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# Real Time Computationally Efficient MIMO System Identification Algorithm

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**Abstract**
MIMO system identification is a fundamental concern in a variety of applications. Various iterative and recursive MIMO system identification algorithms exist in the literature. The iterative algorithms suffer from high computational cost due to large matrix computations and recursive algorithms suffer from slow convergence speed. This paper proposes a fast recursive exact least squares algorithm for MIMO system identification with fast convergence speed and low computational cost. The recursions of the algorithm give rise to a lattice structure. The lattice structure is less sensitive to round-off errors, coefficients variations and can also be used for model order reduction. Although in this work we estimate an FIR (finite impulse response) model of the MIMO system, the framework can also be used for IIR (infinite impulse response) models, and for estimating linear periodically time varying (LPTV) and multivariable systems. The theory proposed is validated using simulation results.

**Keywords** MIMO system identification · Parameter estimation · Multirate synthesis filter bank · Exact least squares algorithm · Lattice filter

## 1 Introduction

System identification refers to the operation of mathematical modelling of dynamic systems from the measured input, $\mathbf{u}(n) \in \mathbb{R}^L$, and output data, $\mathbf{y}(n) \in \mathbb{R}^M$. The system can either be described by a state space model or a transfer function model [1–3], depending on the particular application. In this work we estimate a given MIMO system using a transfer function model.

Various system identification methods exist in the literature like prediction error method (PEM) [3] which provides asymptotically optimal estimates for the given model if the noise is Gaussian, cost function is quadratic and the model orders are correct. However, the non-convex optimization function leads to low computational efficiency. To improve the computational efficiency of PEM, algorithms like Subspace method [4] and instrumental variable method [5] have been introduced which avoid the non-convexity by providing proper initialization. Steiglitz-McBride method [6] uses iterative least squares to avoid the non-convexity of PEM.

To improve the computational performance of system identification methodologies, various algorithms based on stochastic gradient(SG) and least squares (LS) have also been proposed [7–11]. The SG algorithms have slower convergence speed than recursive least squares (RLS) algorithm, however, they are computationally more efficient than RLS as there is no need to compute the covariance matrix at each recursion [12]. Han and Ding [12] used multi-innovation identification theory to improve the convergence rate of SG algorithm for MIMO system identification. Ding and Chen [13] proposed a hierarchical stochastic gradient method (HSG) for a class of multivariable discrete time systems by decomposing the given system into several sub-models with smaller dimensions and fewer variables. The algorithm computes several smaller covariance matrices.

Recently several authors [14–18] have proposed filtering and auxiliary model based recursive and iterative least squares algorithms for system identification. In these works, it was noticed that since the iterative least squares (ILS) algorithms use all the available data to estimate the

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parameters at each iteration, their accuracy is high but are not computationally efficient as it requires computation of larger matrices and their inverses. In contrast, the recursive least squares (RLS) algorithm computes the parameter estimates with only the data available till the present time, $n$, making them less accurate but more computationally efficient. Therefore, the major drawback of the above mentioned algorithms, [14, 15, 19–21], is the high computational cost of ILS and low convergence rate of RLS.

In this work, we propose a fast, time, and order recursive exact least squares algorithm for MIMO system identification. The algorithm is developed by first converting an $L$-input $M$-output system (consisting of $L \times M$ subsystems) into a SISO system with only $M$-subsystems. Here, the input signal space of each subsystem consist of all $L$ input signals, thus the cross-correlation of the input signals is not neglected while estimating the subsystem. The SISO form is found to be structurally equivalent to a multirate synthesis filter bank. Therefore, the problem of estimating the MIMO system parameters becomes identical to obtaining the synthesis filters of a multirate filter bank. The relationship of MIMO system with multirate filter bank can be used to develop multirate system identification algorithms and linear periodically time varying (LPTV) system identification algorithms.

The MIMO system parameters are obtained in two stages: first, by providing the required geometrical framework we present a fast least squares algorithm and obtain a lattice structure. Here the recursions consist of only scalar computations unlike RLS. Once the lattice coefficients have converged, MIMO filter coefficients are estimated in the second stage using a time and order recursive algorithm. Although in this work we estimate a finite impulse response(FIR) model of the MIMO system, the framework can also be used for infinite impulse response (IIR) model. Also, the proposed algorithm can be used for multivariable system identification.

Salient contribution of this work are as follows:

1. FIR model of the given MIMO system is first converted to a SISO multirate filter and hence the problem of MIMO system identification can be interpreted as estimating synthesis filters of multirate filter bank,
2. A computationally efficient, time and order recursive exact least squares algorithm for MIMO system identification has been proposed to achieve fast convergence of estimated parameters,
3. A lattice structure is presented which can be used for model order reduction

This paper is organized as follows: Section 2 presents the problem addressed in this work and propose the SISO form of the given MIMO system. Section 3, first, provides the required geometrical framework which is then used to develop a fast least squares algorithm for MIMO system identification. Simulation results are presented in Section 4 to validate the efficacy of the proposed algorithm. Conclusions are presented in Section 5.

## 2 Methodology

### 2.1 Notations

In this work, random variables are denoted by lower-case letters, vectors of random variable by boldfaced lower-case letters and matrices of random variables with capital letters. $A^T$ denotes the transpose of matrix $A$. The norm of a vector $\mathbf{x}$ is denoted by $|| \ \mathbf{x} \ ||$, which is the positive square-root of the inner-product of $\mathbf{x}$ with itself, where $\mathbf{x}$ belongs to a Hilbert space.

### 2.2 Problem Statement

Consider the following FIR model of a MIMO system:

$$\mathbf{y}(n) = H(z)\mathbf{u}(n) + \mathbf{v}(n) \tag{1}$$

$$= \hat{\mathbf{y}}(n) + \mathbf{v}(n) \tag{2}$$

where, $\mathbf{y}(n) = \left[y_0(n), y_1(n), \cdots, y_{M-1}(n)\right]^T \in \mathbb{R}^M$ is the measured output vector, $\mathbf{u}(n) = [u_0(n), u_1(n), \cdots, u_{L-1}(n)]^T \in \mathbb{R}^L$ is system input vector, $\mathbf{v}(n) = \left[v_0(n), v_1(n), \cdots, v_{M-1}(n)\right]^T$ is the measurement noise, $H(z)$ is the MIMO system to be estimated, represented as:

$$H(z) = \begin{bmatrix} H_{0,0} & H_{0,1} & \cdots & H_{0,L-1} \\ H_{1,0} & H_{1,1} & \cdots & H_{1,L-1} \\ \vdots & \vdots & \cdots & \vdots \\ H_{M-1,0} & H_{M-1,1} & \cdots & H_{M-1,L-1} \end{bmatrix}$$

where, $H_{ij}$'s are polynomials in $z^{-1}$, which is the delay operator, $0 \leq i \leq M-1$, $0 \leq j \leq L-1$ and $\hat{\mathbf{y}}(n)$ is the response of the estimated system i.e. $H(z)\mathbf{u}(n)$. The time domain representation of the estimated system can be written as follows:

$$\begin{bmatrix} \hat{y}_0(n) \\ \hat{y}_1(n) \\ \vdots \\ \hat{y}_{M-1}(n) \end{bmatrix} = \begin{bmatrix} h_{0,0}(0) & h_{0,1}(0) & \cdots & h_{0,L-1}(0) \\ h_{1,0}(0) & h_{1,1}(0) & \cdots & h_{1,L-1}(0) \\ \vdots & \vdots & \cdots & \vdots \\ h_{M-1,0}(0) & h_{M-1,1}(0) & \cdots & h_{M-1,L-1}(0) \end{bmatrix} \begin{bmatrix} u_0(n) \\ u_1(n) \\ \vdots \\ u_{L-1}(n) \end{bmatrix}$$
$$+ \cdots + \begin{bmatrix} h_{0,0}(N-1) & h_{0,1}(N-1) & \cdots & h_{0,L-1}(N-1) \\ h_{1,0}(N-1) & h_{1,1}(N-1) & \cdots & h_{1,L-1}(N-1) \\ \vdots & \vdots & \cdots & \vdots \\ h_{M-1,0}(N-1) & h_{M-1,1}(N-1) & \cdots & h_{M-1,L-1}(N-1) \end{bmatrix}$$
$$\times \begin{bmatrix} u_0(n-N+1) \\ u_1(n-N+1) \\ \vdots \\ u_{L-1}(n-N+1) \end{bmatrix} \tag{3}$$

as $H_{ij}(z) = \sum_{n=0}^{N-1} h_{ij}(n)z^{-n}$ for $0 \leq j \leq L-1$ and $0 \leq i \leq M-1$.

