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The characteristic finite difference streamline diffusion method for convection-dominated diffusion problems ☆

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ABSTRACT

In this paper, we consider the characteristic finite difference streamline diffusion method for two-dimensional convection-dominated diffusion problems. The scheme is combined the method of characteristics with the finite difference streamline diffusion (FDSD) method to create the characteristic FDSD (C-FDSD) procedures. Stability analysis and error estimate of the C-FDSD method are deduced. The scheme not only realizes the purpose of lowering the time-truncation error, using larger time step for solving the convection-dominated diffusion problems, but also keeps the favorable stability and high precision of the FDSD method. Finally, numerical experiments are presented to illustrate the availability of the scheme.

1. Introduction

It is well known that the convection-dominated diffusion problem has strongly hyperbolic nature, the solution often develops sharp fronts that are nearly shocks. So the finite difference method (FDM) or the finite element method (FEM) applied to the problem do not work well when it is convection-dominated. Therefore to construct an effective numerical method for solving such a problem is not easy. In 1982, Douglas and Russell [1] considered combining the method of characteristics with finite element or finite difference techniques to overcome oscillation and faults likely to occur in the traditional FDM or FEM. There are many related approximation techniques for the convection-dominated diffusion equations [2,3]. For example, Tabata and his coworkers [4–7] have studied the upwind schemes based on triangulation for the convection–diffusion problem. Yuan has presented a characteristic finite element alternating direction method with moving meshes [8] and an upwind finite difference fractional step method [9], respectively. In problems with significant convection, these approaches of characteristics have much smaller time-truncation errors compared to FDM or FEM. Moreover, these schemes of characteristics will permit the use of larger time steps, with corresponding improvements in efficiency, at no cost in accuracy [10–14].

Streamline diffusion (SD) method for the convection-dominated diffusion problem is introduced by Hughes and Brooks [15]. Afterwards, Johnson [16] and Hansbo [17,18] develop a space–time FEM based on combining the method of

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characteristics with streamline diffusion stabilization, respectively. The method is based on space–time elements approximately aligned with the characteristics in space–time. Furthermore, for the time-dependent convection–diffusion problems and Navier–Stokes equations, the SD method is based on a general space–time finite element mesh, with the basis functions continuous in space but discontinuous in time, then it will increase the number of dimension comparing with the completely discrete Galerkin FEM [19–24]. Due to its favorable stability and higher precision, the SD method has been developed rapidly in theory and practice in the past years [25,26,3]. The finite difference streamline diffusion method (FDSD) is proposed firstly by Sun and his coworkers [27,28] in 1998, that is, using the SD method discrete only in space variables and using the finite difference discrete in time variables. Compared with the SD method, the FDSD method not only simplifies the computational work but also keeps the good stability and high accuracy. So it is developed and used widely by many authors [29–33].

In this paper, the method of characteristics is combined with the FDSD method to create the C-FDSD scheme for solving two-dimensional convection-dominated diffusion problems. In addition, we derive the stability analysis and error estimate in the L^2-norm and L^∞-norm. The main results for the C-FDSD method introduced in Section 2, are contained in Theorem 3.1 of Section 3 and in Theorem 4.1 of Section 4. Then this method keeps the favorable property of having much smaller time-truncation errors and using larger time steps, the favorable stability and high precision of FDSD method is also kept. Finally, we give numerical experiments for a convection-dominated diffusion problem. Compared with the standard Galerkin FEM, the characteristic Galerkin FEM and the FDSD method, the C-FDSD method is more effective.

The remainder of the paper is organized as follows. In Section 2, the C-FDSD method for solving the time-dependent convection-dominated diffusion problem is presented and in Section 3 we give stability analysis of the new scheme. Section 4 is devoted to error estimate for the new scheme. Numerical experiments confirming the theoretical results are provided in Section 5. Finally, conclusions are given in the last section.

2. C-FDSD finite element method

Let Ω be an open bounded domain with piecewise smooth boundary $\partial\Omega$ in R^2. $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ denote the standard Sobolev space [34] respectively. The norm of $v \in W^{m,p}(\Omega)$ will be denoted by $\|v\|_{m,p,\Omega} = \left(\sum_{j\leqslant m}\|D^j v\|^p_{L^p(\Omega)}\right)^{1/p}$, for $1 \leqslant p < \infty$ and $\|v\|_{m,\infty,\Omega} = \max_{j\leqslant m} \operatorname{ess\,sup}|D^j v|$, for $p = \infty$. We shall write $W^{m,2}(\Omega) = H^m(\Omega),\ \|v\|_{H^m(\Omega)} = \|v\|_m$ and $\|v\|_{L^2(\Omega)} = \|v\|_0 = \|v\|$.

Assume that Ω is the square domain $(0,1)\times(0,1)$, $T > 0$ represents a finite time. Denote $Q = \Omega \times (0,T]$. Now we consider the following time-dependent convection-dominated diffusion problem

$$\frac{\partial u}{\partial t} - \nabla\cdot(a(x,t)\nabla u) + \beta(x,t)\cdot\nabla u + \sigma(x,t)u = f(x,t), \quad (x,t)\in Q, \tag{2.1a}$$
$$u(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T], \tag{2.1b}$$
$$u(x,0) = u_0(x), \quad x\in\Omega, \tag{2.1c}$$

where $x = (x_1,x_2)$, $\beta(x,t) = (\beta_1(x,t),\beta_2(x,t))$. Here the coefficients $a(x,t)$, $\beta_1(x,t)$, $\beta_2(x,t) \in L^\infty(W^{1,\infty}(\Omega))$, $\sigma(x,t)\in L^\infty(Q)$. The prescribed external force $f(x,t)\in L^2(Q)$ and the initial velocity $u_0(x)\in H^{r+1}(\Omega)\cap H_0^1(\Omega),\ r\geqslant 1$ and there exist positive constants $a_0, a_1, \bar{a}, \tilde{a}, b_1, \sigma_1, k$ such that

$$a_0 = \inf_Q a(x,t), \quad a_1 = \sup_Q a(x,t), \quad q_0 = \frac{a_1}{a_0},$$
$$\bar{a} = \sup_Q \|\nabla a(x,t)\|, \quad \tilde{a} = \left\|\frac{\partial a}{\partial t}\right\|_{L^2(L^\infty(\Omega))}, \quad \sigma_1 = \sup_Q|\sigma(x,t)|,$$
$$b_1 = \sup_Q\|\nabla\cdot\beta(x,t)\|, \quad k = \sup_Q\|\beta(x,t)\| = \sup_Q\left(\sum_{i=1}^2|\beta_i(x,t)|^2\right)^{\frac{1}{2}}, \quad a_0 \ll k.$$

We also assume the solution $u(x,t)$ of problem (2.1a)–(2.1c) satisfies:

$$u\in L^\infty(0,T;H^{r+1}(\Omega)\cap H_0^1(\Omega)), \tag{2.2a}$$
$$\frac{\partial u}{\partial t}\in L^2(0,T;H^{r+1}(\Omega)), \tag{2.2b}$$
$$\frac{\partial^2 u}{\partial t^2}\in L^2(0,T;L^2(\Omega)). \tag{2.2c}$$

We assume that $\overline{\Omega}$ is partitioned by a quasi-uniform triangulation $T_h = \{\kappa\}$ with the mesh parameter h_i, where h_i is the largest diameter of each element in T_h. Set $h = \max\{h_i\}$ and $0 < h \leqslant h_0 < 1$. Denote

$$V_h = \{v|v\in H_0^1(\Omega)\cap C(\overline{\Omega}),\ v|_\kappa\in P_r(\kappa),\ \forall\kappa\in T_h\},$$

where $P_r(\kappa)$ is a polynomial set of degree $\leqslant r$ on each element κ, $r\geqslant 1$. Let

$$\psi = \sqrt{1+\beta_1^2+\beta_2^2} \tag{2.3}$$

and $\tau = \tau(x)$ be a unit vector in the direction $(1, \beta_1, \beta_2)$, then we have

$$\psi \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \beta_1 \frac{\partial}{\partial x_1} + \beta_2 \frac{\partial}{\partial x_2}. \tag{2.4}$$

