centrifugal force.

Looking further into Eq. (15), the terms associated with Ω^2 (underscored by double lines) are responsible for softening of the beam due to rotation and hence are said to impart spin-softening effect. In contrast to hardening/stiffening, softening leads to decrease in frequency (which has been discussed in section 3). Out of these terms, those associated with the square of the nonlocal parameter η are due to the size-dependent effect of spin-softening. It can be seen that those terms without being associated with the square of the nonlocal parameter η are carrying a negative sign, and thus are found to impart diminishing effect on the stiffness of the beam. The spin-softening effect arises due to the fact that a rotating coordinate frame (and hence accelerating even through it is rotating with the beam with constant angular speed Ω) has been considered to formulate the problem [17]. In this regard, the expression of kinetic energy, expressed in rotating coordinates through Eq. (4), can be seen where each of the first two terms contain the parameter Ω , which when expanded give rise to spin-softening effect and Coriolis effect.

Further, in Eq. (15), the terms associated with angular speed Ω are due to Coriolis acceleration (underscored by triple lines), and out of these, those associated with the square of the nonlocal parameter η impart size-dependent Coriolis effect. The terms responsible for Coriolis effect show coupling between the axial (u_0) and in-plane transverse (v_0) displacements, and thus make the axial and chord-wise vibratory motions coupled. Coriolis effect is difficult to visualize but is said to develop due to simultaneous actions of accelerated radial motion and rotational motion of the points on the beam. Here the radial motion implies the motion, taking place along a line which is parallel to the plane of rotation ($x-y$) and which joins any point on the beam to the appropriate point on the axis of rotation.

Following Ritz method, the displacement fields are approximated as given below:

$$u_0(x,t) = \sum_{j=1}^{nu} d_j(t) \phi_j^u(x), v_0(x,t) = \sum_{j=1}^{nv} d_{nu+j}(t) \phi_j^v(x), w_0(x,t) = \sum_{j=1}^{nw} d_{nu+nv+j}(t) \phi_j^w(x), \quad (16)$$

where $\phi_j^u(x)$, $\phi_j^v(x)$ and $\phi_j^w(x)$ are sets of orthogonal admissible functions with nu , nv and nw number of functions respectively in each set; $d_j(t)$ is the set of generalized coordinates dependent on time. The sets of functions $\{\phi_j^u, \phi_j^v, \phi_j^w\}$ are numerically generated employing Gram-Schmidt

orthogonalization algorithm, by using the corresponding lowest order admissible functions for each of the displacement fields. The lowest order admissible functions used for the present problem are provided in Table 1 [63]. Substituting (16) in Eq. (15), the governing equations are obtained as follows:

$$[\mathbf{M}]\{\ddot{\mathbf{D}}\} + [\mathbf{G}]\{\dot{\mathbf{D}}\} + [\mathbf{K}]\{\mathbf{D}\} = 0. \quad (17)$$

Here $[\mathbf{M}]$, $[\mathbf{G}]$ and $[\mathbf{K}]$ are mass matrix, gyroscopic matrix and stiffness matrix of the centrifugally stiffened nano-beam, and $\{\mathbf{D}\}$ is the vector of time-dependent generalized coordinates having components $d_j(t)$. It is noted that a dot ($\dot{\cdot}$) over any parameter symbolizes its time derivative and double dots ($\ddot{\cdot}$) over any parameter symbolizes its double time derivative. The gyroscopic matrix ($[\mathbf{G}]$) contributes the Coriolis effect to the equations of motion. It will be seen that both $[\mathbf{M}]$ and $[\mathbf{K}]$ are diagonal matrices. The elements of $[\mathbf{K}]$ are shown by (18a)-(18c), of $[\mathbf{G}]$ are shown by (19a)-(19b), and of $[\mathbf{M}]$ are shown by (20a)-(20c) below. Now a decorative analogy between Eq. (15), and that of (18a)-(18c) and (19a)-(19b) are drawn here. It shows that the terms underscored by a single line are due to the direct effect of centrifugal force, that underscored by double lines are due to the effect of spin-softening, and that underscored by triple lines are due to Coriolis effect.

Diagonal Elements of stiffness matrix:

$$[k_{ij}]_{\substack{i=1,nu \\ j=1,nu}} = EA \int_0^L \frac{d\phi_i^u}{dx} \frac{d\phi_j^u}{dx} dx - \Omega^2 \rho A \int_0^L \phi_i^u \phi_j^u dx - \eta^2 \Omega^2 \rho A \int_0^L \frac{d\phi_i^u}{dx} \frac{d\phi_j^u}{dx} dx, \quad (18a)$$

$$\begin{aligned} [k_{ij}]_{\substack{i=nu+1,nu+nv \\ j=nu+1,nu+nv}} &= EI_z \int_0^L \frac{d^2\phi_{i-nu}^v}{dx^2} \frac{d^2\phi_{j-nu}^v}{dx^2} dx \\ &+ \int_0^L F_c \frac{d\phi_{i-nu}^v}{dx} \frac{d\phi_{j-nu}^v}{dx} dx + \eta^2 \left(\int_0^L F_c \frac{d^2\phi_{i-nu}^v}{dx^2} \frac{d^2\phi_{j-nu}^v}{dx^2} dx + \int_0^L \frac{\partial F_c}{\partial x} \frac{d\phi_{i-nu}^v}{dx} \frac{d^2\phi_{j-nu}^v}{dx^2} dx \right) \\ &- \Omega^2 \left(\rho A \int_0^L \phi_{i-nu}^v \phi_{j-nu}^v dx + \rho I_z \int_0^L \frac{d\phi_{i-nu}^v}{dx} \frac{d\phi_{j-nu}^v}{dx} dx \right) \\ &+ \eta^2 \Omega^2 \left(\rho A \int_0^L \phi_{i-nu}^v \frac{d^2\phi_{j-nu}^v}{dx^2} dx - \rho I_z \int_0^L \frac{d^2\phi_{i-nu}^v}{dx^2} \frac{d^2\phi_{j-nu}^v}{dx^2} dx \right), \end{aligned} \quad (18b)$$

$$\begin{aligned}
\left[k_{ij} \right]_{\substack{i=nu+nv+1, nu+nv+nw \\ j=nu+nv+1, nu+nv+nw}} &= EI_y \int_0^L \frac{d^2 \phi_{i-nu-nv}^w}{dx^2} \frac{d^2 \phi_{j-nu-nv}^w}{dx^2} dx \\
&+ \int_0^L F_c \frac{d \phi_{i-nu-nv}^w}{dx} \frac{d \phi_{j-nu-nv}^w}{dx} dx + \eta^2 \left(\int_0^L F_c \frac{d^2 \phi_{i-nu-nv}^w}{dx^2} \frac{d^2 \phi_{j-nu-nv}^w}{dx^2} dx + \int_0^L \frac{\partial F_c}{\partial x} \frac{d \phi_{i-nu-nv}^w}{dx} \frac{d \phi_{j-nu-nv}^w}{dx^2} dx \right) \quad (18c) \\
&\underline{\underline{-\Omega^2 \rho I_y \int_0^L \frac{d \phi_{i-nu-nw}^w}{dx} \frac{d \phi_{j-nu-nw}^w}{dx} dx - \eta^2 \Omega^2 \rho I_y \int_0^L \frac{d^2 \phi_{i-nu-nv}^w}{dx^2} \frac{d^2 \phi_{j-nu-nv}^w}{dx^2} dx.}}
\end{aligned}$$