The problem addressed in this work can be stated as: For an L input M output system, develop a system identification algorithm such that as soon as the new measured input and output sample is available, the system parameters are adapted in an order recursive manner to ensure that the error, $\mathbf{e}(n)$, between the response of the estimated system, $\hat{\mathbf{y}}(n)$ and the true system, $\mathbf{y}(n)$ is minimized in the least square sense i.e.$\left(\sum_{i=0}^{M-1} || e_i(n) ||^2\right)$, as shown in Fig. 1, and the error is given below:

$$\mathbf{e}(n) = \mathbf{y}(n) - H(z)\mathbf{u}(n) \tag{4}$$

It is pertinent to mention here that the algorithm proposed in this work, does not assume the matrix $H(z)$ to be square. Moreover, the filter orders are not assumed to be known, unlike [22] where the system order is, first, identified using sub-space method. Before we present the derivation of the proposed estimation algorithm, we first convert the given MIMO system, Eq. 3, into a SISO system and establish its equivalence with an $M$-channel multirate synthesis filter bank.

### 2.3 SISO Form of the MIMO System

The MIMO system, as given in Eq. 3, has a vector form representation and thus estimating the parameters in the current state will require matrix computations. To reduce the computational complexity we propose to convert the MIMO system into a single input single output system which can be then estimated using only scalar computations with the proposed algorithm. Equation 3 can be converted into a single input single output form using the following steps:

1. **Serializing the inputs:**
   We define a scalar process, $x(n)$, which is a serialized form of the vector process $\mathbf{u}(n) = [u_0(n)u_1(n) \cdots u_{L-1}(n)]$, as follows:

$$x(Ln - j) \triangleq u_j(n), 0 \leq j \leq L-1. \tag{5}$$

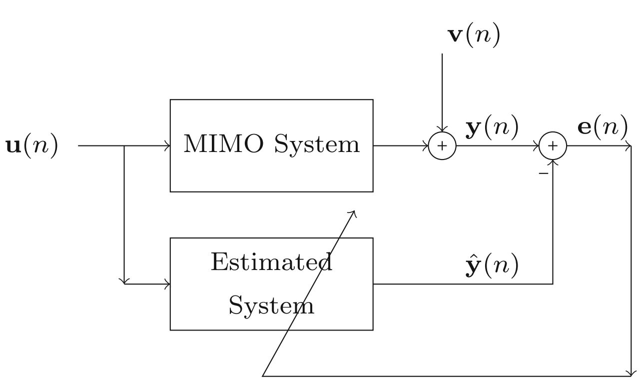


**Figure 1** Block diagram for proposed MIMO System identification.

2. **Restructuring of the filters:**
   The i-th row of Eq. 3 can be written, in terms of $x(n)$, as follows:

$$\hat{y}_i(n) = \left[\, h_{i,0}(0) \;\; h_{i,1}(0) \;\; \cdots \;\; h_{i,L-1}(N-1) \,\right] \begin{bmatrix} x(Ln) \\ x(Ln-1) \\ \vdots \\ x(Ln-LN+1) \end{bmatrix} \tag{6}$$

The above expression can be written in a closed form as:

$$\hat{y}_i(n) = \sum_{j=0}^{L-1} \sum_{k=0}^{N-1} h_{ij}(k)x(L(n-k)-j). \tag{7}$$

Equation 7 represents a system with $L$ filters, each of order $N$. In order to simplify it further, we restructure these $L$ filters to obtain only one filter, of order $LN$, given as:

$$\hat{y}_i(n) = \left[\, f_i(0) \;\; f_i(1) \;\; \cdots \;\; f_i(LN-1) \,\right] \begin{bmatrix} x(Ln) \\ x(Ln-1) \\ \vdots \\ x(Ln-LN+1) \end{bmatrix} \tag{8}$$

The above equation can also be expressed as:

$$\hat{y}_i(n) = \sum_{j=0}^{LN-1} f_i(j)x(Ln-j). \tag{9}$$

The filters $F_i(z) \equiv \sum_{n=0}^{LN-1} f_i(n)z^{-n}$ for $0 \leq i \leq M-1$, are related to MIMO subsystems as:

$$f_i(j + Lk) = h_{ij}(k), \tag{10}$$

for $0 \leq i \leq M-1$, $0 \leq j \leq L-1$ and $0 \leq k \leq N-1$. Equation 9 gives a single input multi output representation which can now be converted into a single input single output system using the following step.

3. **Serializing the output:**
   Using $M$-fold upsamplers and a delay line, as shown in Fig. 2, the $M$ outputs of Eq. 9 are converted to a single output, $\hat{y}(n)$, defined as follows:

$$\hat{y}(Mn - i) \triangleq \hat{y}_i(n), 0 \leq i \leq M-1. \tag{11}$$

The MIMO system represented by Eq. 3 can, now, be written as follows:

$$\hat{y}(Mn - i) = \sum_{j=0}^{LN-1} f_i(j)x(Ln-j), 0 \leq i \leq M-1. \tag{12}$$

Using the SISO form, Eq. 12, the MIMO system identification problem, represented by Eq. 4 can, now, be written as:

$$\mathbf{e}(Mn) = \mathbf{y}(Mn) - F(z)\mathbf{x}(Ln) \tag{13}$$

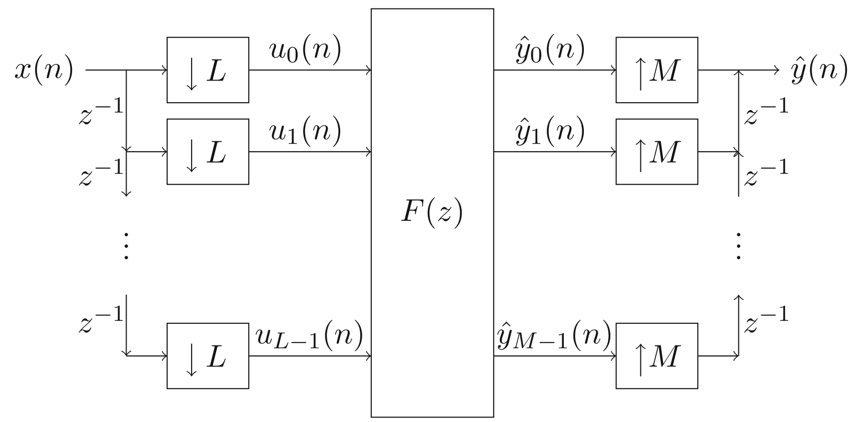


**Figure 2** SISO form of $L \times M$ MIMO system.

where, $F(z) = \left[\, F_0(z)\;\, F_1(z)\;\, \cdots\;\, F_{M-1}(z) \,\right]^T$.

The objective of the MIMO system identification problem, in its new form as given in Eq. 13, is to obtain the parameter matrix $F(z)$, which minimizes $||\; \mathbf{e}(Mn)\; ||^2$, in a computationally efficient manner. In the next section, we present a recursive least squares algorithm for the same, but first we briefly discuss the relationship between the SISO form of MIMO system with an M-channel multirate synthesis filter bank.

**Relationship between a MIMO system and M-channel synthesis filter bank** As discussed in [23], type-II polyphase decomposition and serializing input signals can lead to a SISO form for an $M$-channel multirate synthesis filter bank. Comparing Fig. 3 from [23] and Fig. 2 of this paper, we observe that the synthesis filter bank is a special case of the proposed SISO system, when $L = M$. Therefore, the parameter estimation algorithm proposed in this work can be used to estimate synthesis filters. This relationship between synthesis filter bank and MIMO system will allow us to use various tools and theory of multirate filter banks for MIMO systems. Two such scenarios are discussed below:

It is a well known fact that each branch of multirate synthesis filter bank, consisting of an upsampler followed by a linear time invariant filter, is an LPTV system [24] and sum of LPTV systems is also an LPTV system. As discussed above, the algorithm proposed in this work can be used to estimate multirate synthesis filter bank and therefore it can also be used for estimating LPTV systems.

In various industrial and chemical systems, the input signals are of different bandwidths, however due to conventional system identification frameworks they are all sampled at the same rate. Hence, their frequency information is not completely utilized and the system ends up with redundant samples and higher computational cost. This multirate system identification is an active research problem in system identification literature [25–27]. The relationship of MIMO systems with multirate filter bank and the underlying theory proposed in this work can provide an efficient solution for these cases where each input channel can be sampled at the required rate. This will lead to a non-uniform filter bank which can be realized using a uniform filter bank [24]. The obtained uniform filter bank can, then, be estimated using the proposed algorithm. Also, we observed that the SISO structure with the set of down-samplers and delay chain, as shown in Fig. 2 can be implemented with a commutator switch [28].

In the next section we develop a recursive algorithm for the obtained SISO system.