Then, problem (2.1a)–(2.1c) can be written in the equivalent form

$$\psi \frac{\partial u}{\partial \tau} - \nabla \cdot (a(x,t)\nabla u) + \sigma(x,t)u = f(x,t), \quad (x,t) \in Q, \tag{2.5a}$$

$$u(x,t) = 0, \quad x \in \partial\Omega \times (0,T], \tag{2.5b}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{2.5c}$$

The variational formulation of problem (2.5) reads as follows

$$\begin{cases} \left(\left(\psi \frac{\partial u}{\partial \tau}\right)^n, v + \delta v_\beta^n\right) - \left(\nabla \cdot (a^n \nabla u^n), v + \delta v_\beta^n\right) + (\sigma^n u^n, v + \delta v_\beta^n) = (f^n, v + \delta v_\beta^n), \quad \forall v \in V_h, \ n = 1, 2, \ldots, N, \\ (u^0 - u_0, v) = 0, \quad \forall v \in V_h, \end{cases} \tag{2.6}$$

where $v_\beta^n \triangleq \beta^n \cdot \nabla v$ and $\delta \geqslant 0$ is an appropriate artificial parameter. Let Δt be the time step, $t^n = n\Delta t$, $n = 1, 2, \ldots, N$, $N = [T/\Delta t]$. The characteristic derivative will be approximated basically in the following manner:

$$\psi \frac{\partial u}{\partial \tau} \approx \psi \frac{u(x,t) - u(\bar{x}, t - \Delta t)}{\sqrt{(x - \bar{x})^2 + (\Delta t)^2}} = \frac{u(x,t) - u(\bar{x}, t - \Delta t)}{\Delta t}, \tag{2.7}$$

where $\bar{x} = x - \beta(x,t) \cdot \Delta t$. Denote $u(x, t_n) = u^n$ and $\bar{u}^{n-1} = u(\bar{x}, t_{n-1})$. On level $t = t^n$ rewrite (2.5a) as

$$\frac{u^n - \bar{u}^{n-1}}{\Delta t} - \nabla \cdot (a^n \nabla u^n) + \sigma^n u^n = f^n + E_1^n, \tag{2.8}$$

where $E_1^n = \frac{u^n - \bar{u}^{n-1}}{\Delta t} - \left(\psi \frac{\partial u}{\partial \tau}\right)^n$. Omitting the truncation error from (2.8), the C-FDSD scheme for solving problem (2.1) is defined as:

Find $U^n \in V_h$, $n = 1, 2, \ldots, N$, such that, for all $v \in V_h$

$$\begin{cases} \left(\frac{U^n - \bar{U}^{n-1}}{\Delta t}, v + \delta v_\beta^n\right) - \left(\nabla \cdot (a^n \nabla U^n), v + \delta v_\beta^n\right) + (\sigma^n U^n, v + \delta v_\beta^n) = (f^n, v + \delta v_\beta^n), \\ (U^0 - u_0, v) = 0, \end{cases} \tag{2.9}$$

where $\left(\nabla \cdot (a^n \nabla U^n), \delta v_\beta^n\right) \triangleq \sum_\kappa \left(\nabla \cdot (a^n \nabla U^n), \delta v_\beta^n\right)_\kappa$ and $(w, v) \triangleq \int_\kappa wv dx$.

3. Stability analysis for the C-FDSD method

In this section, we will give the stability result of the C-FDSD method. Firstly we introduce some lemmas in the following.

Lemma 3.1 (Inverse inequality [35]). *There exists a positive constant $\mu > 0$ independent of h, for any $v \in V_h$, such that*

$$\|\nabla v\| \leqslant \mu h^{-1} \|v\|, \quad \|\Delta v\| \leqslant \mu h^{-1} \|\nabla v\|, \tag{3.1}$$

where $\|\Delta v\|^2 = \sum_\kappa \|\Delta v\|_\kappa^2$.

Lemma 3.2 (Interpolation approximation property[36]). *Let $\Pi_h : H^{r+1}(\Omega) \cap C(\overline{\Omega}) \to V_h$ be an interpolation operator, there exists a positive constant c independent of h, for any $v \in H^{r+1}(\Omega) \cap C(\overline{\Omega})$, such that*

$$\|v - \Pi_h v\| + h\|\nabla(v - \Pi_h v)\| \leqslant ch^{r+1} \|v\|_{r+1}. \tag{3.2}$$

Lemma 3.3 ([1]). *Let $\omega \in L^2(\Omega), \bar{\omega} = \omega(x - g(x)\Delta t)$, g and g′ be bounded functions on $\overline{\Omega}$, if the time step Δt is sufficient small, there exists a positive constant k_0, such that*

$$\|\omega - \bar{\omega}\|_{H^{-1}} \leqslant k_0 \|\omega\|_{L^2} \Delta t, \tag{3.3}$$

where k_0 only dependent on $\|g\|_{L^\infty}$ and $\|g'\|_{L^\infty}$, and the norm of $\|\cdot\|_{H^{-1}}$ is defined as follows

$$\|v\|_{H^{-1}} = \sup_{0 \neq \phi \in H^1} \frac{(v, \phi)}{\|\phi\|_{H^1}}.$$

Denoting $\partial_t U^n = \frac{U^n - U^{n-1}}{\Delta t}$, the choice of artificial parameter δ satisfies the following restricting conditions

(H_1) $\delta \max\{k\mu,\ 4k\mu q_0,\ 2\bar{a}\mu k\} \leqslant \frac{h}{4}$,

(H_2) $\delta \max\{k^2,\ \frac{1}{8},\ 2k\bar{a}\} \leqslant \frac{a_0}{4}$,

(H_3) $\delta \max\{k^2\mu^2,\ 2a_1 k\mu^2\} \leqslant \frac{h^2}{4}$.

Throughout this paper, C denotes a generic positive constant independent of n, h, Δt which may be different at different occurrences.

Theorem 3.1. *Let $\{U^n\}$ be the solution of problem (2.9), δ satisfies the assumptions (H_1)–(H_3) and Δt is sufficient small. Then we have*

$$\max_{1\leqslant n\leqslant N} \|U^n\|^2 + \sum_{n=1}^{N} \|U^n - U^{n-1}\|^2 + a_0 \sum_{n=1}^{N} \|\nabla U^n\|^2 \Delta t \leqslant C\left(\sum_{n=1}^{N} \|f^n\|^2 \Delta t + \|\nabla U^0\|^2 + \|U^0\|^2\right). \tag{3.4}$$

Proof. Taking $v = U^n$ in (2.9) then

$$\left(\frac{U^n - U^{n-1}}{\Delta t}, U^n\right) + (a^n \nabla U^n, \nabla U^n) = (f^n, U^n + \delta U^n_\beta) - \left(\frac{U^n - U^{n-1}}{\Delta t}, \delta U^n_\beta\right) - \left(\frac{U^{n-1} - \overline{U}^{n-1}}{\Delta t}, \delta U^n_\beta\right) - \left(\frac{U^{n-1} - \overline{U}^{n-1}}{\Delta t}, U^n\right) + (\nabla \cdot (a^n \nabla U^n), \delta U^n_\beta) - (\sigma^n U^n, U^n + \delta U^n_\beta) \triangleq \sum_{i=1}^{6} S_i. \tag{3.5}$$