Non-zero elements of gyroscopic matrix:

$$\left[g_{ij} \right]_{\substack{i=1, nu \\ j=nu+1, nu+nv}} = -2\Omega \rho A \left(\int_0^L \phi_i^u \phi_{j-nu}^v dx + \eta^2 \int_0^L \frac{d \phi_i^u}{dx} \frac{d \phi_{j-nu}^v}{dx} dx \right), \quad (19a)$$

$$\left[g_{ij} \right]_{\substack{i=nu+1, nu+nv \\ j=1, nu}} = 2\Omega \rho A \left(\int_0^L \phi_{i-nu}^v \phi_j^u dx - \eta^2 \int_0^L \frac{d^2 \phi_{i-nu}^v}{dx^2} \phi_j^u dx \right). \quad (19b)$$

Diagonal elements of mass matrix:

$$\left[m_{ij} \right]_{\substack{i=1, nu \\ j=1, nu}} = \rho A \int_0^L \phi_i^u \phi_j^u dx + \eta^2 \rho A \int_0^L \frac{d \phi_i^u}{dx} \frac{d \phi_j^u}{dx} dx, \quad (20a)$$

$$\begin{aligned}
\left[m_{ij} \right]_{\substack{i=nu+1, nu+nv \\ j=nu+1, nu+nv}} &= \rho A \int_0^L \phi_{i-nu}^v \phi_{j-nu}^v dx + \rho I_z \int_0^L \frac{d \phi_{i-nu}^v}{dx} \frac{d \phi_{j-nu}^v}{dx} dx \\
&+ \eta^2 \left(-\rho A \int_0^L \phi_{i-nu}^v \frac{d^2 \phi_{j-nu}^v}{dx^2} dx + \rho I_z \int_0^L \frac{d^2 \phi_{i-nu}^v}{dx^2} \frac{d^2 \phi_{j-nu}^v}{dx^2} dx \right), \quad (20b)
\end{aligned}$$

$$\begin{aligned}
\left[m_{ij} \right]_{\substack{i=nu+nv+1, nu+nv+nw \\ j=nu+nv+1, nu+nv+nw}} &= \rho A \int_0^L \phi_{i-nu-nv}^w \phi_{j-nu-nv}^w dx + \rho I_y \int_0^L \frac{d \phi_{i-nu-nv}^w}{dx} \frac{d \phi_{j-nu-nv}^w}{dx} dx \\
&+ \eta^2 \left(-\rho A \int_0^L \phi_{i-nu-nv}^w \frac{d^2 \phi_{j-nu-nv}^w}{dx^2} dx + \rho I_y \int_0^L \frac{d^2 \phi_{i-nu-nv}^w}{dx^2} \frac{d^2 \phi_{j-nu-nv}^w}{dx^2} dx \right). \quad (20c)
\end{aligned}$$

2.3. Solution of governing equations

Defining a state vector of dimension $2(nu+nv+nw)$ as $\{\mathbf{X}\} = \{\mathbf{D}^T, \dot{\mathbf{D}}^T\}^T$, Eq. (17) is transformed to the state space to the form given below [29]:

$$[\mathbf{S}]\{\mathbf{X}\} = \{\dot{\mathbf{X}}\} \quad (21)$$

where the transformation matrix $[\mathbf{S}]$ is given by,

$$[\mathbf{S}] = \left[\begin{array}{c|c} 0 & [\mathbf{I}] \\ \hline -[\mathbf{M}]^{-1}[\mathbf{K}] & -[\mathbf{M}]^{-1}[\mathbf{G}] \end{array} \right]. \quad (22)$$

Here $[\mathbf{I}]$ is identity matrix. The general solution of Eq. (21) is assumed as $\{\mathbf{X}\} = \{\mathbf{X}_c\} \exp(\alpha t)$, where $\{\mathbf{X}_c\}$ is the time-independent amplitude of motion. Substituting this into Eq. (21) leads to an eigenvalue problem given by,

$$[\mathbf{S}]\{\mathbf{X}_c\} - \alpha\{\mathbf{X}_c\} = 0. \quad (23)$$

Eq. (23) is solved using a standard eigen-solver. If all the eigenvalues are obtained as pure complex conjugate pairs given by $\alpha = \pm i\omega$, it indicates a stable oscillatory motion where ω is the frequency of oscillation [29]. If at least one of the eigenvalues possesses real part, it indicates unstable motion. In the present work, the results for the stable oscillatory motion are only presented. Also, the mode shapes of vibration are obtained by employing the top half of the eigenvectors $\{\mathbf{X}_c\}$ into Eq. (16).

3. Results and Discussion

The non-dimensional speed (Ω^*) and frequency of vibration (ω^*) are defined as: $\Omega^* = \Omega L^2 [(\rho A)/(EI_y)]^{1/2}$ and $\omega^* = \omega L^2 [(\rho A)/(EI_y)]^{1/2}$. The following parameters are defined for presentation of results: Normalized nonlocal parameter, $\mu = e_0 a/L$; Slenderness parameter, $\xi = L/h$; Section aspect ratio, $\kappa = b/h$; Hub parameter, $\beta = R/L$. The results are presented for $\kappa \geq 1$, and that is why the minimum area moment of inertia (I_y) is chosen in defining the non-dimensional speed and frequency. Unless otherwise mentioned, the results are presented for the following material property and parametric values: $E = 210$ GPa, $\rho = 7850$ kg/m³, $\eta^2 = 4$ nm², $\mu = 0.2$, $\xi = 100$, $\kappa = 2$, $\beta = 0.25$.

The comparison of free vibration frequencies with Ref. [17] for a local ($\eta = 0$) rotating beam are provided in Tables 2 and 3 for the first three flap-wise and chord-wise modes respectively. The results for Tables 2 and 3 are generated for $\kappa = 1$ and $\beta = 0$. Tables 2 and 3 show

very good matching of the results. The non-dimensional speed-frequency behavior for the first three flap-wise modes, each for different normalized nonlocal parameters (μ) are compared with Ref. [68] and the comparison plots are shown in Figs. 2(a)-(c). Fig. 2 shows that for $\mu=0.3$, the stable solution does not extend to the full range of speeds as obtained for the lower value of μ ($=0.2$) or for the local beam ($\mu=0$). The results for Fig. 2 are generated for $\beta=1.0$. Fig. 2 shows excellent matching of the results. The above comparison studies thus validate the present model.