## 3 Algorithm

In this section we present a Hilbert space framework required to obtain the geometrical interpretation of Eq. 13. We can then develop the required time and order recursive algorithm.

### 3.1 Hilbert Space Framework

Using (13), we write the estimation error of the i-th output, for time upto $Mn - i$, with the pre-windowed assumption, in matrix form as follows:

$$\left[\, 0\; \dots\; e^p(M-i)\; \dots\; e^p(Mn-i) \,\right] = \left[\, 0\; \dots\; y(M-i)\; \dots\; y(Mn-i) \,\right] - \left[\, f_i(0)\;\; f_i(1)\; \dots\; f_i(p-1) \,\right] \begin{bmatrix} 0 & \dots & x(L) & \dots & x(Ln) \\ 0 & \dots & x(L-1) & \dots & x(Ln-1) \\ \vdots & & & & \\ 0 & \dots & x(L-p+1) & \dots & x(Ln-p+1) \end{bmatrix}, \quad 0 \le i \le M-1. \tag{14}$$

Since we are interested in an order recursive solution to the problem, we introduce a variable '$p$' as superscript to denote the order of estimation. We can write the above equation, in matrix form, as follows:

$$\mathbf{e}^p(Mn-i) \equiv \mathbf{y}(Mn-i) - \mathbf{f}_i^p X_p^{Ln}, \tag{15}$$

**Table 1** Definition of auxiliary quantities arising in the proposed algorithm.

| $\nu P^{\perp}[V_{1:p}]\mathbf{w}^T$ | $\nu$ | $V_{1:p}$ | $\mathbf{w}$ | Comment |
|---|---|---|---|---|
| $e^p(Mn-i)$ | $\mathbf{y}(Mn-i)$ | $X_p^{Ln}$ | $\pi$ | p-th order Forward prediction error of $i^{th}$ MIMO channel |
| $\alpha^p(Ln+j)$ | $\mathbf{x}(Ln+j)$ | $X_p^{Ln+j-1}$ | $\pi$ | p-th order Forward prediction error of $j^{th}$ band joint process estimator |
| $\beta^p(Ln+j)$ | $\mathbf{x}(Ln-j-p)$ | $X_p^{Ln+j}$ | $\pi$ | p-th order Backward prediction error of $j^{th}$ band joint process estimator |
| $\delta_p^i(Ln+j)$ | $\pi$ | $X_p^{Ln+j}$ | $\pi$ | $j^{th}$ band likelihood variable |

We define $\mathbf{e}^p(Mn-i)$ as the minimum norm error vector obtained when $\mathbf{y}(Mn-i)$ is projected on the space spanned by $\{\mathbf{x}(Ln), \mathbf{x}(Ln-1), \cdots, \mathbf{x}(Ln-p+1)\}$. Using the least squares theory, $\mathbf{e}^p(Mn-i)$ can be written as:

$$\mathbf{e}^p(Mn-i) = \mathbf{y}(Mn-i) - \mathbf{y}(Mn-i)\left[X_p^{Ln}\right]^T\left[X_p^{Ln}\left[X_p^{Ln}\right]^T\right]^{-1}X_p^{Ln},$$
$$= \mathbf{y}(Mn-i)P^{\perp}\left[X_p^{Ln}\right], \qquad 0 \le i \le M-1. \tag{16}$$

where, $P$ denotes the projection operator and $P^{\perp} = (I-P)$, denotes the projection operator corresponding to the orthogonal complement space. Hence, the optimum filter coefficients, $\mathbf{f_i^p}$, are:

$$\mathbf{f_i^p} = \mathbf{y}(Mn-i)\left[X_p^{Ln}\right]^T\left[X_p^{Ln}\left[X_p^{Ln}\right]^T\right]^{-1}. \tag{17}$$

The above set of equations, Eqs. 16 and 17, give us the geometrical framework required to develop the recursive least squares algorithm. We now re-state the problem statement addressed in this work in the Hilbert space setting.

### 3.2 Problem Statement

With Eqs. 16 and 17 at our disposal the problem statement can now be re-formulated as follows:"Given a set of pre-windowed records of input, $\{\mathbf{u}(n) \in \mathbb{R}^L, 0 \le n \le T-1\}$, and output, $\{\mathbf{y}(n) \in \mathbb{R}^M, 0 \le n \le T-1\}$ and the system parameters at time T-1, estimate the SISO system, $F(z)$, in an order recursive manner as soon as the input and output samples are available at time T. The SISO system is estimated to minimize the error between the estimated output and the measured output in the exact least squares sense."

We now present the least squares algorithm to address the MIMO system identification problem as stated above.

**Table 2** Autocorrelation and cross-correlation coefficients.

| $\nu$ | $\mathbf{w}$ | $\nu\mathbf{w}^{\mathbf{T}}$ |
|---|---|---|
| $\alpha^{\mathbf{p}}(Ln+j)$ | $\alpha^{\mathbf{p}}(Ln+j)$ | $R_p^{\alpha}(Ln+j)$ |
| $\beta^{\mathbf{p}}(Ln+j-1)$ | $\beta^{\mathbf{p}}(Ln+j-1)$ | $R_p^{\beta}(Ln+j-1)$ |
| $\alpha^{\mathbf{p}}(Ln+j)$ | $\beta^{\mathbf{p}}(Ln+j-1)$ | $\Delta_{\alpha,\beta}^p(Ln+j)$ |
| $\beta^{\mathbf{p}}(Ln+j-1)$ | $\alpha^{\mathbf{p}}(Ln+j)$ | $\Delta_{\beta,\alpha}^p(Ln+j-1)$ |
| $\mathbf{e}^{\mathbf{p}}(Mn-i)$ | $\beta^{\mathbf{p}}(Ln)$ | $\Delta_{e,\beta}^p(Mn-i)$ |

### 3.3 Exact Least Squares Algorithm

The proposed algorithm is developed in two stages: first we obtain prediction error, Eq. 16, in a recursive manner which gives rise to a lattice structure. In the second stage we use these lattice parameters to obtain the system coefficients, Eq. 17, in an order and time recursive manner.

**Stage 1: Development of the lattice structure** We, first, develop a time, and order recursive least squares algorithm to compute error given in Eq. 16. To obtain this recursive algorithm we are interested only in the present (or most recent) error, hence, we post-multiply both the sides of Eq. 16 by pinning vector defined as $\pi^T = \left[0 \cdots 0\ 1\right]^T$, as shown below:

$$e^p(Mn-i) = \mathbf{y}(Mn-i)P^{\perp}\left[X_p^{Ln}\right]\pi^T.0 \le i \le M-1 \tag{18}$$

Our problem now reduces to the computations of the set of equations given in Eq. 18 in an order recursive manner. This is achieved by substituting proper values in the inner product update formula, Eq. 33. To give a complete readability to this paper we have briefly discussed the inner product update formula in the Appendix.

To obtain $e^{p+1}(Mn-i)$ in a recursive manner, we substitute $\mathbf{v} = \mathbf{y}(Mn-i)$, $V_{1:p} = X_p^{Ln}$, $\mathbf{v}_{p+1} = \mathbf{x}(Ln-p)$ and $\mathbf{w} = \pi$ in Eq. 33. The recursion obtained is given below:

$$e^{p+1}(Mn-i) = e^p(Mn-i) - \mathbf{y}(Mn-i)P^{\perp}\left[X_p^{Ln}\right]\mathbf{x}^T(Ln-p)[\mathbf{x}(Ln-p)$$
$$\times P^{\perp}\left[X_p^{Ln}\right]\mathbf{x}^T(Ln-p)]^{-1}\mathbf{x}(Ln-p)P^{\perp}\left[X_p^{Ln}\right]\pi^T$$
$$= e^p(Mn-i) - \Delta_{e,\beta}^p(Mn-i)R_p^{-\beta}(Ln)\beta^p(Ln), \tag{19}$$

where,

$$\Delta_{e,\beta}^p(Mn-i) \triangleq \mathbf{y}(Mn-i)P^{\perp}\left[X_p^{Ln}\right]\mathbf{x}^T(Ln-p,)$$
$$R_p^{\beta}(Ln) \triangleq \mathbf{x}(Ln-p)P^{\perp}\left[X_p^{Ln}\right]\mathbf{x}^T(Ln-p),$$
$$\beta^p(Ln) \triangleq \mathbf{x}(Ln-p)P^{\perp}\left[X_p^{Ln}\right]\pi^T.$$

$\Delta_{e,\beta}^p(Mn-i)$, $R_p^{-\beta}(Ln)$ and $\beta^p(Ln)$ are the auxiliary quantities that appears in a natural way. This is the typicality of fast algorithms like [29]. $\beta^p(Ln)$ is the backward prediction error, $\Delta_{e,\beta}^p(Mn-i)$ is the cross-correlation of $e^{p+1}(Mn-i)$ and $\beta^p(Ln)$, $R_p^{-\beta}(Ln) = 1/R_p^{\beta}(Ln)$ and