Obviously, we have the following equality

$$\left(\frac{U^n - U^{n-1}}{\Delta t}, U^n\right) = \frac{1}{2\Delta t}\left(\|U^n\|^2 - \|U^{n-1}\|^2 + \|U^n - U^{n-1}\|^2\right).$$

At first, we estimate the left hand-side of (3.5). It is easy to derive that

$$\left(\frac{U^n - U^{n-1}}{\Delta t}, U^n\right) + (a^n \nabla U^n, \nabla U^n) \geqslant \frac{\|U^n\|^2 - \|U^{n-1}\|^2 + \|U^n - U^{n-1}\|^2}{2\Delta t} + a_0 \|\nabla U^n\|^2. \tag{3.6}$$

Now we estimate the terms of the right-hand side in (3.5) respectively. It follows from the Young inequality, Lemmas 3.1, 3.3 and assumptions (H_1)–(H_3)

$$S_1 = (f^n, U^n + \delta U^n_\beta) \leqslant 3\|f^n\|^2 + \frac{1}{8}\|U^n\|^2 + \frac{a_0}{16}\|\nabla U^n\|^2,$$

$$S_2 = -\left(\frac{U^n - U^{n-1}}{\Delta t}, \delta U^n_\beta\right) \leqslant \|\partial_t U^n\|^2 + \frac{a_0}{16}\|\nabla U^n\|^2,$$

$$S_3 = -\left(\frac{U^{n-1} - \overline{U}^{n-1}}{\Delta t}, \delta U^n_\beta\right) \leqslant \delta \|\beta^n \nabla U^n\|_1 \left\|\frac{U^{n-1} - \overline{U}^{n-1}}{\Delta t}\right\|_{-1} \leqslant k_0^2 \|U^{n-1}\|^2 + \frac{\delta^2 k^2}{4}\|\nabla U^n\|_1^2 \leqslant k_0^2 \|U^{n-1}\|^2 + \frac{a_0}{16}\|\nabla U^n\|^2,$$

$$S_4 = -\left(\frac{U^{n-1} - \overline{U}^{n-1}}{\Delta t}, U^n\right) \leqslant \|\nabla U^n\|_1 \left\|\frac{U^{n-1} - \overline{U}^{n-1}}{\Delta t}\right\|_{-1} \leqslant \frac{\delta^2 k^2}{4}\|\nabla U^n\|_1^2 + \frac{k_0^2}{\delta^2 k^2}\|U^{n-1}\|^2 \leqslant \frac{k_0^2}{\delta^2 k^2}\|U^{n-1}\|^2 + \frac{a_0}{16}\|\nabla U^n\|^2,$$

$$S_5 = (\nabla \cdot (a^n \nabla U^n), \delta U^n_\beta) = (\nabla a^n \cdot \nabla U^n, \delta U^n_\beta) + (a^n \nabla \cdot (\nabla U^n), \delta U^n_\beta) \leqslant (q_0 a_0 \delta k \mu h^{-1} + \delta k \bar{a})\|\nabla U^n\|^2 \leqslant \frac{a_0}{8}\|\nabla U^n\|^2.$$

Using the Cauchy–Schwarz inequality, we obtain

$$S_6 = -(\sigma^n U^n, U^n + \delta U^n_\beta) \leqslant \sigma_1 \|U^n\|^2 + \frac{\sigma_1^2}{2}\|U^n\|^2 + \frac{\delta^2 k^2}{2}\|\nabla U^n\|^2 \leqslant \left(\sigma_1 + \frac{\sigma_1^2}{2}\right)\|U^n\|^2 + \frac{a_0}{8}\|\nabla U^n\|^2.$$

Substituting (3.6), $(S_1$–$S_6)$ into (3.5), multiplying two sides by $2\Delta t$ and summing up for $n = 1, 2, \ldots, m$ $(m \leqslant N)$, we have

$$\|U^m\|^2 + \sum_{n=1}^{m} \|U^n - U^{n-1}\|^2 + a_0 \sum_{n=1}^{m} \|\nabla U^n\|^2 \Delta t \leqslant C\Delta t \sum_{n=1}^{m} \left(\|f^n\|^2 + \|U^n\|^2 + 2\|\partial_t U^n\|^2\right) + \|U^0\|^2. \qquad \Box \tag{3.7}$$

Lemma 3.4. *Let $\{U^n\}$ be the solution of problem (2.9) and δ satisfies the assumptions (H_1)–(H_3), then*

$$\sum_{n=1}^{m}\|\partial_t U^n\|^2\Delta t + a_0\|\nabla U^m\|^2 \leqslant C\Delta t\sum_{n=1}^{m}\left(\|f^n\|^2+\|\nabla U^n\|^2+\|U^n\|^2\right)+\|\nabla U^0\|^2. \tag{3.8}$$

Proof. Taking $v = \partial_t U^n$ in (2.9) then

$$(\partial_t U^n,\partial_t U^n) = (f^n,\partial_t U^n+\delta\beta^n\cdot\nabla(\partial_t U^n)) - \left(\frac{U^{n-1}-\overline{U}^{n-1}}{\Delta t},\partial_t U^n\right) - \left(\frac{U^{n-1}-\overline{U}^{n-1}}{\Delta t},\delta\beta^n\cdot\nabla(\partial_t U^n)\right)$$
$$+(\nabla\cdot(a^n\nabla U^n),\delta\beta^n\cdot\nabla(\partial_t U^n)) - (a^n\nabla U^n,\nabla(\partial_t U^n)) - (\sigma^n U^n,\partial_t U^n+\delta\beta^n\cdot\nabla(\partial_t U^n)) \triangleq \sum_{i=1}^{6}T_i. \tag{3.9}$$

Obviously the left-hand side of (3.9) is $(\partial_t U^n,\partial_t U^n) = \|\partial_t U^n\|^2$.

Now we estimate the terms of the right-hand side in (3.9) respectively. It follows from the Young inequality, Lemmas 3.1, 3.3 and assumptions (H_1)–(H_3)

$$T_1 = (f^n,\partial_t U^n+\delta\beta^n\cdot\nabla\partial_t U^n) \leqslant 4\|f^n\|^2+\frac{1}{8}\|\partial_t U^n\|^2+\frac{\delta^2k^2}{8}\|\nabla(\partial_t U^n)\|^2 \leqslant 4\|f^n\|^2+\frac{5}{32}\|\partial_t U^n\|^2,$$

$$T_2 = -\left(\frac{U^{n-1}-\overline{U}^{n-1}}{\Delta t},\partial_t U^n\right) \leqslant \|\partial_t U^n\|_1\left\|\frac{U^{n-1}-\overline{U}^{n-1}}{\Delta t}\right\|_{-1} \leqslant \frac{k_0^2}{\delta k^2}\|U^{n-1}\|^2+\frac{\delta^2k^2}{4}\|\partial_t U^n\|_1^2 \leqslant \frac{k_0^2}{\delta k^2}\|U^{n-1}\|^2+\frac{1}{16}\|\partial_t U^n\|^2,$$

$$T_3 = -\left(\frac{U^{n-1}-\overline{U}^{n-1}}{\Delta t},\delta\beta^n\cdot(\partial_t U^n)\right) \leqslant \|\partial_t U^n\|_1\left\|\frac{U^{n-1}-\overline{U}^{n-1}}{\Delta t}\right\|_{-1} \leqslant \frac{k_0^2}{2k^2}\|U^{n-1}\|^2+\frac{k^2}{2}\|\delta\beta^n\cdot\nabla(\partial_t U^n)\|_1^2$$
$$\leqslant \frac{k_0^2}{2k^2}\|U^{n-1}\|^2+\frac{1}{32}\|\partial_t U^n\|^2,$$

$$T_4 = (\nabla\cdot(a^n\nabla U^n),\delta\beta^n\cdot\nabla(\partial_t U^n)) = (\nabla a^n\cdot\nabla U^n,\delta\beta^n\cdot\nabla(\partial_t U^n)) + (a^n\nabla\cdot(\nabla U^n),\delta\beta^n\cdot\nabla(\partial_t U^n))$$
$$\leqslant (\delta k a_1\delta\mu^2h^{-2}+\delta\bar{a}\mu k h^{-1})\|\nabla U^n\|\|\partial_t U^n\| \leqslant \frac{1}{10}\|\nabla U^n\|^2+\frac{5}{32}\|\partial_t U^n\|^2.$$