The speed-frequency behavior for the first six modes are shown in Figs. 3(a)-(e), each corresponding to different normalized nonlocal parameters (μ), where Fig. 3(a) shows both the first flap-wise and chord-wise modes in order to illustrate the mode-switching phenomenon between them. Figs. 3(b)-(d) represent the second flap-wise, second chord-wise and third flap-wise modes respectively. Fig. 3(e) mostly exhibits fourth flap-wise modes but it also represents third chord-wise mode for $\mu=0$ and 0.05 at higher speed regions due to its mode-switching with the preceding fourth flap-wise modes. It is further seen that $\mu=0.25$ and 0.3 does not provide stable solutions for the full range of speeds considered (similar kind of phenomenon is also reported in Refs. [67,68]). With increasing μ values (i.e., increasing nonlocal effect as the beam length becomes comparable to the nonlocal parameter $\eta = e_0 a$), the first flap-wise and chord-wise modes (Fig. 3(a)) exhibit hardening type behavior leading to increasing frequency values. Also, in this case, the speed-frequency plots diverge with increasing speeds. However, all the higher modes presented, (Figs. 3(b)-3(e)) exhibit softening type behavior with increasing μ values, leading to decreasing frequency values. Also, in such cases, the speed-frequency plots seem to converge somewhat with increasing speed, which is more prominent for the flap-wise second mode (Fig. 3(b)). The normalized mode shape plots for the flap-wise and chord-wise motions, for each of the first, second and third modes are depicted in Figs. 4(a)-(c) respectively for $\mu=0.2$ at $\Omega^*=10$. It clearly shows the difference between the flap-wise and chord-wise motions for the second and third modes though no such difference exists for the first modes.

Figs. 5(a) and (b) present the variation of normalized frequency ($\bar{\omega}$) with Ω^* for the first and second chord-wise modes respectively, each for different μ . Here $\bar{\omega}$ is defined as the ratio of the frequencies without and with the consideration of spin-softening effect i.e., $\bar{\omega} = \omega_{\text{without spin-soft.}} / \omega_{\text{with spin-soft.}}$. Fig. 5 exhibits that the spin-softening effect reduces the

chord-wise vibration frequencies and this effect tends to increase with increasing speed. It is also seen that the effect of spin-softening decreases with increase in normalized nonlocal parameter (μ) for the first chord-wise mode, and this is reversed for the second chord-wise mode. The spin-softening effect is seen to be more prominent for the first mode corresponding to the second mode. It is to be mentioned that the spin-softening effect is not found to be present for the flap-wise modes (also reported in Ref. [27] for local rotating beams) and hence is not illustrated here.

The speed-frequency behavior for the first axial vibration mode is shown in Fig. 6(a) for different normalized nonlocal parameters (μ) with slenderness parameter $\xi=10$. Fig. 6(b) illustrates the normalized mode shape for $\mu=0.2$ at $\Omega^*=3.75$ and it is shown to confirm the axial vibratory mode which exhibits zero slope at the free end. The plots of Fig. 6(a) are limited to $\Omega^*=3.75$ which is the maximum speed for $\mu=0.2$, yielding stable solution. Fig. 6(a) shows that the nonlocal effect induces softening on the axial mode as the frequency decreases with increasing μ which is unlike that found for the first flap-wise and chord-wise modes (Fig. 3(a)). It also shows that the axial vibration frequency increases with speed showing stiffening effect. This stiffening effect with increasing speed is indirectly contributed by the centrifugal stiffening through Coriolis effect due to coupling of axial-chord-wise motions. To illustrate the Coriolis effect **on the axial mode**, Fig. 7(a) is presented which shows the variation of normalized frequency ($\bar{\omega}$) with Ω^* for local ($\mu=0$) and nonlocal ($\mu=0.2$) beams. Here $\bar{\omega}$ is defined as the ratio of the frequencies without and with the consideration of Coriolis effect i.e., $\bar{\omega} = \omega_{\text{without Coriolis}} / \omega_{\text{with Coriolis}}$. Fig. 7(a) reveals that the Coriolis effect increases the axial vibration frequency and that effect is monotonically enhanced with increasing speed (also reported for local beams in Ref. [87]). Fig. 7(a) further reveals that the nonlocal effect has negligible influence on the change in frequency due to Coriolis effect. Though the chord-wise vibration is influenced by the Coriolis effect due to coupling of axial-chord-wise motions, its effect is found to be negligible for the range of speeds considered and hence it is not reported here. It must be mentioned that $\xi=10$ is chosen for Figs. 6 and 7 as the Coriolis effect becomes more pronounced with lower slender parameter [66]. **To illustrate spin-softening effect on the axial mode, Fig. 7(b) is presented, showing variation of normalized frequency ($\bar{\omega}$) with Ω^* for local ($\mu=0$) and nonlocal ($\mu=0.2$) beams. It reveals that the spin-softening effect reduces the frequency of vibration for the axial mode and this effect**

gradually increases with increasing speed. It also shows that, though marginally, the spin-softening has more effect on the non-local beam compared to the local beam. The present study is restricted to the first axial mode only, because the frequencies associated with the higher axial modes would be very high, and the present Ritz based method of solution may not be an appropriate methodology to correctly capture that.

The speed-frequency behavior for the first six modes are shown in Figs. 8(a)-(e), each corresponding to different hub parameters (β), where Fig. 8(a) shows both the first flap-wise and chord-wise modes in order to illustrate the mode-switching. Figs. 8(b)-(d) represent the second flap-wise, second chord-wise and third flap-wise modes respectively. Fig. 8(e) shows fourth flap-wise mode but also represents third chord-wise mode for $\beta=0.5, 1.0$ and 2.0 at higher speed regions due to its mode-switching with the preceding fourth flap-wise modes. Fig. 8 clearly exhibits enhanced centrifugal stiffening with increasing hub parameter for both the flap-wise and chord-wise modes, leading to increase in frequency. The speed-frequency behavior for different section aspect ratios (κ) are shown in Figs. 9(a)-(c), each showing the first six modes of vibration. Fig. 9(a) shows that for a square section beam, the chord-wise and flap-wise modes occur alternatively with spin-softening being the cause of reducing the chord-wise frequencies with respect to the corresponding flap-wise frequencies. Higher section aspect ratios (Figs. 9(b) and (c)) exhibit a completely different mode-wise speed-frequency behavior. For $\kappa=2$, mode-switching occurs between the first two modes, whereas for $\kappa=4$, it occurs between the fifth and sixth modes. The reason behind the influence of changing κ is the change in relative flexural rigidities for the flap-wise and chord-wise motions.

