Course on Symmetry, Topology and Entanglement

Homework 2

- **Instructor**: Kargarian - Vaezi
- **Semester**: Fall 2022
- **Handed out**: Monday 02-08-1401
- **Due date**: Monday 23-08-1401 in class

1. *Duality* – In this exercise we will see that the Coulomb gas (CG) model of the vortices of XY model is *dual* to the Sine-Gordon (SG) model. Given a configuration of vortices, in class we derived the following expression for the free energy

$$F[\{n_{\mathbf{r}}\}] = \varepsilon_c \sum_{\mathbf{r}} n_{\mathbf{r}}^2 - 2\pi^2 J \sum_{\mathbf{r}\neq\mathbf{r}'} n_{\mathbf{r}} n_{\mathbf{r}'} V(\mathbf{r}-\mathbf{r}'), \tag{1}$$

where ε_c is the energy of the vortex core. Note that in class I used parameter $K = \beta J$ instead of J.

(a) Fourier transform and show that

$$F = \frac{2\pi^2 J}{L^2 a^4} \sum_{\mathbf{k}} |n_{\mathbf{k}}|^2 V(\mathbf{k}), \quad V(\mathbf{k}) = \frac{1}{k^2}. \tag{2}$$

(b) Use the following identity to determine γ and α.

$$e^{-\beta \frac{2\pi^2 J}{L^2 a^4} |n_{\mathbf{k}}|^2} = A_{\mathbf{k}} \int d\phi \; e^{-\alpha|\phi|^2 + \gamma\phi}, \quad \beta = 1/k_B T. \tag{3}$$

(c) Show that the partition function in real spaces reads as

$$Z_{CG} = A \left(\prod_{\mathbf{r}} \sum_{n_{\mathbf{r}}=-\infty}^{\infty} \right) \left(\int \prod_{\mathbf{r}} d\phi_{\mathbf{r}} \right) e^{-S}, \tag{4}$$

where

$$S = \frac{1}{8\pi^2} \frac{a^2}{\beta J} \sum_{\mathbf{r}} |\nabla\phi|^2 - i \sum_{\mathbf{r}} \phi_{\mathbf{r}} n_{\mathbf{r}} + \beta\varepsilon_c \sum_{\mathbf{r}} n_{\mathbf{r}}^2. \tag{5}$$

(d) Perform the sum over the vortex configuration gently and show that the partition can be written as

$$Z_{CG} = A \int \prod_{\mathbf{r}} d\phi_{\mathbf{r}} e^{-\frac{1}{8\pi^2}\frac{a^2}{\beta J} \sum_{\mathbf{r}} |\nabla\phi|^2 + \sum_{\mathbf{r}} 2y\cos\phi_{\mathbf{r}}}, \quad y = e^{-\beta\varepsilon_c}. \tag{6}$$

(e) The functional energy of the Sine-Gordon model is given by

$$\beta F_{SG}[\phi] = \int d^2 r \left[\frac{g}{2} |\nabla\phi|^2 - J\cos\phi_{\mathbf{r}} \right]. \tag{7}$$

Show that the partition of function of two models are dual to each other, i.e.,

$$Z_{CG}(\beta J, \beta\varepsilon_c) = A Z_{SG}\left(\frac{1}{4\pi^2 \beta J}, \frac{2}{a^2} e^{-\beta\varepsilon_c} \right) \tag{8}$$

(f) What does it mean?

2. *Local U(1) symmetry* – Consider a field theory with the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(D^\mu \phi^*)(D_\mu \phi) - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{9}$$

$$V(\phi) = -\frac{t^2}{2}\phi^*\phi + \frac{u}{4}(\phi^*\phi)^2 \tag{10}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor.

(a) Show that the theory is invariant under local U(1) gauge transformation $\phi \to \phi' = e^{i\eta(\mathbf{r},t)}\phi$ provided the normal derivatives are replaced by the covariant ones, i.e., in the Lagrangian we have

$$D_\mu \phi = (\partial_\mu - igA_\mu), \tag{11}$$

where g is the coupling to the electromagnetic field.

(b) Following the steps I did in class to study the Goldestone modes, show the Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}\partial^\mu \rho \partial_\mu \rho + \frac{1}{2}(\partial_\mu \sigma - g\phi_0 A_\mu)^2 \left(1 + \frac{\rho}{\phi_0}\right)^2 - V(\rho) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{12}$$

(c) Show that the σ field can be removed from the Lagrangian.

(d) Derive the equation of motion, or using Fourier transform, and show that the electromagnetic field becomes massive. It is the so-called Meissner effect in superconductors.