**Table 3** Substitution table, for $0 \leq i \leq M-1, 0 \leq j \leq L-1$.

| | $v$ | $V_{1:p}$ | $v_{p+1}$ | $\mathbf{w}$ |
|---|---|---|---|---|
| 1 | $\mathbf{x}(Ln+j)$ | $X_p^{Ln+j-1}$ | $\pi$ | $\mathbf{x}(Ln+j-1-p)$ |
| 2 | $\mathbf{x}(Ln+j-p-1)$ | $X_p^{Ln+j-1}$ | $\pi$ | $\mathbf{x}(Ln+j)$ |
| 3 | $\mathbf{x}(Ln+j-p-1)$ | $X_p^{Ln+j-1}$ | $\pi$ | $\mathbf{x}(Ln+j-p-1)$ |
| 4 | $\mathbf{x}(Ln+j)$ | $X_p^{Ln+j-1}$ | $\pi$ | $\mathbf{x}(Ln+j)$ |
| 5 | $\pi$ | $X_p^{Ln+j-1}$ | $\mathbf{x}(Ln+j-p-1)$ | $\pi$ |
| 6 | $\mathbf{x}(Ln+j)$ | $X_p^{Ln+j-1}$ | $\mathbf{x}(Ln+j-p-1)$ | $\pi$ |
| 7 | $\mathbf{x}(Ln+j-p-1)$ | $X_p^{Ln+j-1}$ | $\mathbf{x}(Ln+j)$ | $\pi$ |
| 8 | $\mathbf{y}(Mn-i)$ | $X_p^{Ln}$ | $\pi$ | $\mathbf{x}(Ln-p)$ |
| 9 | $\mathbf{y}(Mn-i)$ | $X_p^{Ln}$ | $\mathbf{x}(Ln-p)$ | $\pi$ |

$R_p^{\beta}(Ln)$ is the autocorrelation of $\beta^p(Ln)$. The product of auxiliary quantities $\Delta_{e,\beta}^p(Mn-i).R_p^{-\beta}(Ln)$ is known lattice reflection coefficient. For complete derivation of the algorithm, all the auxiliary quantities arising in the development are defined in Table 1 and their correlations are given in Table 2. One has to find recursions for these auxiliary quantities in order to complete the cycle. This cycle, then, can be repeated when new inputs become available at the next instance.

To obtain the update relations, Table 3 consists of the quantities to be substituted in the inner product update formula and the recursions obtained are given in Table 4. For example, to compute order updates of $\beta^p(Ln)$ substitute the values given in row (7) of Table 3 in the inner product update formula (33), we get:

$$\beta^{p+1}(Ln) = \beta^p(Ln-1) - \Delta_{\beta,\alpha}^p(Ln-1)R_p^{-\alpha}(Ln)\alpha^p(Ln), \quad (20)$$

where, $\Delta_{\beta,\alpha}^p(Ln-1)$, $R_p^{-\alpha}(Ln)$ and $\alpha^p(Ln)$ are the auxiliary quantities.

We present one more example to illustrate the update recursion for correlation auxiliary quantities. Time Update relation for $\Delta_{e,\beta}^p(Mn-i)$ is given as the eighth row entry of

**Table 4** The fast recursive exact least squares algorithm.

| | Set all $\Delta s$ and $Rs$ to 0 at $n=0$, $e^0(Mn-i)=y(Mn-i)$<br>$\alpha^0(Ln+j)=\beta^0(Ln+j)=x(Ln+j)$<br>For $p=1$ to LN , $i=0$ to $M-1$, $j=0$ to $L-1$, $n=1$ to $T$ | No. of multiplications per recursion |
|---|---|---|
| 1 | $\Delta_{\alpha,\beta}^p(Ln+j)=\Delta_{\alpha,\beta}^p(L(n-1)+j))+$<br>$\alpha^p(Ln+j).\delta_p^{-1}(Ln+j-1).\beta^p(Ln+j-1)$ | 2 |
| 2 | $\Delta_{\beta,\alpha}^p(Ln+j-1)=\Delta_{\beta,\alpha}^p(L(n-1)+j-1)+$<br>$\beta^p(Ln+j-1).\delta_p^{-1}(Ln+j-1).\alpha^p(Ln+j)$ | 1 |
| 3 | $R_p^{\beta}(Ln+j-1)=R_p^{\beta}(L(n-1)+j-1)+$<br>$\beta_p(Ln+j-1).\delta_p^{-1}(Ln+j-1).\beta_p(Ln+j-1)$ | 1 |
| 4 | $R_p^{\alpha}(Ln+j)=R_p^{\alpha}(L(n-1)+j)+$<br>$\alpha^p(Ln+j).\delta_p^{-1}(Ln+j).\alpha^p(Ln+j)$ | 2 |
| 5 | $\delta_{p+1}(Ln+j-1)=\delta_p(Ln+j-1)-$<br>$\beta^p(Ln+j-1).R_p^{-\beta}(Ln+j-1).\beta^p(Ln+j-1)$ | 2 |
| 6 | $\alpha^{p+1}(Ln+j)=\alpha^p(Ln+j)-$<br>$\Delta_{\alpha,\beta}^p(Ln+j).R_p^{-\beta}(Ln+j-1).\beta^p(Ln+j-1)$ | 2 |
| 7 | $\beta^{p+1}(Ln+j)=\beta^p((Ln+j-1)-\Delta_{\beta,\alpha}^p(Ln+j-1)R_p^{-\alpha}(Ln+j).$<br>$\alpha^p(Ln+j)$ | 2 |
| 8 | $\Delta_{e,\beta}^p(Mn-i)=\Delta_{e,\beta}^p(M(n-1)-i)+e^p(Mn-i).\delta_p^{-1}(Ln).\beta^p(Ln)$ | 1 |
| 9 | $e^{p+1}(Mn-i)=e^p(Mn-i)-\Delta_{e,\beta}^p(Mn-i).R_p^{-\beta}(Ln).\beta^p(Ln)$ | 1 |

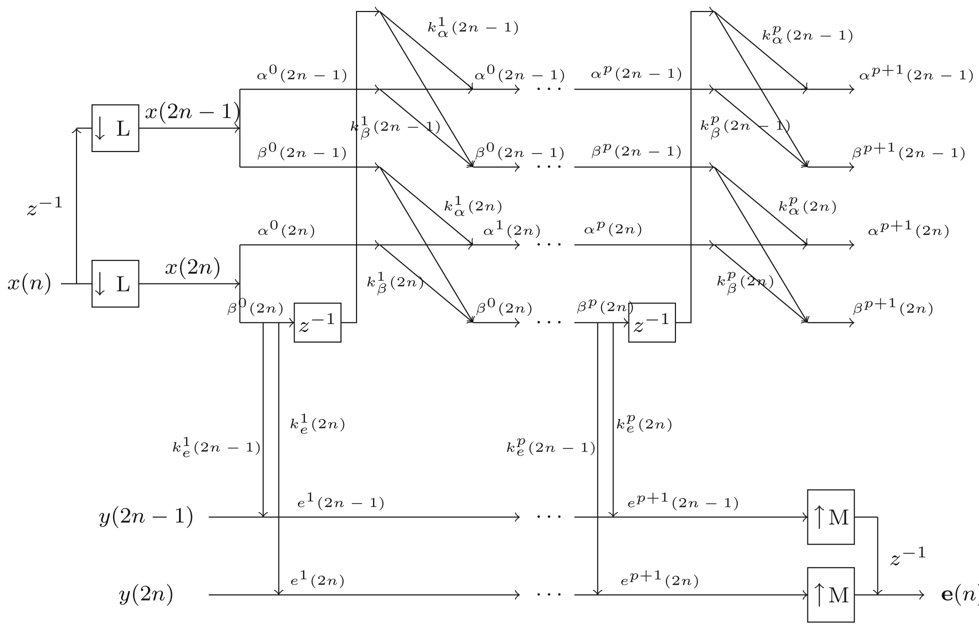


**Figure 3** Lattice-Ladder structure for the MIMO system identification when $L = M = 2$.

Table 4 which is obtained by substituting eighth row entries of Table 3 in Eq. 33, as shown below:

$$\begin{aligned}\Delta^p_{e,\beta}(M(n-1)-i) &= \mathbf{y}(Mn-i)P^{\perp}\left[X_p^{L(n-1)}\right]\mathbf{x}(Ln-p)^T\\ &= \mathbf{y}(Mn-i)P^{\perp}\left[X_p^{Ln}\right]\mathbf{x}(Ln-p)^T - \mathbf{x}(Mn-i)\\ &\quad\times P^{\perp}\left[X_p^{Ln}\right]\pi^T\left[\pi P^{\perp}[X_p^{Ln}]\pi^T\right]^{-1}\pi P^{\perp}\left[X_p^{Ln}\right]\mathbf{x}(Ln-p)^T\\ &= \Delta^p_{e,\beta}(Mn-i) - e^p(Mn-i)\delta_p^{-1}\beta^p(Ln)\end{aligned} \tag{21}$$