It is easy to derive that

$$T_5 = -(a^n\nabla U^n,\nabla(\partial_t U^n)) = -\left(a^n\nabla U^n,\nabla\left(\frac{U^n-U^{n-1}}{\Delta t}\right)\right) \leqslant -a_0\left(\nabla U^n,\nabla\left(\frac{U^n-U^{n-1}}{\Delta t}\right)\right) \leqslant -\frac{a_0}{2\Delta t}\left(\|\nabla U^n\|^2-\|\nabla U^{n-1}\|^2\right),$$

$$T_6 = -(\sigma^n U^n,\partial_t U^n+\delta\beta^n\cdot\nabla(\partial_t U^n)) \leqslant 5\sigma_1^2\|U^n\|^2+\frac{1}{16}\|\partial_t U^n\|^2+\frac{\delta k^2}{4}\|\partial_t U^n\|_1^2 \leqslant 5\sigma_1^2\|U^n\|^2+\frac{3}{32}\|\partial_t U^n\|^2.$$

Combining (T_1–T_6), we get

$$\frac{1}{2}\|\partial_t U^n\|^2+\frac{a_0}{2\Delta t}\left(\|\nabla U^n\|^2-\|\nabla U^{n-1}\|^2\right) \leqslant C\left(\|f^n\|^2+\|\nabla U^n\|^2+\|U^n\|^2+\|U^{n-1}\|^2\right). \tag{3.10}$$

Multiplying (3.10) by $2\Delta t$ and summing up for $n = 1, 2, \ldots, m$ ($m \leqslant N$) to have

$$\sum_{n=1}^{m}\|\partial_t U^n\|^2\Delta t + a_0\|\nabla U^m\|^2 \leqslant C\Delta t\sum_{n=1}^{m}\left(\|f^n\|^2+\|\nabla U^n\|^2+\|U^n\|^2\right)+\|\nabla U^0\|^2. \qquad \square \tag{3.11}$$

It follows from (3.7), (3.11) and Lemma 3.4, we get

$$\|U^m\|^2+\sum_{n=1}^{m}\|U^n-U^{n-1}\|^2+a_0\sum_{n=1}^{N}\|\nabla U^n\|^2\Delta t \leqslant C\left\{\sum_{n=1}^{m}(\|f^n\|^2+\|U^n\|^2)\Delta t+\|\nabla U^0\|^2+\|U^0\|^2\right\}. \tag{3.12}$$

By the Gronwall's inequality, we have

$$\max_{1\leqslant n\leqslant N}\|U^n\|^2+\sum_{n=1}^{N}\|U^n-U^{n-1}\|^2+a_0\sum_{n=1}^{N}\|\nabla U^n\|^2\Delta t \leqslant C\left(\sum_{n=1}^{N}\|f^n\|^2\Delta t+\|\nabla U^0\|^2+\|U^0\|^2\right). \tag{3.13}$$

4. Error estimation for the C-FDSD method

Let $\omega(t)$ be an arbitrary differentiable map $[0,T] \to V_h : \omega(t) = \Pi_h u(t)$. Set

$$\xi^n = U^n - \omega^n, \quad \eta^n = u^n - \omega^n, \quad e^n = u^n - U^n = \eta^n - \xi^n,$$

where U^n is the solution of problem (2.9), u^n is the weak solution of problem (2.1). From Lemma 3.2, we have

$$\|\eta^n\| + h\|\nabla \eta^n\| \leqslant ch^{r+1}\|u^n\|_{r+1}. \tag{4.1}$$

For any $v \in V_h$, from (2.6) and (2.9) we can get

$$\begin{aligned}\left(\frac{\xi^n - \xi^{n-1}}{\Delta t}, v\right) + (a^n\nabla\xi^n, \nabla v) = {} & \left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, v\right) + \left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, \delta v^n_\beta\right) + \left(\frac{\eta^{n-1} - \bar\eta^{n-1}}{\Delta t}, v\right) + \left(\frac{\eta^{n-1} - \bar\eta^{n-1}}{\Delta t}, \delta v^n_\beta\right) \\ & - \left(\frac{\xi^{n-1} - \bar\xi^{n-1}}{\Delta t}, v\right) - \left(\frac{\xi^{n-1} - \bar\xi^{n-1}}{\Delta t}, \delta v^n_\beta\right) - \left(\frac{\xi^n - \xi^{n-1}}{\Delta t}, \delta v^n_\beta\right) + (a^n\nabla\eta^n, \nabla v) \\ & + (\nabla\cdot(a^n\nabla\xi^n), \delta v^n_\beta) - (\nabla\cdot(a^n\nabla\eta^n), \delta v^n_\beta) + (\sigma^n\eta^n, v + \delta v^n_\beta) - (\sigma^n\xi^n, v + \delta v^n_\beta) \\ & + (E^n_1, v + \delta v^n_\beta) \triangleq \sum_{i=1}^{13} S_i,\end{aligned} \tag{4.2}$$

where $E^n_1 = \frac{u^n - \bar u^{n-1}}{\Delta t} - \left(\psi \frac{\partial u}{\partial \tau}\right)^n$.

Theorem 4.1. *Let $\{u^n\}$ and $\{U^n\}$ be the solution of problems (2.6) and (2.9) respectively. δ satisfies the assumptions (H_1)–(H_3) and Δt is sufficient small. Then the following error estimation holds*

$$\max_{1\leqslant n\leqslant N}\|e^n\|^2 + \sum_{n=1}^{N}\|e^n - e^{n-1}\|^2 + a_0\sum_{n=1}^{N}\|\nabla e^n\|^2\Delta t \leqslant C(h^{2r} + (\Delta t)^2). \tag{4.3}$$

Proof. Taking $v = \xi^n$ in (4.2), then we estimate the two sides of it.

At first, we estimate the left hand-side of (4.2). It is easy to see that

$$\left(\frac{\xi^n - \xi^{n-1}}{\Delta t}, \xi^n\right) + (a^n\nabla\xi^n, \nabla\xi^n) \geqslant \frac{1}{2\Delta t}\left(\|\xi^n\|^2 - \|\xi^{n-1}\|^2 + \|\xi^n - \xi^{n-1}\|^2\right) + a_0\|\nabla\xi^n\|^2. \tag{4.4}$$

