Question 1

Consider a variant of the Bertrand price competition model from the lecture slides. There are $N \geq 2$ sellers each selling a homogenous good. All the sellers set their prices simultaneously and the marginal cost of production for every firm is $c > 0$. There is a unit mass of buyers (or, equivalently, a representative buyer). Throughout, I use the female pronoun for the buyer and the male pronoun for the seller. Each buyer consumes at most one unit of the good and values it at $v > c$ — i.e., she will not buy if the price is greater than v but will buy one unit from the seller selling the good at the lowest price otherwise. (This means that if there is only one seller, his monopolist price will be $p = v$, and every buyer will buy one unit from him. If there are $N \geq 2$ sellers, then the Bertrand competition argument will lead to every seller setting $p = c$ as the only pure strategy Nash equilibrium.)

The difference here from the Bertrand model is that a buyer must first visit a seller before she knows what is that seller's price, and she incurs a search cost for doing so. More specifically, assume that each buyer first arrives at the store of a randomly chosen seller and learns that seller's price. We assume that there is no cost of doing so — i.e., the buyer learns the first price for free.[1] Subsequently, the buyer has three options: first, she can choose to accept that seller's price and buy one unit from him; second, she can leave the market completely; third, she can pay a search cost s to learn the price of another randomly chosen seller before deciding what to do. The buyer can repeatedly choose the third option and learn as many prices as she wishes at a cost of s for each price. Thus, if she pays $(N-1)s$, she will know the prices of all N firms in the market. (For simplicity, we will assume that all the sellers choose their prices at the start and cannot alter their price in response to any buyer's search behavior.)

(If $s = 0$, this is the Bertrand competition setup, and the unique pure strategy Nash equilibrium is every seller setting a price of c, every buyer will learn all the prices and then randomly buy from a seller.)

Assume that $s > 0$, but it can be arbitrarily small.

We are interested in finding a symmetric pure strategy equilibrium — i.e., every firm charges a common price p^*. Show that there is a unique symmetric pure strategy equilibrium and find that p^*.

[1] This is without loss of generality because, even if there is a search cost here, this cost is sunk and will not affect the buyer's subsequent decision, which is the interest of the analysis here.

Question 2

There are two firms competing via quantity in a market, which has an inverse demand function of

$$p = 1 - q_1 - q_2,$$

where q_1 and q_2 denote the quantities of firms 1 and 2, respectively. Both firms have a marginal cost of production of $c < 1$.

Part A: Suppose that the firms choose their quantity simultaneously (i.e., Cournot competition). What is the Nash equilibrium?

Part B: Suppose that firm 1 chooses its quantity q_1 first. Subsequently, firm 2 observes q_1 and chooses its quantity q_2 (i.e., Stackelberg competition). What is the subgame perfect Nash equilibrium?

Part C: Let q_i^S denote the quantity that firm i produces in the equilibrium outcome in part B. Show that against q_2^S, if firm 1 can go back in time and change its output level (which firm 1 can't!), firm 1 would want to decrease its output level. Given this, why did firm 1 still produce the (high) quantity q_1^S?

Part D: Continue to assume that firm 1 chooses its quantity q_1 first. However, instead of firm 2 observing q_1 directly, firm 2 now observes a number x. With a probability of $1 - \varepsilon$, this number x will be q_1, but with a probability of ε, this number x is a randomly drawn positive number (i.e., a noise). Note that firm 2 does not know if the number that it observes is q_1 or a noise. Find the pure strategy Nash equilibrium of this game for each $\varepsilon > 0$.

Question 3: Mechanism Design and Enforcement without Verifiability

A buyer and a seller want to contract to trade a good that has not yet been produced. There is some randomness about the quality of the good that is beyond the control of the seller. For simplicity, let us assume that after the good has been produced, the good could be of either high or low quality. Both players' utility can be measured in dollars. A high quality good is worth \$60 to the buyer and a low quality good is worth \$40 to the buyer. Thus, if a trade of a high quality good occurs at price p, the buyer's utility is $\$60 - p$, and the seller's utility is p. If a trade of a low quality good occurs at price p, the buyer's utility is $\$40 - p$, and the seller's utility is p. If there is no trade, both parties' utility is zero.

Suppose (for some exogenous reasons) that the two parties would like to trade at $p = \$30$ if the good is of high quality and at $p = \$20$ if the good is of only low quality. Both parties can objectively observe the quality of the good after it is produced. However, the two parties cannot simply write a contract that specifies the (desired) arrangement above because the court lacks the expertise to verify the quality and hence, cannot enforce it. In particular, one needs to determine what should happen if the buyer claims that the quality is low but the seller claims that the quality is high, which is a commonly observed kind of disagreement in practice.

Below, we consider two "mechanisms" to enforce the desired trade outcome that ensure that both parties will not try to lie about the quality, even though they are the only two people in the world who know what the true quality is.

Design 1: Static Nash Implementation

Consider the following mechanism: after the good is produced, both the buyer and the seller *simultaneously* report whether the quality of the good is high or low to the court. The following outcome is then implemented:

- If both the buyer and the seller report that the quality is *high*, trade takes place at $p = \$30$ — i.e., the buyer pays the seller \$30 and gets the good.
- If both the buyer and the seller report that the quality is *low*, trade takes place at $p = \$20$ — i.e., the buyer pays the seller \$30 and gets the good.
- If the two parties' report disagree with each other, there is no trade — i.e., the court destroys the good.