4. Conclusions

An advanced mathematical model is presented to study free flap-wise and chord-wise vibrations as well as the axial vibration of rotating nano-beams. The model is based on Eringen's nonlocal elasticity theory and Euler-Bernoulli beam theory, and incorporates spin-softening and Coriolis effects. This type of study and advanced model are reported for the first time through this work. The non-dimensional speed-frequency behaviors are presented for different normalized nonlocal parameters, hub parameters and section aspect ratios. The spin-softening and Coriolis

effects are individually illustrated and discussed. The followings conclusions are drawn based on the presented results:

- (i) The nonlocal effect induces hardening type behavior for the first flap-wise and chord-wise modes whereas it induces softening type behavior for the higher modes for both the flap-wise and chord-wise motions.
- (ii) The nonlocal effect induces softening type behavior for the first axial mode.
- (iii) The spin-softening effect reduces the frequency of vibration of the chord-wise modes. It is diminished by the nonlocal effect for the first chord-wise mode but aggravated by it for the second chord-wise mode.
- (iv) The Coriolis effect increases the vibration frequency for the axial mode with speed through its coupling with the chord-wise motion.
- (v) The hub parameter and section aspect ratio exhibit significant influence on the flap-wise and chord-wise modes.

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Declaration of Conflict of Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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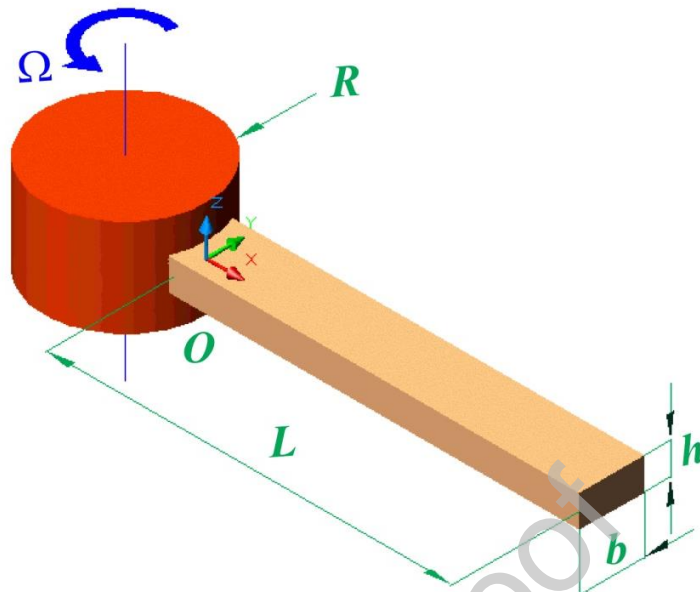
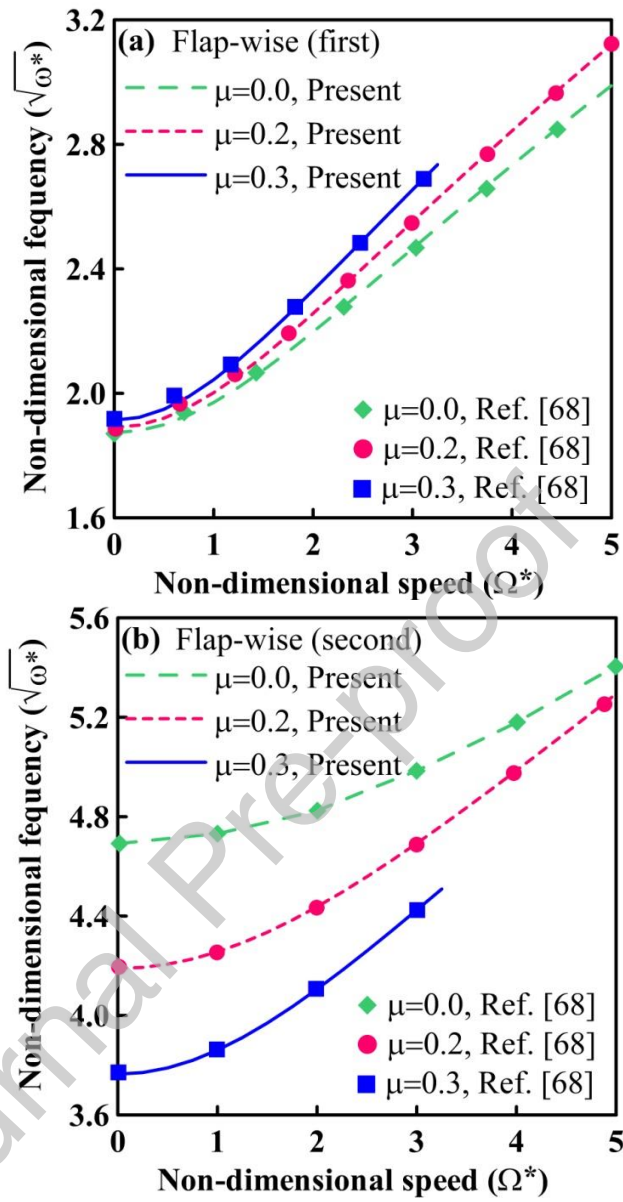


Fig. 1: Schematic diagram of a rotating beam with dimensions and coordinate axes.



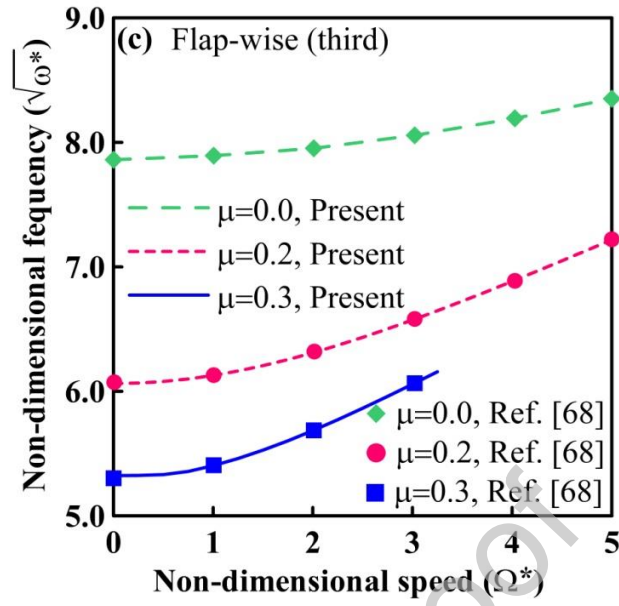
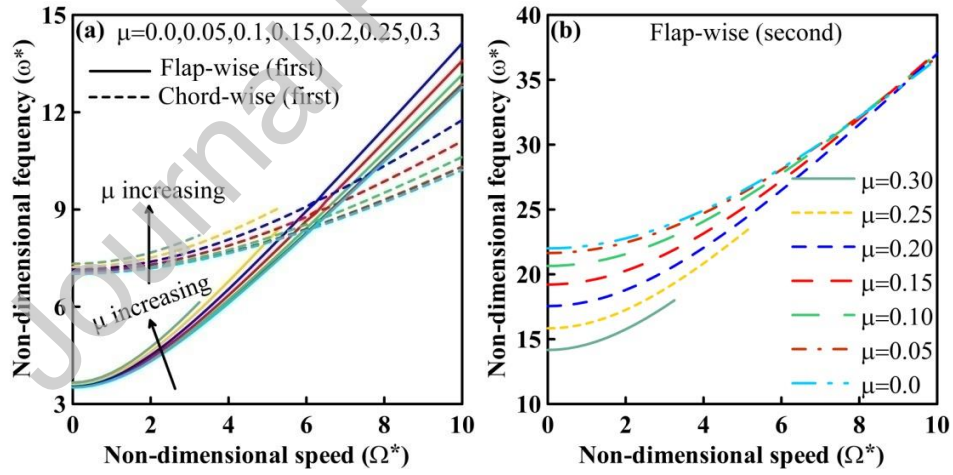


Fig. 2: Comparison of non-dimensional speed-frequency behavior for different normalized nonlocal parameters for flap-wise modes: (a) First mode, (b) Second mode, (c) Third mode.