Now we estimate the terms of the right-hand side in (4.2) respectively. It follows from the Young inequality, Lemmas 3.1, 3.3 and assumptions (H_1)–(H_3)

$$S_1 = \left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, \xi^n\right) \leqslant \frac{C}{\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t_{n-1},t_n;L^2)} + \frac{1}{16}\|\xi^n\|^2,$$

$$S_2 = \left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, \delta\beta^n\cdot\nabla\xi^n\right) \leqslant \frac{C}{\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t_{n-1},t_n;L^2)} + \frac{\delta k^2}{4}\|\nabla\xi^n\|^2 \leqslant \frac{C}{\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t_{n-1},t_n;L^2)} + \frac{a_0}{16}\|\nabla\xi^n\|^2,$$

$$\begin{aligned}S_3 = \left(\frac{\eta^{n-1} - \bar\eta^{n-1}}{\Delta t}, \xi^n\right) &\leqslant \|\xi^n\|_1\cdot\left\|\frac{\eta^{n-1} - \bar\eta^{n-1}}{\Delta t}\right\|_{-1} \leqslant \|\xi^n\|_1\cdot k_0\|\eta^{n-1}\| \leqslant \frac{\delta k^2}{4}\|\nabla\xi^n\|^2 + \frac{k_0^2}{\delta k^2}\|\eta^{n-1}\|^2 \\ &\leqslant \frac{k_0^2}{\delta k^2}\|\eta^{n-1}\|^2 + \frac{a_0}{16}\|\nabla\xi^n\|^2,\end{aligned}$$

$$\begin{aligned}S_4 = \left(\frac{\eta^{n-1} - \bar\eta^{n-1}}{\Delta t}, \delta\beta^n\cdot\nabla\xi^n\right) &\leqslant \|\delta\beta^n\cdot\nabla\xi^n\|_1\cdot\left\|\frac{\eta^{n-1} - \bar\eta^{n-1}}{\Delta t}\right\|_{-1} \leqslant \|\delta\beta^n\cdot\nabla\xi^n\|_1\cdot k_0\|\eta^{n-1}\| \leqslant \frac{k^2}{4}\|\delta\beta^n\nabla\xi^n\|_1^2 + \frac{k_0^2}{k^2}\|\eta^{n-1}\|^2 \\ &\leqslant \frac{k_0^2}{k^2}\|\eta^{n-1}\|^2 + \frac{a_0}{64}\|\nabla\xi^n\|^2,\end{aligned}$$

$$\begin{aligned}S_5 = -\left(\frac{\xi^{n-1} - \bar\xi^{n-1}}{\Delta t}, \xi^n\right) &\leqslant \|\xi^n\|_1\cdot\left\|\frac{\xi^{n-1} - \bar\xi^{n-1}}{\Delta t}\right\|_{-1} \leqslant \|\xi^n\|_1\cdot k_0\|\xi^{n-1}\| \leqslant \frac{\delta k^2}{4}\|\nabla\xi^n\|^2 + \frac{k_0^2}{\delta k^2}\|\xi^{n-1}\|^2 \\ &\leqslant \frac{k_0^2}{\delta k^2}\|\xi^{n-1}\|^2 + \frac{a_0}{16}\|\nabla\xi^n\|^2,\end{aligned}$$

$$S_6 = -\left(\frac{\xi^{n-1}-\bar{\xi}^{n-1}}{\Delta t}, \delta\beta^n\cdot\nabla\xi^n\right) \leqslant \|\delta\beta^n\cdot\nabla\xi^n\|_1\cdot\left\|\frac{\xi^{n-1}-\bar{\xi}^{n-1}}{\Delta t}\right\|_{-1} \leqslant \|\delta\beta^n\cdot\nabla\xi^n\|_1\cdot k_0\|\xi^{n-1}\| \leqslant \frac{k^2}{4}\|\delta\beta^n\nabla\xi^n\|_1^2 + \frac{k_0^2}{k^2}\|\xi^{n-1}\|^2$$
$$\leqslant \frac{k_0^2}{k^2}\|\xi^{n-1}\|^2 + \frac{a_0}{64}\|\nabla\xi^n\|^2.$$

Using the definition of $\partial_t\xi^n$ and assumption (H_2), we have

$$S_7 = -\left(\frac{\xi^{n-1}-\xi^{n-1}}{\Delta t}, \delta\beta^n\cdot\nabla\xi^n\right) \leqslant \|\partial_t\xi^n\|^2 + \frac{\delta^2k^2}{4}\|\nabla\xi^n\|^2 \leqslant \|\partial_t\xi^n\|^2 + \frac{a_0}{16}\|\nabla\xi^n\|^2.$$

By the Young inequality, Lemma 3.1 and assumptions (H_1)–(H_3), it follows that

$$S_8 = (a^n\nabla\eta^n, \nabla\xi^n) \leqslant Ca_0\|\nabla\eta^n\|^2 + \frac{a_0}{64}\|\nabla\xi^n\|^2,$$
$$S_9 = (\nabla\cdot(a^n\nabla\xi^n), \delta\xi^n_\beta) = (\nabla a^n\cdot\nabla\xi^n, \delta\beta^n\cdot\nabla\xi^n) + (a^n\nabla\cdot(\nabla\xi^n), \delta\beta^n\cdot\nabla\xi^n) \leqslant (\delta q_0a_0k\mu h^{-1} + \delta k\bar{a})\|\nabla\xi^n\|^2 \leqslant \frac{3a_0}{16}\|\nabla\xi^n\|^2,$$
$$S_{10} = -(\nabla\cdot(a^n\nabla\eta^n), \delta\xi^n_\beta) \leqslant \frac{9a_0}{16}\|\nabla\eta^n\|^2 + \frac{a_0}{64}\|\nabla\xi^n\|^2,$$
$$S_{11} = (\sigma^n\eta^n, \xi^n+\delta\xi^n_\beta) \leqslant 4\sigma_1^2\|\eta^n\|^2 + \frac{1}{32}\|\xi^n\|^2 + \frac{\delta^2k^2}{8}\|\nabla\xi^n\|^2 + C\sigma_1^2\|\eta^n\|^2 \leqslant C\|\eta^n\|^2 + \frac{1}{16}\|\xi^n\|^2,$$
$$S_{12} = -(\sigma^n\xi^n, \xi^n+\delta\xi^n_\beta) \leqslant C\|\xi^n\|^2 + \frac{1}{16}\|\xi^n\|^2,$$
$$S_{13} = (E_1^n, \xi^n+\delta\xi^n_\beta) \leqslant \frac{1}{32}\|\xi^n\|^2 + \frac{\delta^2k^2}{8}\|\nabla\xi^n\|^2 + C\Delta t \leqslant \frac{1}{16}\|\xi^n\|^2 + C\Delta t.$$

Combining (S_1–S_{13}) and (4.4), we have

$$\frac{1}{2\Delta t}\left(\|\xi^n\|^2 - \|\xi^{n-1}\|^2\right) + \frac{1}{2\Delta t}\|\xi^n-\xi^{n-1}\|^2 + \frac{a_0}{2}\|\nabla\xi^n\|^2$$
$$\leqslant C\left(\frac{1}{\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t_{n-1},t_n;L^2)} + \|\eta^n\|^2 + \|\nabla\eta^n\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + \|\eta^{n-1}\|^2\right) + \|\partial_t\xi^n\|^2. \quad (4.5)$$

Multiplying (4.5) by $2\Delta t$, and summing up for $n = 1, 2, \ldots, m$ ($m \leqslant N$), we obtain

$$\|\xi^m\|^2 + \sum_{n=1}^m\|\xi^n-\xi^{n-1}\|^2 + a_0\sum_{n=1}^m\|\nabla\xi^n\|^2\Delta t$$
$$\leqslant C\left(\sum_{n=1}^m\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t_{n-1},t_n;L^2)} + \sum_{n=1}^m\|\nabla\eta^n\|^2\Delta t + \sum_{n=1}^m\|\eta^n\|^2\Delta t + \sum_{n=1}^m\|\xi^n\|^2\Delta t\right) + 2\sum_{n=1}^m\|\partial_t\xi^n\|^2\Delta t + \|\xi^0\|^2. \quad (4.6)$$

Lemma 4.1. *Let δ satisfy the assumptions (H_1)–(H_3), then*

$$\sum_{n=1}^m\|\partial_t\xi^n\|^2\Delta t \leqslant C\left(\sum_{n=1}^m\|\nabla\xi^n\|^2\Delta t + \sum_{n=1}^m\|\xi^n\|^2\Delta t + (\Delta t)^2 + h^{2r}\Delta t + h^{2r+2}\right). \quad (4.7)$$