Therefore, we get:

$$\Delta^p_{e,\beta}(Mn-i) = \Delta^p_{e,\beta}(M(n-1)-i) + e^p(Mn-i)\delta_p^{-1}\beta^p(Ln) \tag{22}$$

Similarly, recursions for other auxiliary quantities can also be obtained by substituting values from Table 3 in Eq. 33. Table 4 is a concise presentation of the algorithm. It is observed that the above recursions lead to a lattice structure, as shown in Fig. 3, where $k^p_\beta(Ln-i) = \Delta^p_{\beta,\alpha}(Ln-1).R_p^{-\alpha}(Ln)$, $k^p_\alpha(Ln-i) = \Delta^p_{\alpha,\beta}(Ln+j).R_p^{-\beta}(Ln+j-1)$ and $k^p_e(Mn-i) = \Delta^p_{e,\beta}(Mn-i).R_p^{-\beta}(Ln)$ are the lattice reflection coefficients.

**Stage 2: Least Squares Estimation of MIMO parameters** Since the optimum MIMO coefficients given in Eq. 17 are not directly available from the algorithm presented above we now propose a fast, order and time recursive algorithm for the estimation of these coefficients from the lattice reflection parameters. It is pertinent to mention here that the filter coefficients are computed after the lattice parameters have converged. The derivation of the required recursive algorithm essentially involves use of pseudo-inverse update formula (35) [30], with judicious choices of variable substitutions. We have briefly discussed the pseudo-inverse update formula (35) in Appendix.

**Table 5** Definition of the auxiliary quantities arising in the parameter estimation algorithm, for $0 \le i \le M-1$ and $0 \le j \le L-1$.

| $\mathbf{e} = \mathbf{z} + \Theta V_{1:p}$ | $\mathbf{z}$ | $V_{1:p}$ | $\Theta \equiv -\mathbf{z}V_{1:p}^T(V_{1:p}V_{1:p}^T)^{-1}$ |
|---|---|---|---|
| $\mathbf{e}^p(Mn-i)$ | $\mathbf{y}(Mn-i)$ | $X_p^{Ln}$ | $\mathbf{f_i^p}$ |
| $\alpha^\mathbf{p}(Ln+j)$ | $\mathbf{x}(Ln+j)$ | $X_p^{Ln+j-1}$ | $\mathbf{c_j^p}$ |
| $\beta^\mathbf{p}(Ln+j)$ | $\mathbf{x}(Ln+j-p)$ | $X_p^{Ln+j}$ | $\mathbf{d_j^p}$ |

To concisely present the algorithm, the parameter vectors involved in the algorithm are defined in Table 5. To obtain the MIMO system parameters $\mathbf{f}_i^{p+1}$ from $\mathbf{f}_i^p$ substitute $\mathbf{z} = \mathbf{y}(Mn-i)$, $V_{1:p} = X_p^{Ln}$ and $\nu_{p+1} = \mathbf{x}(Ln+j-p)$ in Eq. 35:

$$\mathbf{f}_i^{p+1}(n) = \begin{bmatrix}\mathbf{f}_i^p(n)\\ 0\end{bmatrix} - \Delta^p_{e,\beta}(Mn-i)R_p^{-\beta}(Ln)\begin{bmatrix}\mathbf{d}^p_{L-1}(n)\\ 1\end{bmatrix},\quad 0 \le i \le M-1 \tag{23}$$

where $\Delta^p_{e,\beta}(Mn-i).R_p^{-\beta}(Ln)$, are the coefficients of the lattice-ladder structure and $\mathbf{d}^p_{L-1}$ is the backward filter coefficients of the $L$-th channel as defined in Table 5. To obtain the required $\mathbf{f}_i's$ recursively, $\mathbf{d}^p_{L-1}$ recursions are also required. The order updates of $\mathbf{d}_i^p(n)$ for $0 \le i \le L-1$ are obtained by substituting $\mathbf{z} = \mathbf{x}(Ln+j)$, $V_{1:p} = X_p^{Ln+j-1}$ and $\nu_{p+1} = \mathbf{x}(Ln+j-p-1)$ in Eq. 35:

$$\mathbf{d}_j^{p+1}(n) = \begin{bmatrix}0\\ \mathbf{d}^p_{j-1}(n)\end{bmatrix} - \Delta^p_{\beta,\alpha}(Ln+j-1)R_p^{-\alpha}(Ln+j)\begin{bmatrix}1\\ \mathbf{c}_j^p(n)\end{bmatrix},\quad 0 \le j \le L-1. \tag{24}$$

**Table 6** Parameter estimates and error for Example 1, k denotes the number of recursions.

| | k | $\beta_{11}$ | $\beta_{12}$ | $\beta_{21}$ | $\beta_{22}$ | $\delta\%$ |
|---|---|---|---|---|---|---|
| F-AM-HSG | 10 | 0.7402 | 0.6953 | 0.6979 | 0.5623 | 35.72 |
| | 50 | 0.9851 | 0.7241 | 0.8377 | 0.8058 | 18.52 |
| | 100 | 1.0775 | 0.7296 | 0.9285 | 0.8501 | 12.17 |
| | 300 | 1.1352 | 0.7825 | 1.0208 | 0.8880 | 6.10 |
| | 500 | 1.1799 | 0.7852 | 1.0575 | 0.9199 | 3.20 |
| | 1000 | 1.1754 | 0.8178 | 1.0801 | 0.9684 | 1.78 |
| AM-HSG | 10 | 1.0191 | 0.2212 | 0.9739 | 0.0589 | 53.32 |
| | 50 | 1.1221 | 0.3478 | 0.9243 | 0.4163 | 35.91 |
| | 100 | 1.1352 | 0.4277 | 1.0189 | 0.5246 | 28.60 |
| | 300 | 1.1655 | 0.6061 | 1.0402 | 0.6921 | 16.64 |
| | 500 | 1.1664 | 0.6556 | 1.0679 | 0.7306 | 14.18 |
| | 1000 | 1.1618 | 0.7093 | 1.0723 | 0.7868 | 9.86 |
| FF-AM-HSG | 10 | 0.7188 | 0.1434 | 0.6652 | 0.1209 | 60.88 |
| | 50 | 0.9972 | 0.5250 | 0.9342 | 0.6630 | 23.74 |
| | 100 | 1.0388 | 0.5553 | 0.9652 | 0.7386 | 19.39 |
| | 300 | 1.1032 | 0.6720 | 1.0154 | 0.8155 | 11.57 |
| | 500 | 1.1261 | 0.7071 | 1.0336 | 0.8479 | 8.84 |
| | 1000 | 1.1476 | 0.7339 | 1.0601 | 0.8751 | 6.41 |
| Proposed algorithm | 10 | 1.5705 | 1.0534 | 1.2709 | 1.2970 | 28.42 |
| | 50 | 1.2039 | 0.8000 | 1.0904 | 0.9800 | 1.83 |
| | 100 | 1.1676 | 0.8543 | 1.1067 | 0.9695 | 2.51 |
| | 300 | 1.2336 | 0.8422 | 1.1296 | 0.9382 | 2.50 |
| | 500 | 1.2306 | 0.8337 | 1.1284 | 0.9658 | 2.27 |
| | 1000 | 1.2158 | 0.8287 | 1.1090 | 0.9647 | 1.14 |
| True values | | 1.20000 | 0.82000 | 1.10000 | 0.95000 | |

Similarly, to obtain the order update of $\mathbf{c}_j^p(n)$, substitute $\mathbf{z} = \mathbf{x}(Ln+j-p-1)$, $V_{1:p} = X_p^{Ln+j-1}$ and $v_{p+1} = \mathbf{x}(Ln+j)$ in Eq. 35 to get:

$$\mathbf{c}_j^{p+1}(n) = \begin{bmatrix} \mathbf{c}_j^p(n) \\ 0 \end{bmatrix} - \Delta_{\alpha,\beta}^p(Ln+j)R_p^{-\beta}(Ln+j-1)\begin{bmatrix} \mathbf{d}_{j-1}^p(n) \\ 1 \end{bmatrix}, \quad 0 \le j \le L-1. \tag{25}$$

Therefore, to obtain the MIMO parameters from the lattice filter coefficients, Eqs. 23, 24 and 25 are the required recursions. Having discussed the complete time and order recursive algorithm, we, now, present some simulation results to validate the proposed algorithm.