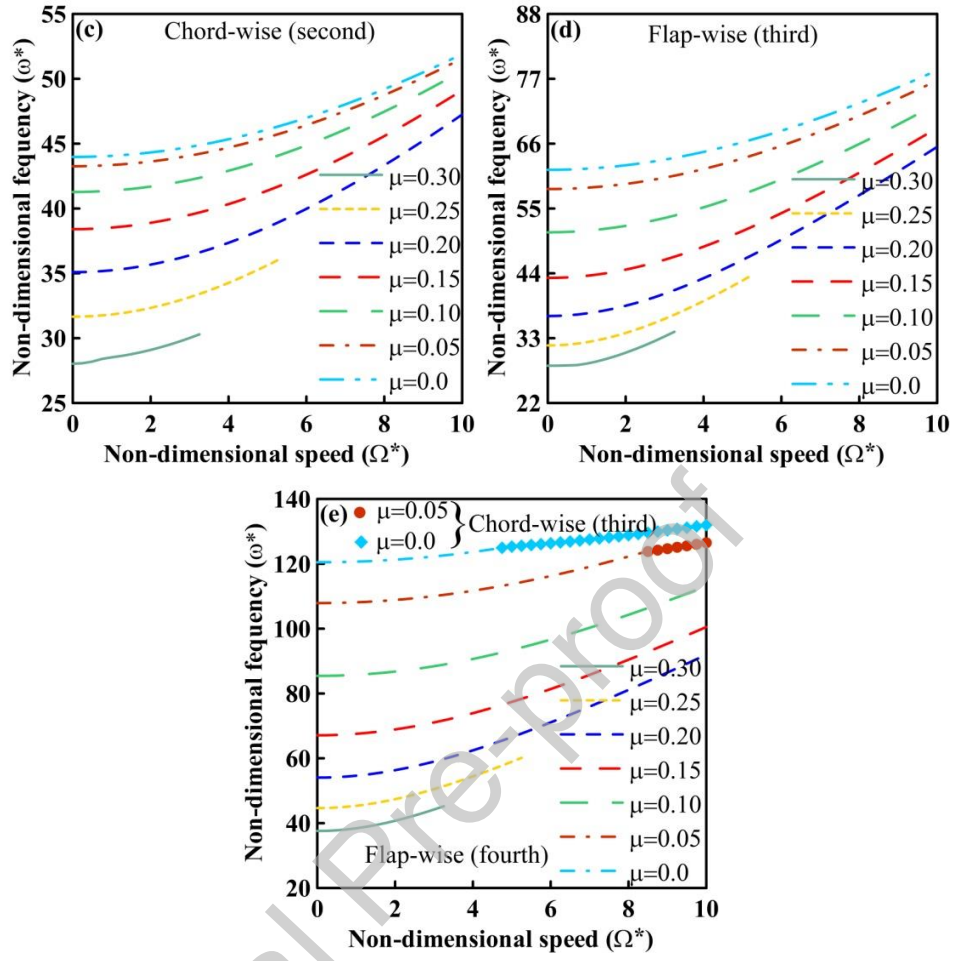
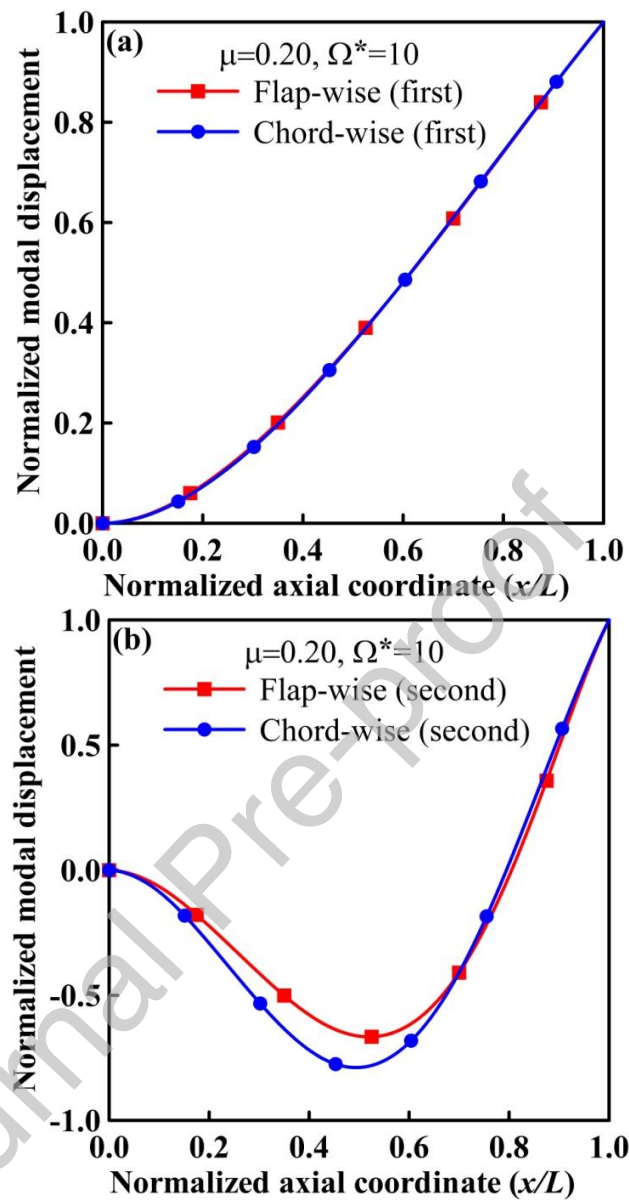


Fig. 3: Non-dimensional speed-frequency behavior for different normalized nonlocal parameters: (a) First two modes, (b) Third mode, (c) Fourth mode, (d) Fifth mode, (e) Sixth mode.