Proof. Taking $v = \partial_t\xi^n$ in (4.2) then

$$\left(\frac{\xi^n-\xi^{n-1}}{\Delta t}, \partial_t\xi^n\right) = \left(\frac{\eta^n-\eta^{n-1}}{\Delta t}, \partial_t\xi^n\right) + \left(\frac{\eta^n-\eta^{n-1}}{\Delta t}, \delta\beta^n\cdot\nabla(\partial_t\xi^n)\right) + \left(\frac{\eta^{n-1}-\bar{\eta}^{n-1}}{\Delta t}, \partial_t\xi^n\right)$$
$$+ \left(\frac{\eta^{n-1}-\bar{\eta}^{n-1}}{\Delta t}, \delta\beta^n\cdot\nabla(\partial_t\xi^n)\right) - \left(\frac{\xi^{n-1}-\bar{\xi}^{n-1}}{\Delta t}, \partial_t\xi^n\right) - \left(\frac{\xi^{n-1}-\bar{\xi}^{n-1}}{\Delta t}, \delta\beta^n\cdot\nabla(\partial_t\xi^n)\right)$$
$$- \left(\frac{\xi^n-\xi^{n-1}}{\Delta t}, \delta\beta^n\cdot\nabla(\partial_t\xi^n)\right) - (a^n\nabla\xi^n, \nabla(\partial_t\xi^n)) + (a^n\nabla\eta^n, \nabla(\partial_t\xi^n))$$
$$+ (\nabla\cdot(a^n\nabla\xi^n), \delta\beta^n\cdot\nabla(\partial_t\xi^n)) - (\nabla\cdot(a^n\nabla\eta^n), \delta\beta^n\cdot\nabla(\partial_t\xi^n)) + (\sigma^n\eta^n, \partial_t\xi^n + \delta\beta^n\cdot\nabla(\partial_t\xi^n))$$
$$- (\sigma^n\xi^n, \partial_t\xi^n + \delta\beta^n\cdot\nabla(\partial_t\xi^n)) + \left(E_1^n, \partial_t\xi^n + \delta\beta^n\cdot\nabla(\partial_t\xi^n)\right) \triangleq \sum_{i=1}^{14}T_i. \quad (4.8)$$

Obviously the left-hand side of (4.8) is $(\partial_t\xi^n, \partial_t\xi^n) = \|\partial_t\xi^n\|^2$.

Now we estimate the terms of the right-hand side in (4.8) respectively. It follows from the Young inequality, Lemmas 3.1, 3.3 and assumptions (H_1)–(H_3)

$$T_1 = \left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, \partial_t \xi^n\right) \leqslant \frac{C}{\Delta t}\left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^2(t_{n-1},t_n;L^2)} + \frac{1}{64}\|\partial_t \xi^n\|^2,$$

$$T_2 = \left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, \delta\beta^n \cdot \nabla(\partial_t \xi^n)\right) \leqslant \frac{C}{\Delta t}\left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^2(t_{n-1},t_n;L^2)} + \frac{1}{64}\|\partial_t \xi^n\|^2,$$

$$T_3 = \left(\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, \partial_t \xi^n\right) \leqslant \|\partial_t \xi^n\|_1 \left\|\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}\right\|_{-1} \leqslant \|\partial_t \xi^n\|_1 \cdot k_0\|\eta^{n-1}\| \leqslant \frac{4k_0^2}{\delta k^2}\|\eta^{n-1}\|^2 + \frac{\delta k^2}{16}\|\nabla(\partial_t \xi^n)\|^2$$
$$\leqslant \frac{4k_0^2}{\delta k^2}\|\eta^{n-1}\|^2 + \frac{1}{64}\|\partial_t \xi^n\|^2,$$

$$T_4 = \left(\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, \delta\beta^n \cdot \nabla(\partial_t \xi^n)\right) \leqslant \|\delta\beta^n \cdot \nabla(\partial_t \xi^n)\|_1 \left\|\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}\right\|_{-1} \leqslant \|\delta\beta^n \cdot \nabla(\partial_t \xi^n)\|_1 \cdot k_0\|\eta^{n-1}\|$$
$$\leqslant \frac{k_0^2}{k^2}\|\eta^{n-1}\|^2 + \frac{k^2}{4}\|\delta\beta^n \cdot \nabla(\partial_t \xi^n)\|_1^2 \leqslant \frac{k_0^2}{k^2}\|\eta^{n-1}\|^2 + \frac{\delta^2 k^4}{4}\|\nabla\partial_t \xi^n\|_1^2 \leqslant \frac{k_0^2}{k^2}\|\eta^{n-1}\|^2 + \frac{1}{64}\|\partial_t \xi^n\|^2,$$

$$T_5 = -\left(\frac{\xi^{n-1} - \bar{\xi}^{n-1}}{\Delta t}, \partial_t \xi^n\right) \leqslant \|\partial_t \xi^n\|_1 \left\|\frac{\xi^{n-1} - \bar{\xi}^{n-1}}{\Delta t}\right\|_{-1} \leqslant \|\partial_t \xi^n\|_1 \cdot k_0\|\xi^{n-1}\| \leqslant \frac{4k_0^2}{\delta k^2}\|\xi^{n-1}\|^2 + \frac{\delta k^2}{16}\|\partial_t \xi^n\|_1^2$$
$$\leqslant \frac{4k_0^2}{\delta k^2}\|\eta^{n-1}\|^2 + \frac{1}{64}\|\partial_t \xi^n\|^2,$$

$$T_6 = -\left(\frac{\xi^{n-1} - \bar{\xi}^{n-1}}{\Delta t}, \delta\beta^n \cdot \nabla(\partial_t \xi^n)\right) \leqslant \|\delta\beta^n \cdot \nabla(\partial_t \xi^n)\|_1 \left\|\frac{\xi^{n-1} - \bar{\xi}^{n-1}}{\Delta t}\right\|_{-1} \leqslant \|\delta\beta^n \cdot \nabla(\partial_t \xi^n)\|_1 \cdot k_0\|\xi^{n-1}\|$$
$$\leqslant \frac{k_0^2}{k^2}\|\xi^{n-1}\|^2 + \frac{k^2}{4}\|\delta\beta^n \cdot \nabla(\partial_t \xi^n)\|_1^2 \leqslant \frac{k_0^2}{k^2}\|\xi^{n-1}\|^2 + \frac{\delta^2 k^4}{4}\|\nabla\partial_t \xi^n\|_1^2 \leqslant \frac{k_0^2}{k^2}\|\xi^{n-1}\|^2 + \frac{1}{64}\|\partial_t \xi^n\|^2.$$