**Computational Complexity** Third column of Table 4 gives the number of real multiplications required at each step. Therefore, at every time instance the proposed algorithm required $(12L+2M)p$ real multiplications to compute $p$-order lattice coefficients for an $L$-input $M$-output system. Since $p = LN$ in this case, the complexity of the algorithm becomes $(14L+4M)LN$. For the second stage, the filter coefficients are computed from the converged lattice parameters using (23), (24) and (25). To obtain the order update of coefficient vector $\mathbf{f}_i^{p+1}$ from $\mathbf{f}_i^p$ we require $p$ real multiplication. Therefore to obtain $\mathbf{f}_i^{LN}$, $\mathbf{d}_j^{LN}$ and $\mathbf{c}_j^{LN}$ for $0 \le i \le M-1$ and $0 \le j \le L-1$, we require $LN(LN-1)/2 * M + LN(LN-1) * L$ real multiplications.

In contrast, if we compute (17) non-recursively, the number of computations required are $(LNK + L^2N^2K + L^3N^3 + L^2N^2) * M$. Where the data instances available are denoted by $K$. We have used the fact that the complexity of matrix multiplication is $\mathcal{O}(n^2)$ and matrix inversion in $\mathcal{O}(n^3)$.

## 4 Simulations

In this section, efficacy of the proposed MIMO system identification algorithm is validated and its performance is compared with existing methodologies. The examples discussed here considers MIMO systems, example 1 and 2, as well as SISO systems, example 3 and 4.

**Table 7** Parameter estimates for example 2, k denotes the number of recursions.

| Methodology | k | $b_{11}$ | $b_{12}$ | $b_{21}$ | $b_{22}$ | $\delta(\%)$ |
|---|---|---|---|---|---|---|
| MISG | 100 | 0.6546 | –0.5942 | –0.3445 | 1.4069 | 20.74 |
| | 200 | 0.7658 | –0.5361 | –0.3221 | 1.3057 | 14.05 |
| | 500 | 0.9024 | –0.4599 | –0.3876 | 1.2348 | 5.93 |
| | 1000 | 0.9334 | –0.4873 | –0.4017 | 1.2240 | 4.45 |
| | 2000 | 0.9527 | –0.4776 | –0.3979 | 1.2329 | 3.04 |
| | 3000 | 0.9612 | –0.4776 | –0.3940 | 1.2106 | 2.40 |
| | 4000 | 0.9697 | –0.4827 | –0.3886 | 1.2074 | 1.87 |
| | 5000 | 0.9834 | –0.4819 | –0.3925 | 1.2015 | 1.48 |
| RLS | 100 | 0.9034 | –0.4478 | –0.3310 | 1.3878 | 11.55 |
| | 200 | 0.9601 | –0.4096 | –0.3177 | 1.2238 | 6.78 |
| | 500 | 1.0284 | –0.3802 | –0.4114 | 1.1685 | 6.36 |
| | 1000 | 1.0122 | –0.4665 | –0.4196 | 1.1812 | 2.58 |
| | 2000 | 1.0047 | –0.4621 | –0.4029 | 1.2207 | 2.31 |
| | 3000 | 1.0012 | –0.4662 | –0.3951 | 1.1892 | 1.86 |
| | 4000 | 1.0072 | –0.4766 | –0.3856 | 1.1888 | 1.57 |
| | 5000 | 1.0225 | –0.4776 | –0.3934 | 1.1830 | 1.85 |
| Proposed algorithm | 100 | 1.0357 | —0.4195 | –0.3623 | 1.1341 | 6.89 |
| | 200 | 0.9765 | –0.5434 | —0.4203 | 1.2118 | 3.24 |
| | 500 | 0.9581 | —0.4983 | –0.3774 | 1.1952 | 2.84 |
| | 1000 | 0.9800 | –0.5373 | –0.4365 | 1.1873 | 3.39 |
| | 2000 | 1.0042 | –0.5103 | –0.4219 | 1.2013 | 1.46 |
| | 3000 | 1.0065 | –0.5065 | –0.4006 | 1.2145 | 1.02 |
| | 4000 | 1.0100 | –0.5021 | –0.4014 | 1.2124 | 0.96 |
| | 5000 | 1.0111 | –0.4952 | –0.4053 | 1.2098 | 0.97 |
| True values | | 1.00000 | —0.50000 | –0.40000 | 1.20000 | |

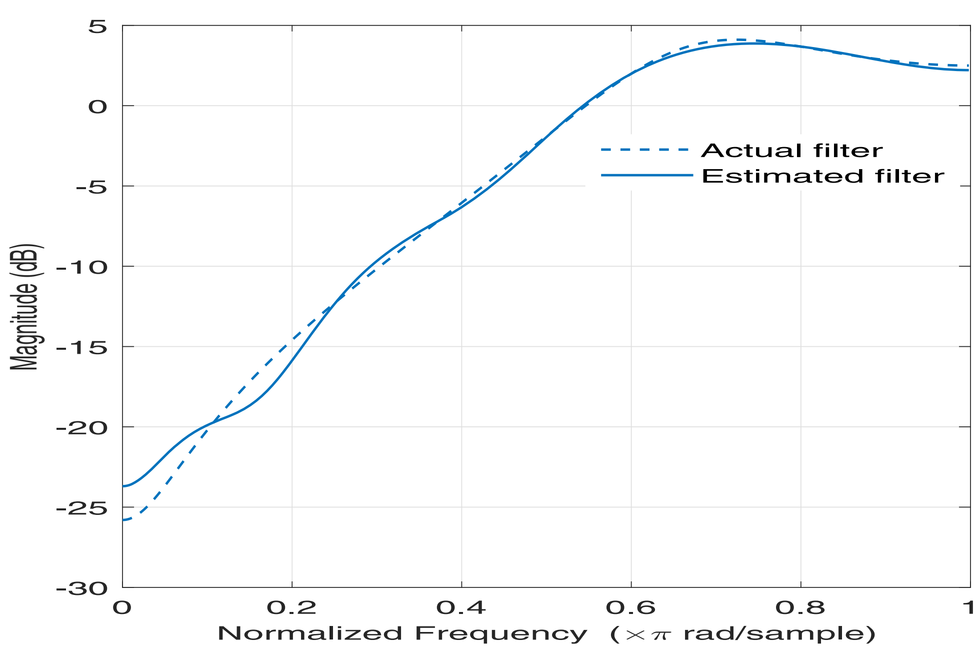


**Figure 4** Frequency response of FIR approximation of IIR filter given in Eq. 28.

**Table 8** Parameter estimates using F-AM-RLS [16] and the proposed algorithm for the system given in Example 3.

| Algorithm | k | $b_1$ | $b_2$ | $\delta\%$ |
|---|---|---|---|---|
| F-AM-RLS | 100 | 0.4392 | –0.5637 | 2.45 |
| | 200 | 0.4504 | –0.5732 | 3.27 |
| | 500 | 0.4601 | –0.5748 | 3.77 |
| | 1000 | 0.4542 | –0.5645 | 2.12 |
| Proposed algorithm | 100 | 0.4330 | –0.5401 | 2.77 |
| | 200 | 0.4323 | –0.5593 | 2.81 |
| | 500 | 0.4457 | –0.5619 | 1.78 |
| | 1000 | 0.4485 | –0.5519 | 0.34 |
| True values: | | 0.45000 | –0.55000 | |

*Example 1* Wang and Ding [18] considered a two-input two-output Box-Jenkins system as given below,

$$\begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix} = \frac{\beta(z)}{\alpha(z)} \begin{bmatrix} u_1(n) \\ u_2(n) \end{bmatrix} + \frac{D(z)}{C(z)} \begin{bmatrix} v_1(n) \\ v_2(n) \end{bmatrix}, \qquad (26)$$

$$\alpha(z) = 1 + 0.13z^{-1}, \beta(z) = \begin{bmatrix} 1.20z^{-1} & 0.82z^{-1} \\ 1.10z^{-1} & 0.95z^{-1} \end{bmatrix}$$

$$C(z) = 1 + 0.10z^{-1}, D(z) = 1 - 0.23z^{-1}$$

For the purpose of simulation studies the inputs $u_1(n)$ and $u_2(n)$ are considered as zero mean and unit variance excitation sequences, $v_1(n)$ and $v_2(n)$ are white noise signals with zero mean and variances $\sigma_1^2 = \sigma_2^2 = 0.40^2$. Since the proposed algorithm estimates FIR system model, we assume $\alpha(z) = 1$ and estimate the matrix $\beta(z)$.