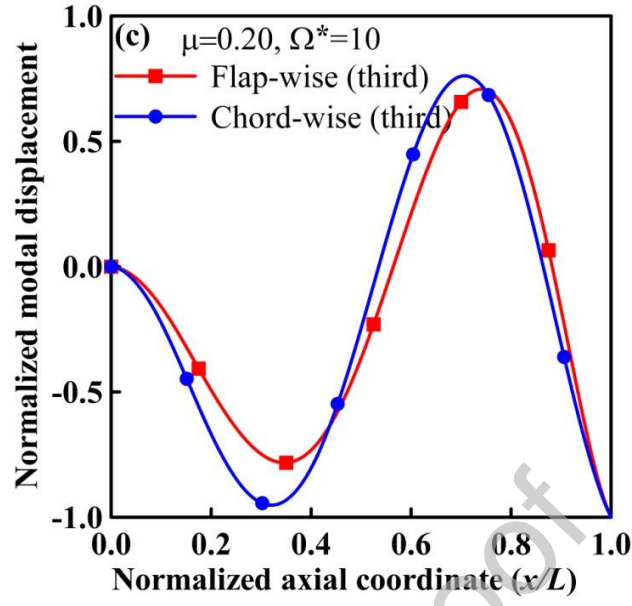
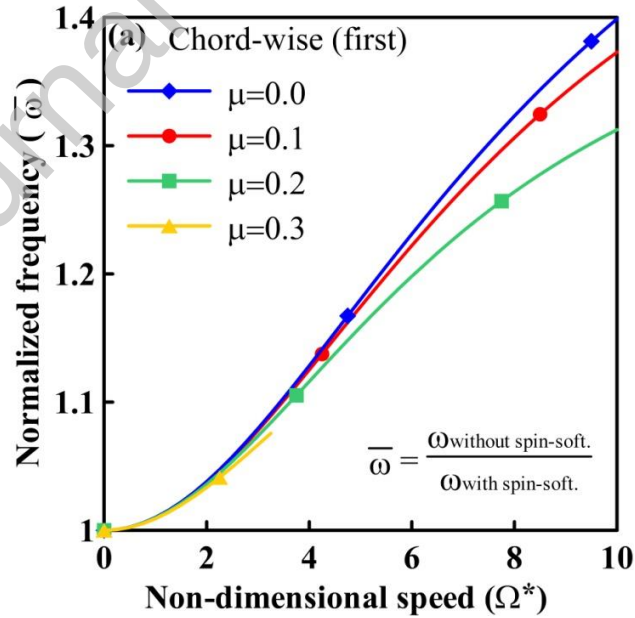


Fig. 4: Normalized mode shape plots: (a) First flap-wise and chord-wise modes, (b) Second flap-wise and chord-wise modes, (c) Third flap-wise and chord-wise modes.



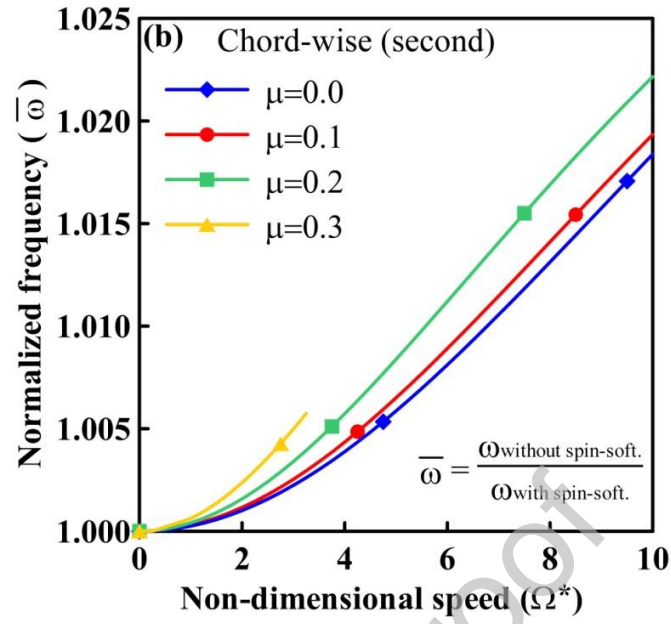
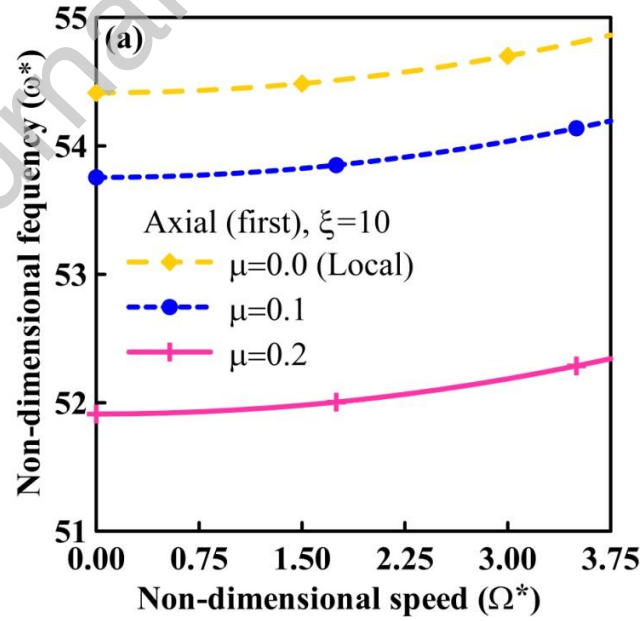


Fig. 5: Effect of normalized nonlocal parameter on spin-softening: (a) First chord-wise mode, (b) Second chord-wise mode.



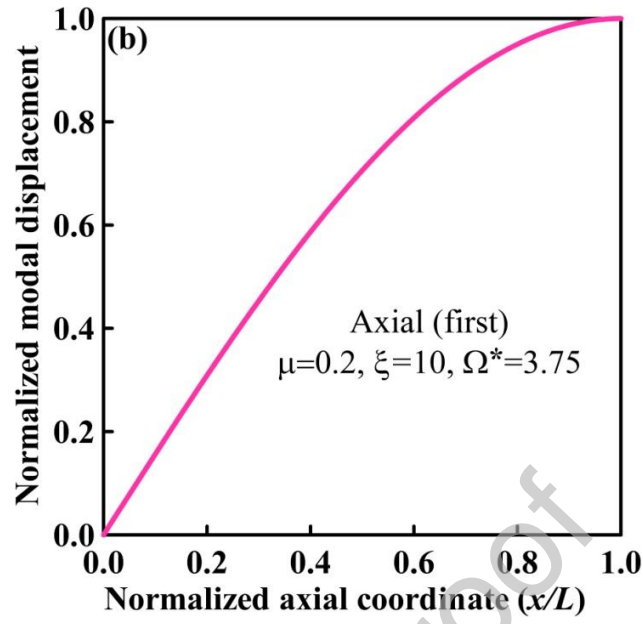
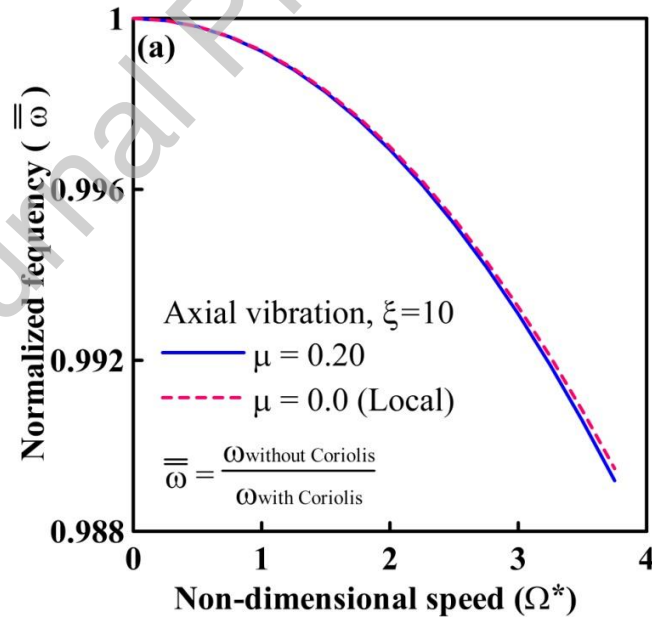


Fig. 6: Axial vibration behavior: (a) Non-dimensional speed-frequency behavior for different normalized nonlocal parameters, (b) Normalized mode shape.



