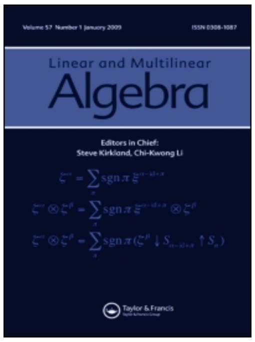

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# Toeplitz matrices are unitarily similar to symmetric matrices

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**ABSTRACT**

We prove that Toeplitz matrices are unitarily similar to complex symmetric matrices. Moreover, two $n \times n$ unitary matrices that uniformly turn all $n \times n$ Toeplitz matrices via similarity to complex symmetric matrices are explicitly given, respectively. When $n \leq 3$, we prove that each complex symmetric matrix is unitarily similar to some Toeplitz matrix, but the statement is false when $n > 3$.

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## 1. Introduction

Denote by $\mathbb{C}_{n\times n}$ the set of all $n \times n$ complex matrices and let $\mathbb{U}(n)$ be the group of $n \times n$ unitary matrices. It is well-known that every $A \in \mathbb{C}_{n\times n}$ is similar to a complex symmetric matrix (cf. [1, Theorem 4.4.24]). Every normal matrix is unitarily similar to a diagonal matrix which is clearly symmetric. One may ask whether every matrix is unitarily similar to a symmetric matrix. This is true when $n = 2$ [2]. However, it is not true [3, Example 7] when $n \geq 3$. See [4,5] for related works.

Toeplitz matrices arise in solutions to differential and integral equations, spline functions, and problems and methods in physics, mathematics, statistics, and signal processing. They are one of the most well-studied matrices. Toeplitz matrices are also generalized as Toeplitz operators acting on the vector Hardy space [6]. Recently, Chien and Nakazato [7] proved that every Toeplitz matrix is unitarily similar to a complex symmetric matrix.

In this paper, we will use two methods to constructively prove that each Toeplitz matrix is unitarily similar to a complex symmetric matrix. Moreover, there are two unitary matrices that uniformly turn all $n \times n$ Toeplitz matrices into symmetric matrices via similarity and they will be given explicitly. Thus, this is an improvement on the result from [7]. We also study the problem of whether every symmetric matrix is unitarily similar to a Toeplitz matrix. When $n \leq 3$, it is true (see Theorem 3.6). However, the answer is negative in general. Proofs are given for the case $n = 4$ and the case $n = 5$ in Section 5 and Section 6, respectively.

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## 2. Kippenhahn polynomials

We introduce some invariants of the unitary similarity. The $k$-th numerical range of $A \in \mathbb{C}_{n\times n}$ is the set

$$\Lambda_k(A) := \{z \in \mathbb{C} : PAP = zP \text{ for some } k - \text{dimensional orthogonal projection } P\}$$

$1 \leq k \leq n$ (cf. [8]). When $k = 1$, $\Lambda_k(A)$ is reduced to the classical numerical range defined as

$$W(A) := \{\xi^* A\xi : \xi \in \mathbb{C}^n, \xi^*\xi = 1\},$$

which has been systematically and intensively studied in the literature (cf. [1,9–11]). The $k$-th numerical range $\Lambda_k(A)$ is completely determined by the following ternary form:

$$F_A(x, y, z) = \det(x\Re(A) + y\Im(A) + zI_n),$$

where $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/(2i)$. Kippenhahn [11] proved this result when $k = 1$. More precisely, $W(A)$ is the convex hull of the real affine part of the dual curve of $F_A(x, y, z) = 0$. In [12], it was proved that the equations $\Lambda_k(A) = \Lambda_k(B)$ $(1 \leq k \leq n)$ for $n \times n$ matrices $A, B$ hold only if those Kippenhahn polynomials satisfy $F_A = F_B$. A matrix $A$ and its transpose $A^T$ have the common ternary form $F_A = F_{A^T}$. Helton and Spitovsky [13] showed that for every $A \in \mathbb{C}_{n\times n}$ there exists a complex symmetric $B \in \mathbb{C}_{n\times n}$ satisfying $F_B(x, y, z) = F_A(x, y, z)$, hence $\Lambda_k(A) = \Lambda_k(B)$. Their result depends on a theorem in [14] which answers affirmatively to the conjectures raised in [9,15], namely, for a hyperbolic ternary form $F(x, y, z)$, there exist real symmetric matrices $H$ and $K$ such that $F(x, y, z) = F_{H+iK}(x, y, z)$. The result of [14] provides us motivation to study the class of matrices which are unitarily similar to symmetric matrices. In [16], a method to construct symmetric matrices $H, K$ starting from a hyperbolic form $F(x, y, z)$ is explicitly given when the curve $F(x, y, z) = 0$ has genus 0 or 1.

