Course on Symmetry, Topology and Entanglement

Homework 1

- **Instructor**: Kargarian - Vaezi
- **Semester**: Fall 2022
- **Hand out**: Monday 02-08-1401
- **Due date**: Monday 16-08-1401 in class

1. In this exercise you will examine the low-energy spectrum, *the magnon*, of a ferromagnet in various spatial dimensions $d = 1, 2, 3$. We assume that spins $\mathbf{S}$ are located on translationally invariant cubic lattice sites, a lattice with coordination number $Z = 2d$. Each spin interacts with its nearest neighbors via the exchange coupling

$$H = -J \sum_{<i,j>} \mathbf{S}_i \cdot \mathbf{S}_j, \qquad J > 0, \tag{1}$$

where sum over i, j runs over lattice sites.

Hint: the solutions of the most parts below can be found in standard textbooks on condensed matter and magnetism. Feel free to use them but write the solutions in your own words.

(a) Make sure you understand the structure of the lattice and the way spin interact with each other.

(b) Refresh your mind with the structure of orthogonal group $SO(3)$, a group of rotation matrices (very briefly!).

(c) Suppose spins are classical moments pointing in a direction $\mathbf{S} = S\mathbf{n}$, where $\mathbf{n}$ is a unite vector in $\mathbb{R}^3$. Show that the Hamiltonian (1) remains invariant under any global rotation applied to all spins simultaneously. This means that the Hamiltonian is *isotropic*.

(d) Determine the lowest energy state of the system.

(e) Determine the ground state energy per spin.

(f) Now assume that the spins are quantum mechanical objects obeying the angular momentum algebra $[S^i, S^j] = i\epsilon_{ijk}S^k$ ($\hbar = 1$). Refresh your mind with the structure of $SU(2)$ group (very briefly!).

(g) Show that the quantum Hamiltonian is also invariant under a global unitary rotation.

(h) Can you tell what the ground state is?

(i) What is the ground state degeneracy?

- Answer: uncountably many; the ground state manifold is continuous. Try to understand this.

(j) To explore the the low-energy excitations known as spin-wave modes above the ground state, we use the standard spin-wave theory known as Holstein-Primakoff transformation. The transformation is given by

$$S^+ = \sqrt{2S}\sqrt{1 - \frac{b^\dagger b}{2S}}\; b, \quad S^- = \sqrt{2S}b^\dagger\sqrt{1 - \frac{b^\dagger b}{2S}}, \quad S^z = S - b^\dagger b, \tag{2}$$

where $S^\pm = S^x \pm S^y$ and $b, b^\dagger$ are annihilation and creation operators of bosons satisfying $[b, b^\dagger] = 1$. Show that spin operators in (2) satisfy the angular momentum algebra.

(k) Rewrite the Hamiltonian (1) in terms of boson operators up to quadratic terms, i.e. the product of two bosonic operators. What does this approximation mean?

(l) Using the translation invariance of the system, we Fourier transform the boson operators to momentum space

$$b_i = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_i} b_{\mathbf{k}}, \tag{3}$$

where N is the number of lattice sites. Note that the vectors $\mathbf{k}$ and $\mathbf{r}$ could have one, two and three components depending on the spatial dimension. Show that $[b_{\mathbf{k}}, b^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$.

(m) Write the Hamiltonian in momentum space and show that the spin-wave modes are described by non-interacting bosons known as magnon.

(n) Find the dispersion of spin-wave excitations and explore the limit of vanishing momentum $\mathbf{k} \to 0$. Is the dispersion gapped or gapless?

(o) Calculate the average expectation value of S^z at finite temperature for $d = 1, 2, 3$. Does the dimensionality matter?

(p) Lets break the full rotational symmetry of (1) by adding a small magnetic field $h > 0$ along the z direction $-h \sum_i S_i^z$. Find the magnon spectrum and observe if it is gapless or gapped.

(q) Again calculate the average expectation value of S^z at finite temperature for $d = 1, 2, 3$. Does the dimensionality matter?

(r) Instead of applying a magnetic field, we could assume that the exchange interaction is intrinsically anisotropic in the z direction. In this case the Hamiltonian becomes

$$H = -J \sum_{<i,j>} \left(S_i^x S_j^x + S_i^y S_j^y + \eta S_i^z S_j^z \right), \tag{4}$$

where the parameter $\eta > 1$ measure the amount of anisotropy. Is the magnon spectrum gapped or gapless?

(s) In the cases (o) and (q) described above, what is the rotational symmetry of the magnetic system?

(t) Through this example, what are the main ingredients in the low-energy effective theory?

- Answer: symmetry and dimension. Try to convince yourself.

2. Every week, Monday through Friday, people working in condensed matter physics and related area post their latest discoveries on arXiv (https://arxiv.org/list/cond-mat/new). The postings include new submissions, cross-lists, and replacements. Downloading is free and available everywhere. You may visit the webpage until the due date of this assignment and select one paper of your own interest. It could be purely theoretical, experimental, or combination of both. You don't need to read the paper carefully and reproduce the results anymore. All you have to do is to grab the main points reported and the methods employed to resolve the problem. Prepare a half-page summary including things that you don't understand, the title and the arXiv id number.