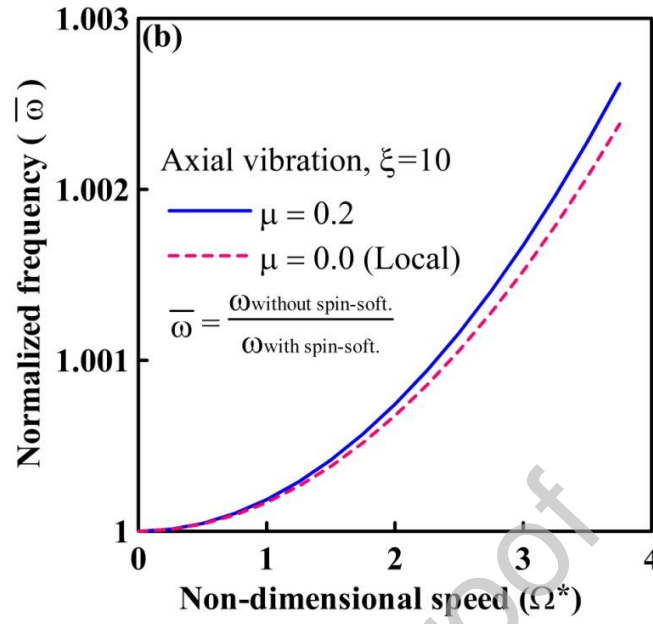
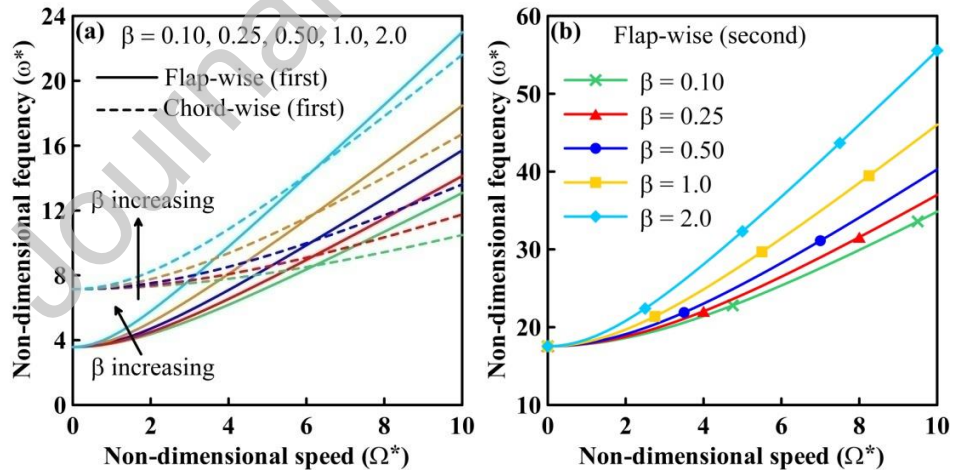


Fig. 7: Various effects on axial vibration with variation of normalized nonlocal parameter: (a) Coriolis effect, (b) Spin-softening effect.



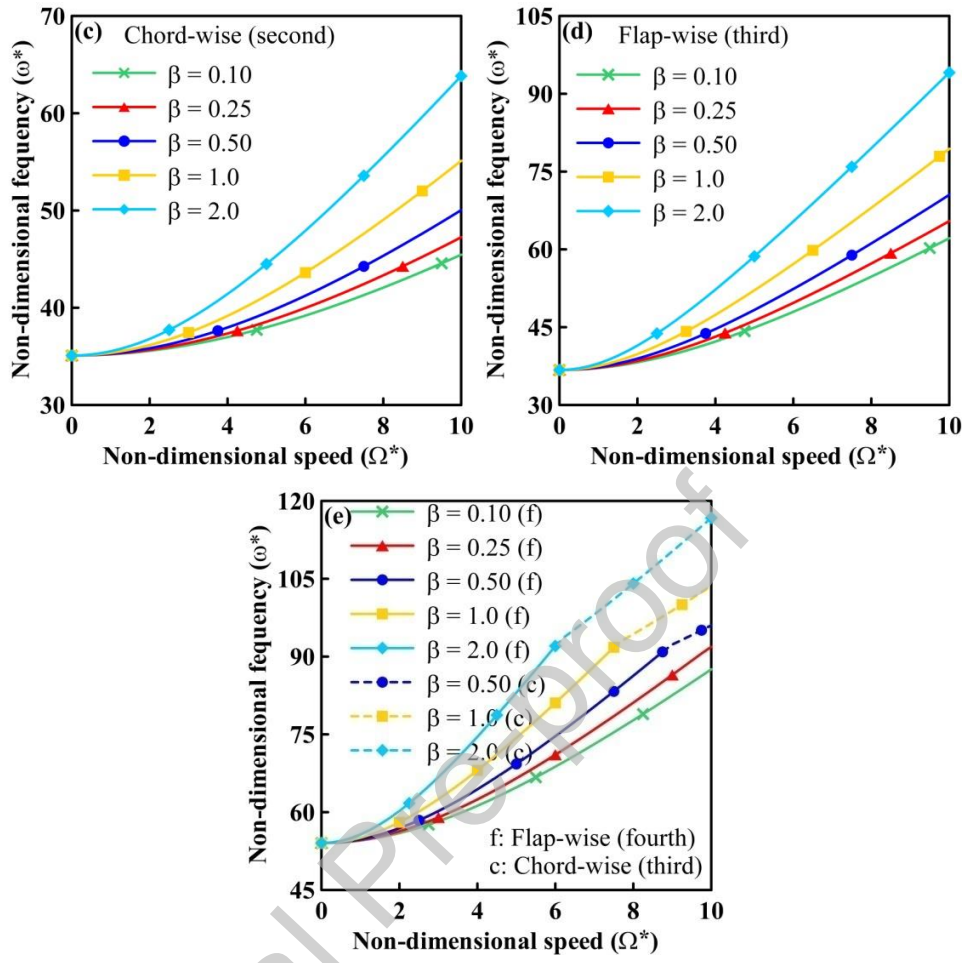


Fig. 8: Non-dimensional speed-frequency behavior for different hub parameters: (a) First two modes, (b) Third mode, (c) Fourth mode, (d) Fifth mode, (e) Sixth mode.