By the definition of $\partial_t \xi^n$, the Young inequality, Lemma 3.1 and assumption (H_3), we get

$$T_7 = -\left(\frac{\xi^n - \xi^{n-1}}{\Delta t}, \delta\beta^n \cdot \nabla(\partial_t \xi^n)\right) \leqslant \frac{1}{4}\|\partial_t \xi^n\|^2 + \delta^2 k^2\|\nabla(\partial_t \xi^n)\|^2 \leqslant \frac{1}{16}\|\partial_t \xi^n\|^2 + \frac{1}{4}\|\partial_t \xi^n\|^2,$$
$$T_8 = -(a^n \nabla\xi^n, \nabla(\partial_t \xi^n)) \leqslant -a_0\left(\nabla\xi^n, \nabla\left(\frac{\xi^n - \xi^{n-1}}{\Delta t}\right)\right) \leqslant -\frac{a_0}{2\Delta t}\left(\|\nabla\xi^n\|^2 - \|\nabla\xi^{n-1}\|^2\right),$$
$$T_9 = (a^n \nabla\eta^n, \nabla(\partial_t \xi^n)) \leqslant a_1\|\eta^n\|\|\nabla(\partial_t \xi^n)\| \leqslant \frac{4a_1^2}{\delta k^2}\|\nabla\eta^n\|^2 + \frac{\delta k^2}{16}\|\nabla(\partial_t \xi^n)\|^2 \leqslant \frac{a_1^2}{4\delta k^2}\|\nabla\eta^n\|^2 + \frac{1}{64}\|\partial_t \xi^n\|^2,$$
$$T_{10} = (\nabla \cdot (a^n \nabla\xi^n), \delta\beta^n \cdot \nabla(\partial_t \xi^n)) = (\nabla a^n \cdot \nabla\xi^n, \delta\beta^n \cdot \nabla(\partial_t \xi^n)) + (a^n \nabla \cdot (\nabla\xi^n), \delta\beta^n \cdot \nabla(\partial_t \xi^n))$$
$$\leqslant (\delta q_0 a_0 k\mu^2 h^{-2} + \delta k\bar{a}\mu h^{-1})\|\nabla\xi^n\|\|\partial_t \xi^n\| \leqslant \|\nabla\xi^n\|^2 + \frac{1}{64}\|\partial_t \xi^n\|^2,$$

$$T_{11} = -(\nabla \cdot (a^n \nabla\eta^n), \delta\beta^n \cdot \nabla(\partial_t \xi^n)) \leqslant \|\nabla\eta^n\|^2 + \frac{1}{64}\|\partial_t \xi^n\|^2.$$

We can easily derive that

$$T_{12} = (\sigma^n \eta^n, \partial_t \xi^n + \delta\beta^n \cdot \nabla(\partial_t \xi^n)) \leqslant 40\sigma_1^2\|\eta^n\|^2 + \frac{1}{64}\|\partial_t \xi^n\|^2,$$
$$T_{13} = -(\sigma^n \xi^n, \partial_t \xi^n + \delta\beta^n \cdot \nabla(\partial_t \xi^n)) \leqslant 40\sigma_1^2\|\xi^n\|^2 + \frac{1}{64}\|\partial_t \xi^n\|^2,$$
$$T_{14} = \left(E_1^n, \partial_t \xi^n + \delta\beta^n \cdot \nabla(\partial_t \xi^n)\right) \leqslant C\Delta t + \frac{1}{64}\|\partial_t \xi^n\|^2.$$

Combining (T_1–T_{14}), it follows that

$$\frac{1}{2}\|\partial_t\xi^n\|^2+\frac{a_0}{2\Delta t}\left(\|\nabla\xi^n\|^2-\|\nabla\xi^{n-1}\|^2\right)\leqslant C\left(\frac{1}{\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t_{n-1},t_n;L^2)}+\|\nabla\eta^n\|^2+\|\nabla\xi^n\|^2+\|\xi^{n-1}\|^2+\|\eta^{n-1}\|^2+\Delta t\right). \tag{4.9}$$

Multiplying (4.9) by $2\Delta t$, and summing up for $n = 1, 2, \ldots, m$ ($m \leqslant N$), we have

$$\begin{aligned}\sum_{n=1}^{m}\|\partial_t\xi^n\|^2\Delta t+a_0\|\nabla\xi^m\|^2\\ \leqslant C\left(\sum_{n=1}^{m}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t_{n-1},t_n;L^2)}+\sum_{n=1}^{m}\|\nabla\eta^n\|^2\Delta t+\sum_{n=1}^{m}\|\eta^n\|^2\Delta t+\sum_{n=1}^{m}\|\nabla\xi^n\|^2\Delta t+\sum_{n=1}^{m}\|\xi^n\|^2\Delta t+(\Delta t)^2+\|\nabla\xi^0\|^2\right)\\ \leqslant C\left(\sum_{n=1}^{m}\|\nabla\xi^n\|^2\Delta t+\sum_{n=1}^{m}\|\xi^n\|^2\Delta t+(\Delta t)^2+h^{2r}\Delta t+h^{2r+2}\right).\qquad\square\end{aligned} \tag{4.10}$$

It follows from (4.6), (4.10) and Lemma 4.1, we get

$$\begin{aligned}\|\xi^n\|^2+\sum_{n=1}^{m}\|\xi^n-\xi^{n-1}\|^2+a_0\sum_{n=1}^{m}\|\nabla\xi^n\|^2\Delta t\\ \leqslant C\left\{\sum_{n=1}^{m}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t_{n-1},t_n;L^2)}+\|\nabla\xi^0\|^2+\sum_{n=1}^{m}\left(\|\nabla\eta^n\|^2+\|\eta^n\|^2+\|\nabla\xi^n\|^2+\|\xi^n\|^2\right)\Delta t+\|\xi^0\|^2+(\Delta t)^2+h^{2r}\Delta t+h^{2r+2}\right\}.\end{aligned} \tag{4.11}$$

By the Gronwall's inequality, we have

$$\max_{1\leqslant n\leqslant N}\|\xi^n\|^2+\sum_{n=1}^{N}\|\xi^n-\xi^{n-1}\|^2+a_0\sum_{n=1}^{N}\|\nabla\xi^n\|^2\Delta t\leqslant C\left(h^{2r}+(\Delta t)^2\right). \tag{4.12}$$

Finally, applying the triangle inequality we obtain

$$\max_{1\leqslant n\leqslant N}\|e^n\|^2+\sum_{n=1}^{N}\|e^n-e^{n-1}\|^2+a_0\sum_{n=1}^{N}\|\nabla e^n\|^2\Delta t\leqslant C\left(h^{2r}+(\Delta t)^2\right). \tag{4.13}$$

5. Numerical experiments

In this section, we present some numerical experiments to illustrate the effectiveness of the C-FDSD method. We consider the following convection-dominated diffusion problem:

$$\frac{\partial u}{\partial t}-\varepsilon\Delta u+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+u=f,\quad (x,y,t)\in\Omega\times(0,T], \tag{5.1}$$

$$u(x,y,t)=0,\quad (x,y,t)\in\partial\Omega\times(0,T], \tag{5.2}$$

$$u(x,y,0)=0,\quad (x,y)\in\Omega. \tag{5.3}$$

Table 1
Numerical results obtained with ε = 1.0e−3 at T = 5.

Method	Grid size	Values of Δt and δ	L^2-norm error
Galerkin	$h=\frac{1}{2}$		2.09991e−4
	$h=\frac{1}{4}$	Δt = 0.005	1.37370e−4
	$h=\frac{1}{8}$		4.61192e−5
C-Galerkin	$h=\frac{1}{2}$		1.53427e−4
	$h=\frac{1}{4}$	Δt = 0.005	1.34192e−4
	$h=\frac{1}{8}$		6.08919e−5
FDSD	$h=\frac{1}{2}$	Δt = 0.005	2.10282e−4
	$h=\frac{1}{4}$	δ = 0.000125	1.32975e−4
	$h=\frac{1}{8}$		4.00132e−5
C-FDSD	$h=\frac{1}{2}$	Δt = 0.005	1.33350e− 4
	$h=\frac{1}{4}$	δ = 0.000125	1.05037e−4
	$h=\frac{1}{8}$		1.26409e−5

Table 2
Numerical results obtained with ε = 1.0e−5 at T = 5.