Note that the court is not strategic — it can commit to following through with the enforcement mechanism described above. This fact, together with the mechanism above, is common knowledge to both the buyer and the seller. Given the enforcement mechanism described above, answer the following questions.

Part 1a.

Suppose that the quality of the good is ***high***. Solve for *all* (i.e., in both pure and mixed strategies) the Nash equilibrium(s) in this game.

Part 1b.

Suppose that the quality of the good is ***low***. Solve for *all* (i.e., in both pure and mixed strategies) the Nash equilibrium(s) in this game.

Design 2: Subgame Perfect Implementation

If you have solved the previous question correctly, you should find that under the previous mechanism, the desired outcome for each quality level is always a Nash equilibrium; however, there also always exist other (undesirable) Nash equilibrium(s). Can we design a better mechanism that always implements only the desired outcome at each quality level? Consider the following multi-stage mechanism instead:

1. The buyer is first asked to publicly announce the quality.
   - If the buyer announces that the quality is high, the court enforces that trade takes place at $p = \$30$ and the game ends.
   - If the buyer announces that the quality is low, the game moves to the next stage.
2. Now, the seller is asked whether he agrees with the buyer that the quality is low.
   - If the seller *agrees* that the quality is low, the court enforces that trade takes place at $p = \$20$ and the game ends.
   - If the seller *disagrees* that the quality is low, the game moves to the next stage.
3. Now, the buyer is offered the option to buy the good for $p = \$55$.
   - If the buyer takes the option, the court enforces that trade takes place at $p = \$55$ and the game ends.
   - If the buyer does not want to take the option, the court destroys the good (i.e., there is no trade) and the game ends.

Again, note that the court can commit to following through with the enforcement mechanism described above, and this arrangement is common knowledge to both the buyer and the seller at the start. Given this enforcement mechanism, answer the following questions.

Part 2a.

Suppose that the quality of the good is ***high***. Solve for all the subgame perfect equilibrium.

Part 2b.

Suppose that the quality of the good is ***low***. Solve for all the subgame perfect equilibrium.

Test yourself now!

Amy and Bob are having a dispute over how to split 10 pieces of land. For concreteness, let us label the lands by L1, L2, ... , L10. Out of these 10 pieces of land, n of them are each worth \$100 million (perhaps because there is gold underneath it), whereas the rest of the $10 - n$ pieces of land are each worth only \$100k. (The actual values are not important; the only relevant detail is that some pieces are more valuable than others.)

You have an "equal-split objective" — i.e., you want to distribute these 10 pieces of land to Amy and Bob such that they each get exactly half of the value of all the 10 pieces of land. For example, suppose that $n = 2$, and L3 and L6 are the lands that are worth \$100 million each. In this case, an example of an ideal split is Amy gets L3, Bob gets L6, and on top of that, each of them also gets any 4 of the other 8 pieces. For simplicity, to accommodate the case where n is odd, assume that one is allowed to cut one (and only one) piece of land into two equal halves, and each half is then worth half of the value of the original land's value.

It is common knowledge that both Bob and Amy know the exact valuation of each of the 10 pieces of land, and they are the only two people in the world who has this information. All you know is n is at least 1. You do ***not*** know what is the exact value of n. You also do ***not*** know which are the n pieces of land that are each worth \$100 million.

Part 3.

Design a (simple!) mechanism to implement the equal-split objective.

(You are free to set any rules to determine the split. Note that both Amy and Bob are strategic and want to receive as high a share of the value as possible. Therefore, it's not very likely that you can get them to truthfully tell you the values of the lands by simply asking them! You will be judged by how close you can obtain the equal-split outcome, given that Amy and Bob will strategically act in response to your mechanism. Keep your answer short but clear. It is possible to score full marks for this question with just one or two sentences.)

Question 4

Let us consider a game that that I call "*Picking from the edge*":

> Picture a massive stack of cards, each adorned with a distinct integer representing its score. A third-party randomly draws n cards from the stack, where n is an even integer. These cards are laid out face up (i.e., revealing their scores) on the table in a single row in the order of when they are drawn.[2] Two players take turns picking cards from the table, with Player 1 taking the first move. The winner is determined by who accumulates the higher total score. But there's a catch — not all cards can be picked at any time. In particular, the player picking can only choose between either the left-most card or the right-most card currently left in the row.

It is a priori unclear what is the best way to play this game. Of course, one can always naively choose the higher of the two edge cards available at each turn. But to win for sure, it seems prudent to also think about how to protect a high-scoring card that is currently stuck in the middle of the row from the opponent (whenever such a card exists). It is also a priori unclear if one should prefer to be Player 1 (the first mover) or Player 2 (the second mover).

Nevertheless, given what has been taught in class, we know that one of the players have a strategy that ensures that he wins all the time. Why is this the case?

Identify the player (first mover or second mover) who has a winning strategy and state the winning strategy.

[2] This means that the cards are **not** necessarily ordered from the lowest to the highest score.