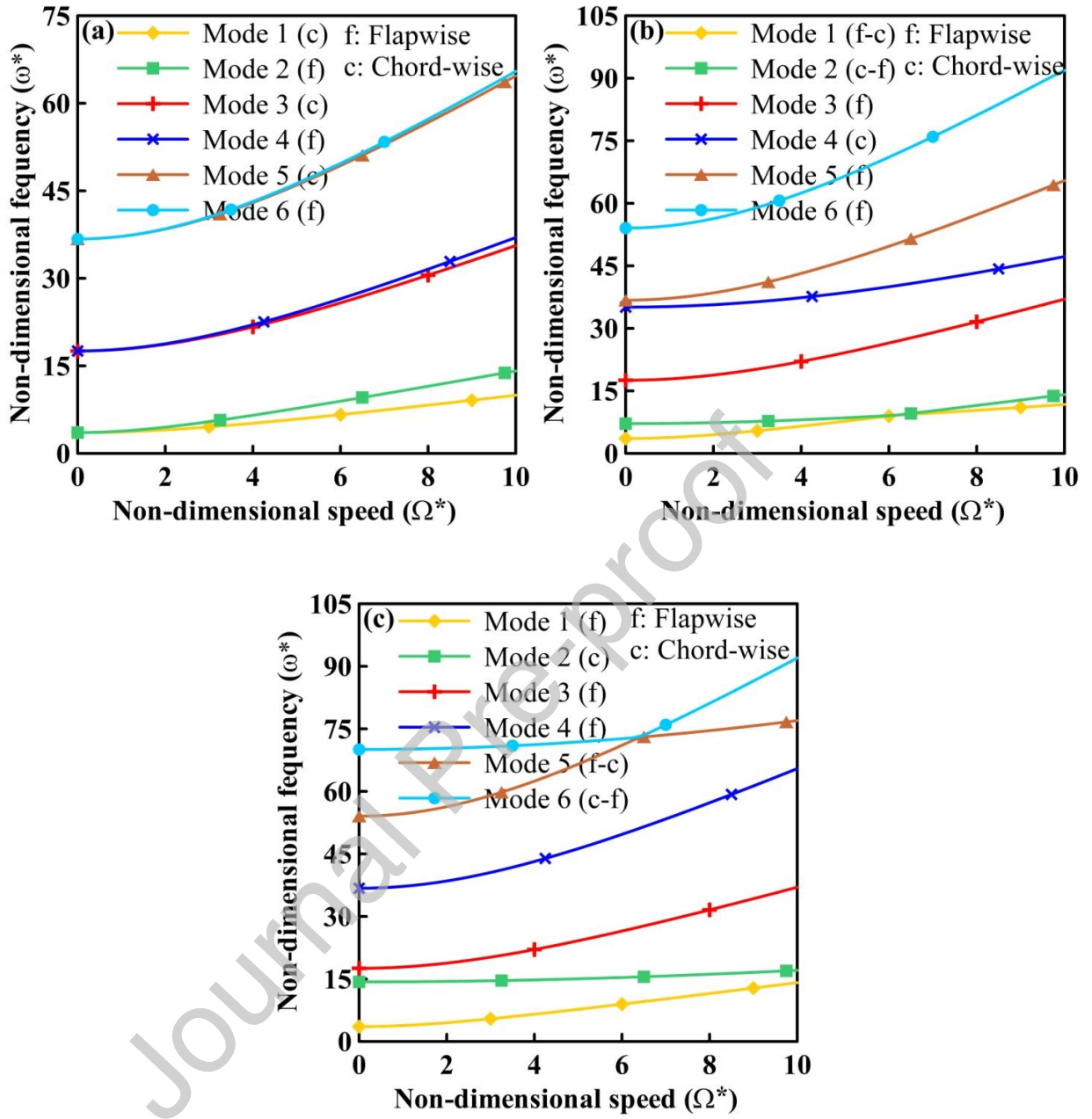


Fig. 9: Non-dimensional speed-frequency behavior for the first six modes: (a) $\kappa=1$, (b) $\kappa=2$, (c) $\kappa=4$.

Table 1: List of lowest order admissible functions.

Boundary conditions	Function
$u_0 _{x=0} = 0, u_0 _{x=L} \neq 0$	$\phi_1^u = x/L$
x/L	
$\frac{d^2 v_0}{dx^2}\bigg _{x=0} \neq 0, \frac{d^2 v_0}{dx^2}\bigg _{x=L} = 0; \frac{d^3 v_0}{dx^3}\bigg _{x=0} \neq 0, \frac{d^3 v_0}{dx^3}\bigg _{x=L} = 0$	$\phi_1^v = (x/L)^2 \{6 - 4(x/L) + (x/L)^2\}$
$w_0 _{x=0} = 0, w_0 _{x=L} \neq 0; \frac{dw_0}{dx}\bigg _{x=0} = 0, \frac{dw_0}{dx}\bigg _{x=L} \neq 0;$	
$\frac{d^2 w_0}{dx^2}\bigg _{x=0} \neq 0, \frac{d^2 w_0}{dx^2}\bigg _{x=L} = 0; \frac{d^3 w_0}{dx^3}\bigg _{x=0} \neq 0, \frac{d^3 w_0}{dx^3}\bigg _{x=L} = 0$	$\phi_1^w = (x/L)^2 \{6 - 4(x/L) + (x/L)^2\}$

Table 2: Comparison of flap-wise free vibration frequencies (ω^*) of a local rotating beam.

Non-dimensional Speed (Ω^*)	Reference	First Mode	Second Mode	Third Mode
0	Present	3.5106	21.9980	61.5648
	Ref. [17]	3.5160	22.0332	61.6885
2	Present	4.1322	22.5786	62.1408
	Ref. [17]	4.1373	22.6136	62.2645
4	Present	5.5798	24.2374	63.8345
	Ref. [17]	5.5850	24.2721	63.9582
8	Present	9.2503	29.9591	70.1599
	Ref. [17]	9.2568	29.9942	70.2848
10	Present	11.1948	33.6031	74.5149
	Ref. [17]	11.2024	33.6393	74.6415

12	Present	13.1615	37.5644	79.4778
	Ref. [17]	13.1703	37.6023	79.6071

Table 3: Comparison of chord-wise free vibration frequencies (ω^*) of a local rotating beam.

Non-dimensional Speed (Ω^*)	Reference	First Mode	Second Mode	Third Mode
0	Present	3.5106	21.9980	61.5648
	Ref. [17]	3.5160	22.0332	61.6885
2	Present	3.6158	22.4898	62.1085
	Ref. [17]	3.6218	22.5250	62.2324
5	Present	4.0648	24.9129	64.8799
	Ref. [17]	4.0739	24.9487	65.0046
10	Present	5.0293	32.0782	73.8388
	Ref. [17]	5.0492	32.1187	73.9687