$\beta(z)$ was estimated in [18] using three recursive algorithms namely, auxiliary model based hierarchical stochastic gradient (AM-HSG) algorithm, filtering based AM-HSG (F-AM-HSG) algorithm, and forgetting factor AM-HSG (FF-AM-HSG) algorithm. These results are compared with the parameter estimates obtained using proposed algorithm in Table 6. The parameter estimation errors is defined as $\delta \triangleq ||\hat{\theta}_k - \theta||/||\theta||$ where $\hat{\theta}_k$ is the estimated value of parameter $\theta$ at k-th recursion.

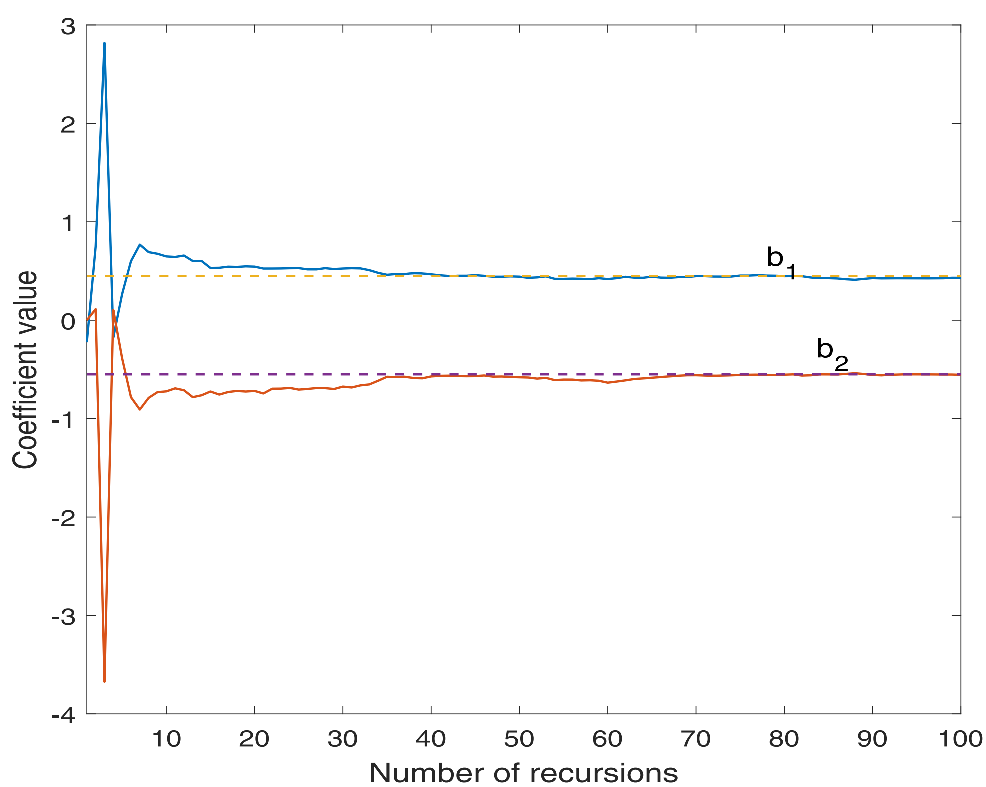


**Figure 5** The transfer function coefficients vs the number of recursions.

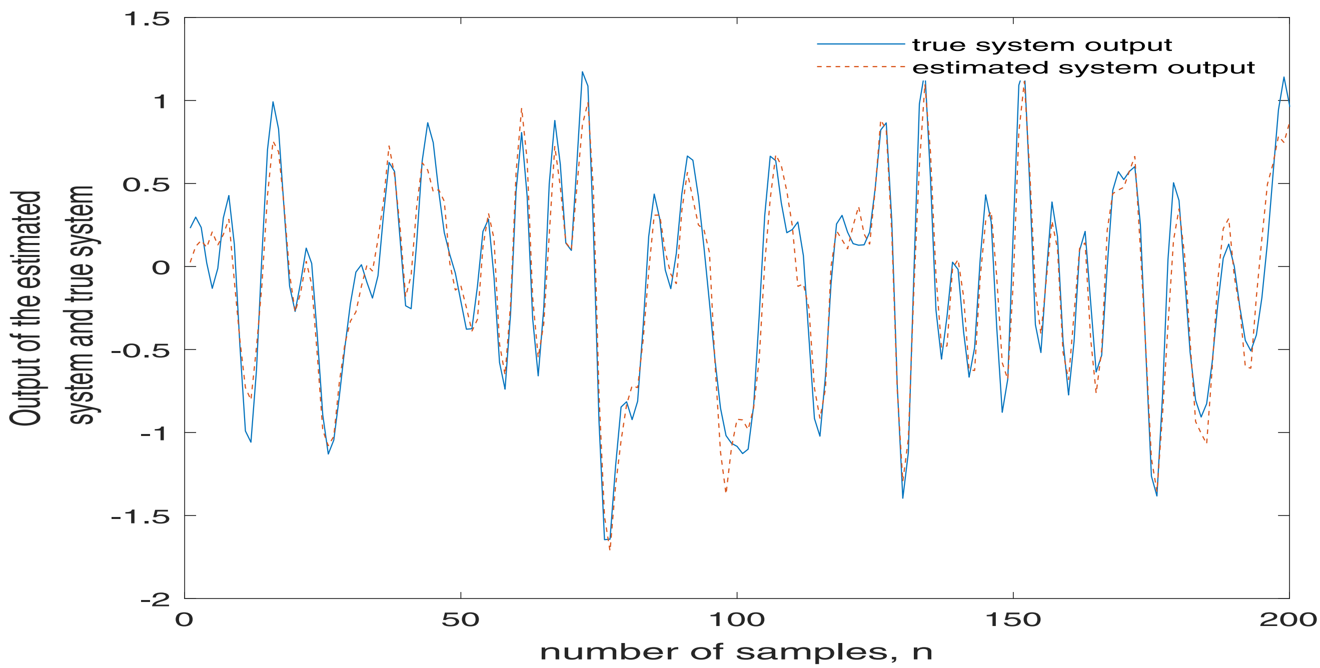


**Figure 6** Output of FIR filter of order 100 vs output of FIR system of order 50 estimated using lattice structure.

*Example 2* Let us consider the following 2-input, 2-output system,

$$\begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix} + \begin{bmatrix} 0.70 & 0.40 \\ -0.30 & 0.80 \end{bmatrix} \begin{bmatrix} y_1(n-1) \\ y_2(n-1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -0.50 \\ -0.40 & 1.20 \end{bmatrix} \begin{bmatrix} u_1(n-1) \\ u_2(n-1) \end{bmatrix} + \begin{bmatrix} v_1(n) \\ v_2(n) \end{bmatrix} \quad (27)$$

As assumed in the previous example, the inputs $u_1(n)$ and $u_2(n)$ are uncorrelated persistent excitation sequences with zero mean and unit variances, and $v_1(n)$ and $v_2(n)$ are zero mean white noise signals with variances $\sigma_1^2 = \sigma_2^2 = 1.00^2$. Han and Ding [12] used stochastic gradient (SG) algorithm and multi-innovation SG (MISG) algorithm for estimating the parameters of system given in Eq. 27, the parameter estimates and their estimation errors are given in Table 7. The parameter estimates obtained using the proposed algorithm are also included in Table 7.

*Example 3* In this example we estimate a SISO system using the proposed recursive algorithm in presence of moving average (MA) noise and compare the obtained parameter estimates with filtering and auxiliary model-based RLS (F-AM-RLS) identification algorithm [16]. Wang [16] estimated the parameters of the following SISO system:

$$y[n] = \frac{B(z)}{A(z)} u[n] + D(z) v[n], \quad (28)$$

where,
$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} = 1 + 0.60 z^{-1} + 0.35 z^{-2}$
$B(z) = b_1 z^{-1} + b_2 z^{-2} = 0.45 z^{-1} - 0.55 z^{-2}$
$D(z) = 1 + d_1 z^{-1} = 1 - 0.80 z^{-1}$.

The input, $u[n]$, is taken as a persistent excitation signal sequence with zero mean and unit variance, and $v[n]$ is a white noise sequence with zero mean and $\sigma^2 = 0.40^2$. The frequency response of FIR system estimated using the proposed algorithm and the actual system $B(z)/A(z)$ is shown in Fig. 4. We have considered the filter parameters after 1000 iterations and the order of the filter is 25.

To compare the accuracy of the parameter estimation obtained using the proposed algorithm with [16], we modify the system by considering $A(z) = 1$. The results so obtained are consolidated in Table 8. From Fig. 5 we illustrate the faster convergence rate of the proposed algorithm.