Method	Grid size	Values of Δt and δ	L^2-norm error
Galerkin	$h = \frac{1}{2}$		2.24474e−4
	$h = \frac{1}{4}$	Δt = 0.05	1.77912e−4
	$h = \frac{1}{8}$		1.61974e−4
FDSD	$h = \frac{1}{2}$	Δt = 0.05	2.24466e−4
	$h = \frac{1}{4}$	δ = 0.000001	1.77862e−4
	$h = \frac{1}{8}$		1.61172e−4
C-FDSD	$h = \frac{1}{2}$	Δt = 0.05	1.48780e−4
	$h = \frac{1}{4}$	δ = 0.000001	6.06253e−5
	$h = \frac{1}{8}$		3.89388e−5

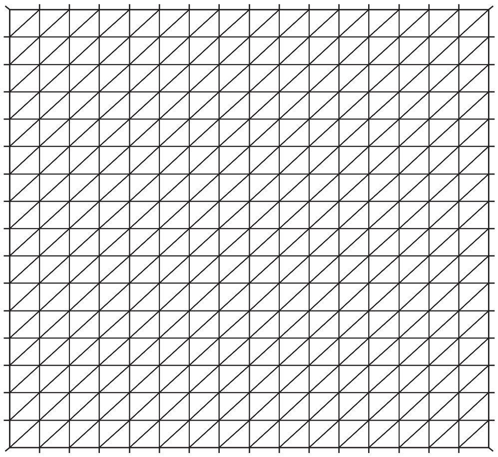

Fig. 1. Mesh with uniform grid size $h = \frac{1}{16}$.

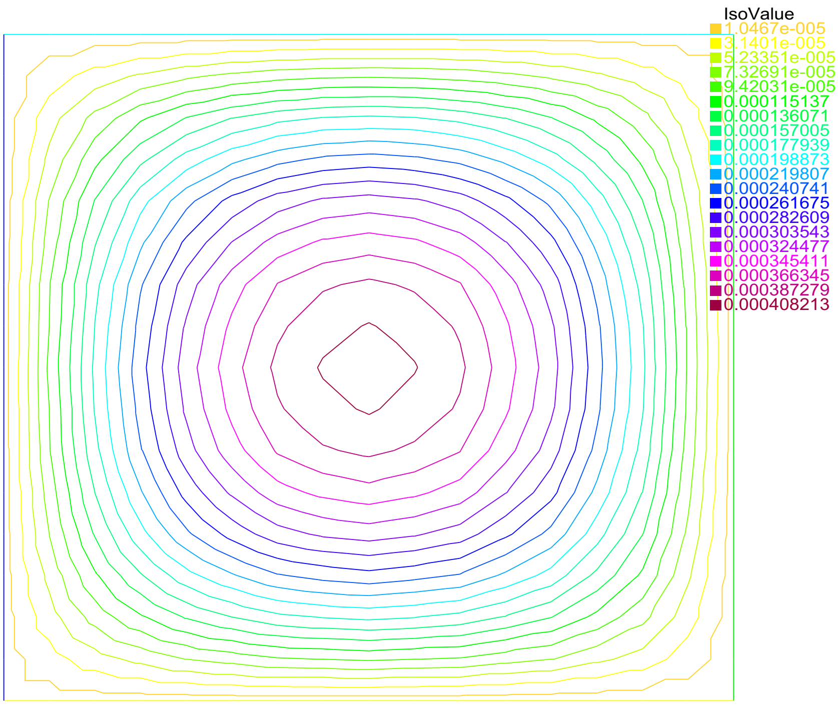

Fig. 2. The isovalue of U when ε = 1.0e − 3, δ = 0.000125, Δt = 0.005.

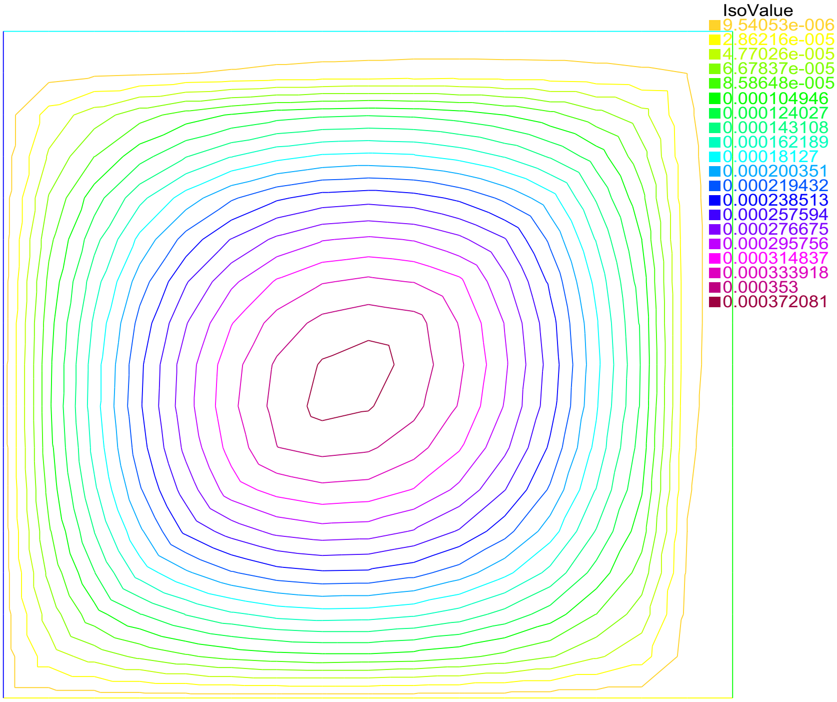

Fig. 3. The isovalue of U when $\varepsilon = 1.0\text{e}{-5}$, $\delta = 0.000001$, $\Delta t = 0.05$.

The spatial domain is $\Omega = [0,1] \times [0,1]$. The time interval is $(0,T] = (0,5]$. The function f is determined by the exact solution $u(x,y,t) = e^{-t}xy(1-x)(1-y)$.

Now we use P_1 conforming finite element, and compare C-FDSD method with FDSD method, C-Galerkin FEM, the standard Galerkin FEM which are based on the regular meshes. In the numerical experiments, the data are chosen as follows.

The domain is partitioned into triangles with the size $h = \frac{1}{N}$ for $N = 2, 4, 8$, respectively. The time step size is $\Delta t = 0.005$ with $T = 5$. The choice of artificial parameter δ is 0.000125. We consider the case $\varepsilon = 1.0\text{e}{-3}$, the absolute error of velocity in L^2-norm and convergence rate are shown in Table 1. From Table 1 we can see the above four kinds of methods are effective but the C-FDSD method is more robust.

If we choose the smaller $\varepsilon = 1.0\text{e}{-5}$, the standard Galerkin FEM and FDSD method are not effective, but the C-FDSD method can be expected work well. In order to confirm the theoretical results, we can choose the larger time step $\Delta t = 0.05$. The numerical results are shown in Table 2.

From Table 2, we see that the C-FDSD method is stable, and the L^2-norm error obtained by the new scheme is smaller than the FDSD method and the standard Galerkin FEM. Furthermore, the C-FDSD method not only realizes the purpose of lowering the error of time, using larger time step for solving the convection-dominated diffusion problems, but also avoids the numerical oscillation and keeps favorable stability and higher precision.

Finally, we present the Figs. 2 and 3 for the isovalue of U, where U is the numerical solution of u, when $\varepsilon = 1.0\text{e}{-3}$, $1.0\text{e}{-5}$ and $\delta = 0.000125, 0.000001$, respectively. The uniform spatial grid size is $h = 1/16$ in Fig. 1, and compute the results at $T = 5$. From Figs. 2 and 3, we can see that when we choose the proper values of parameter δ, we can adopt the larger time step to computation for save operation and complexity. It is noticed that the choice of parameter values δ depended on the assumptions $(H1)$–$(H3)$.

6. Conclusions

In this paper, we provide the characteristic finite difference streamline diffusion method for two-dimensional convection-dominated diffusion problems and deduced error estimates for its full-discrete scheme under some assumptions. It allows to simplifies the computational work and keeps the good stability and high accuracy by choosing the proper parameter values. Obviously, we can easily extend the present analysis to the three-dimensional convection-dominated diffusion problems.

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