*Example 4* The lattice structure obtained from the proposed algorithm can be used to reduce order of the estimated model. Here, we consider a low pass FIR filter of order 100 and cut off as $0.4\pi$. We estimated this system using a lattice filter of order 55, the output of the true system and the estimated system are shown in Fig. 6.

Discussion on simulation results:

1. Example 1 compares the performance of proposed algorithm with AM-HSG, F-AM-HSG, and FF-AM-HSG which are different versions of hierarchical stochastic gradient algorithm. The hierarchical identification algorithms based on the decomposition technique are used for parameter estimation of MIMO systems. HSG algorithm have lower computational complexity than its

RLS counterparts, however the estimation accuracy is poor and convergence rate is relatively slow. The filtering technique has been used in [18] to improve the estimation accuracy through filtering the input-output data of the systems. Forgetting factor has also been used for speeding up the convergence performance. For parameter estimation, algorithms in [18] require the orders/structures of the system. From Table 6, we observe the following:

a) For $k = 10$, the estimation error $\delta$ is minimum for proposed algorithm.
b) The proposed algorithm displays the sharpest decay of estimation error from $k = 10$ to $k = 50$. Therefore, the proposed algorithm converges faster than AM-HSG, F-AM-HSG and FF-AM-HSG.

2. In Example 2 we compare the convergence rate and estimation accuracy of proposed algorithm with MISG algorithm [12] and RLS algorithm. RLS converges faster than MISG but the computation burden of RLS is more than stochastic gradient based algorithms. From Table 7 we observe:

   a) The proposed algorithm offers the best estimation accuracy at $k = 100$ i.e. after hundred recursions.
   b) The convergence speed of the proposed algorithm is better than both RLS and MISG. It should be noted here than unlike RLS, the algorithm developed in this work only performs scalar computations at each recursion.

3. In Example 3 and 4 we discuss the performance of the proposed algorithm for SISO systems. Although the proposed algorithm has been developed to estimate FIR model for a given system we can extend this to obtain IIR systems. Also, the FIR models can be used to approximate IIR system response as shown in Fig. 4.
4. Figure 5 illustrates the convergence property of the proposed algorithm. Since the proposed algorithm is based on exact least squares, the convergence rate is faster compared to RLS and SG algorithms.

## 5 Conclusions

In this work we converted an $L \times M$ MIMO system into a SISO form, which is shown to be a generalized version of an M-channel synthesis filter bank when $L = M$. Thus, the algorithm proposed to identify a MIMO system can also be used to obtain synthesis filters. The proposed algorithm can also estimate LPTV systems when $L = 1$ and $M$ is the period of the system. A computationally efficient, fast, time and order recursive exact least squares algorithm has been developed for the same, with better convergence rate and smaller steady state error than recursive algorithms existing in the literature. Simulation results have been presented to compare the performance of the proposed algorithm with some latest works. The recursions of the proposed algorithm give rise to a lattice structure, which can be used for model order reduction.

In this paper, we estimated an FIR model for the given MIMO system, the algorithm can also be used to identify IIR systems and multivariable systems. In future work, we will estimate the given MIMO system with an IIR transfer function model with ARMA noise.

## Compliance with Ethical Standards

**Conflict of interests** The authors declare that they have no conflict of interest.

## Appendix: Formulae Used in the Algorithm

We call a signal $v(n)$ to be in a pre-windowed form if $v(n) = 0$ for $n < 0$. A given discrete time signal, $\mathbf{v}(Ln-i)$, is defined as $1 \times K$ (K is a fixed number) vector, with $K \gg Ln - i$, :

$$\mathbf{v}(Ln-i) \equiv \begin{bmatrix} 0 & \cdots & 0 & v(-i) & v(L-i) & \cdots & v(Ln-i) \end{bmatrix}$$

and the set of vectors, $\{\mathbf{v}(Ln-k) | i \leq k \leq i+p\}$, forms a $(p+1) \times K$ matrix, denoted as $V_{p+1}^{Ln-i}$, and is given as follows:

$$V_{p+1}^{Ln-i} \equiv \begin{bmatrix} \mathbf{v}(Ln-i) \\ \mathbf{v}(Ln-i-1) \\ \vdots \\ \mathbf{v}(Ln-i-p) \end{bmatrix}. \tag{29}$$

Here the superscript denotes the top row vector used and the subscript $p+1$ denotes number of rows. The span of row-space of $V_{p+1}^{Ln-i}$ is given as:

$$S = span\{\mathbf{v}(Ln-i), \mathbf{v}(Ln-i-1), \cdots, \mathbf{v}(Ln-i-p)\}$$

If the rows vectors are linearly independent, the span $S$ can be written as:

$$S = span\{\mathbf{v}(Ln-i), \mathbf{v}(Ln-i-1), \cdots, \mathbf{v}(Ln-i-p)P^{\perp}\left[V_p^{Ln-i}\right]\}.$$

The projection on span of $V_{p+1}^{Ln-i}$ can be written as:

$$P\left[V_{1+p}^{Ln-i}\right] = P\begin{bmatrix} V_p^{Ln-i} \\ \mathbf{v}(Ln-i-p)P^{\perp}\left[V_p^{Ln-i}\right] \end{bmatrix}$$
$$= P[V_p^{Ln-i}] + P\left[\mathbf{v}(Ln-i-p)P^{\perp}\left[V_p^{Ln-i}\right]\right] \tag{30}$$

Since the projection operator is defined as $P[X] = X^T[XX^T]^{-1}X$, we can re-write the above equation as:

$$P\left[V_{1+p}^{Ln-i}\right] = P[V_p^{Ln-i}] + P^{\perp}\left[V_p^{Ln-i}\right]\mathbf{v}^T(Ln-i-p)[\mathbf{v}(Ln-i-p) \times P^{\perp}\left[V_p^{Ln-i}]\mathbf{v}^T(Ln-i-p)\right]^{-1}\mathbf{v}(Ln-i-p)P^{\perp}\left[V_p^{Ln-i}\right] \quad (31)$$

and the projection onto the orthogonal complement space, $P^{\perp}\left[V_{1+p}^{Ln-i}\right]$, can be written as

$$P^{\perp}\left[V_{1+p}^{Ln-i}\right] = I - P\left[V_{1+p}^{n-1}\right]$$
$$= P^{\perp}[V_p^{Ln-i}] - P^{\perp}\left[V_p^{Ln-i}\right]\mathbf{v}^T(Ln-i-p)[\mathbf{v}(Ln-i-p) \times P^{\perp}\left[V_p^{Ln-i}]\mathbf{v}^T(Ln-i-p)\right]^{-1}\mathbf{v}(Ln-i-p)P^{\perp}\left[V_p^{Ln-i}\right] \quad (32)$$

Pre-multiply (32) with $\mathbf{v}$ and post-multiply with $\mathbf{w}^T$ to get:

$$\mathbf{v}P^{\perp}\left[V_{1+p}^{Ln-i}\right]\mathbf{w}^T = \mathbf{v}P^{\perp}[V_p^{Ln-i}]\mathbf{w}^T - \mathbf{v}P^{\perp}\left[V_p^{Ln-i}\right]\mathbf{v}^T(Ln-i-p) \times \left[\mathbf{v}(Ln-i-p)P^{\perp}\left[V_p^{Ln-i}\right]\mathbf{v}^T(Ln-i-p)\right]^{-1} \times \mathbf{v}(Ln-i-p)P^{\perp}\left[V_p^{Ln-i}\right]\mathbf{w}^T \quad (33)$$

The above equation is known as the inner-product update formula, for more detail refer [29]. We now briefly discuss the projection update formula from [30]. This update relation is used to compute recursions for (17). Let us define $K_{p+1}^{Ln-i}$, the pseudo-inverse of $X_{p+1}^{Ln-i}$ i.e. $X_{p+1}^{Ln-i}K_{p+1}^{Ln-i} = I$, as follows:

$$K_{p+1}^{Ln-i} = \left[X_{p+1}^{Ln-i}\right]^T\left[X_{p+1}^{Ln-i}\left[X_{p+1}^{Ln-i}\right]^T\right]^{-1} \quad (34)$$

Pre-multiply (31) with $z$ and post-multiply by $K_{p+1}^{Ln-i}$, we get:

$$\mathbf{z}K_{p+1}^{Ln-i} = \begin{bmatrix} \mathbf{z}K_p^{Ln-i} \\ 0 \end{bmatrix} + \mathbf{z}P^{\perp}[V_{1:p}]v_{p+1}^T[v_{p+1}P^{\perp}[V_{1:p}]v_{p+1}^T]^{-1}. \begin{bmatrix} -v_{p+1}K_p^{Ln-i} \\ 1 \end{bmatrix}. \quad (35)$$

For more information and derivation of (35) refer [30].

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