## 3. Main results

An $n \times n$ matrix $T = (a_{ij})$ is called a Toeplitz matrix if $a_{ij} = a_{k\ell}$ for every pairs $(i, j)$, $(k, \ell)$ satisfying $i - j = k - \ell$. In this case, $a_{ij}$ is denoted by $a_{i-j}$ for some $a_0, a_{\pm 1}, a_{\pm 2}, \ldots, a_{\pm(n-1)}$. Explicitly,

$$T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{pmatrix}.$$

The $n \times n$ Jordan block $J_n(0)$ corresponding to the zero eigenvalue is a Toeplitz matrix and [1, p.208] $J_n(0)$ is unitarily similar to the symmetric matrix

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & \cdots & 0 & -1 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 1 \\ 0 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 \\ -1 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

via $U = \frac{1}{\sqrt{2}}(I_n + iJ_n) \in \mathbb{U}(n)$, where $J_n$ is the $n \times n$ backward identity.

The property that $J_n(0)$ is unitarily similar to a symmetric matrix is also true for arbitrary Toeplitz matrix $T \in \mathbb{C}_{n\times n}$ and it was proved by Chien and Nakazato [7]. We are going to reprove their result by explicitly giving unitary matrices $U$ that transform all Toeplitz matrices in $\mathbb{C}_{n\times n}$ to symmetric matrices.

A Toeplitz matrix can be viewed as a linear combination of Jordan block $J_n(0)$, the transpose of $J_n(0)$, and their powers. It is not difficult to see that the unitary matrix $U = \frac{1}{\sqrt{2}}(I_n + iJ_n) \in \mathbb{U}(n)$ can also turn all Toeplitz matrices to symmetric matrices.

**Theorem 3.1:** *Every Toeplitz matrix $T$ is unitarily similar to a symmetric matrix $B = (b_{ij})$ via the unitary matrix $U = \frac{1}{\sqrt{2}}(I_n + iJ_n) \in \mathbb{U}(n)$, where $J_n$ is the $n \times n$ backward identity. More specifically,*

$$b_{ij} = \frac{1}{2}(a_{i-j} + a_{j-i}) + \frac{i}{2}(a_{i+j-n-1} - a_{n+1-i-j}).$$

***Proof*:** Since $U^* = \frac{1}{\sqrt{2}}(I_n - iJ_n) \in \mathbb{U}(n)$ and $JTJ = T^T$, we have

$$\begin{aligned} U^*TU &= \frac{1}{2}(I - iJ)T(I + iJ) \\ &= \frac{1}{2}(T + iTJ - iJT + T^T) \\ &= \frac{1}{2}(T + T^T) + \frac{i}{2}(TJ - JT) \end{aligned}$$

Note that $T + T^T$ is symmetric, $TJ$ and $JT$ are Hankel matrices (see [1, 0.9.8]), which are symmetric. Hence, $U^*TU$ is symmetric. □

**Example 3.2:** When $n = 4$, $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$ and

$$U^*TU = \frac{1}{2}\begin{pmatrix} 2a_0 + i(a_{-3} - a_3) & a_1 + a_{-1} + i(a_{-2} - a_2) & a_2 + a_{-2} + i(a_{-1} - a_1) & a_3 + a_{-3} \\ a_1 + a_{-1} + i(a_{-2} - a_2) & 2a_0 + i(a_{-1} - a_1) & a_1 + a_{-1} & a_2 + a_{-2} + i(a_1 - a_{-1}) \\ a_2 + a_{-2} + i(a_{-1} - a_1) & a_1 + a_{-1} & 2a_0 + i(a_1 - a_{-1}) & a_1 + a_{-1} + i(a_2 - a_{-2}) \\ a_3 + a_{-3} & a_2 + a_{-2} + i(a_1 - a_{-1}) & a_1 + a_{-1} + i(a_2 - a_{-2}) & 2a_0 + i(a_3 - a_{-3}) \end{pmatrix}$$

We remark that in Theorem 3.1 the unitary $U$ that uniformly turns all Toeplitz matrices to symmetric matrices via similarity is not unique.

**Theorem 3.3:** *Every Toeplitz matrix $T \in \mathbb{C}_{n\times n}$ is unitarily similar to a symmetric matrix. Moreover, the following $U \in \mathbb{U}(n)$ uniformly turns all Toeplitz matrices in $\mathbb{C}_{n\times n}$ into symmetric matrices via similarity:*

(1) *When $n = 2m$, with $m \geq 1$,*

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & & & & i \\ & \ddots & & & \iddots & \\ & & 1 & i & & \\ & & 1 & -i & & \\ & \iddots & & & \ddots & \\ 1 & & & & & -i \end{pmatrix}.$$

(2) *When $n = 2m + 1$, with $m \geq 1$,*

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & & & & & i \\ & \ddots & & & & \iddots & \\ & & 1 & 0 & i & & \\ & & 0 & \sqrt{2} & 0 & & \\ & & 1 & 0 & -i & & \\ & \iddots & & & & \ddots & \\ 1 & & & & & & -i \end{pmatrix}.$$

***Proof*:** Clearly $U$ is unitary and we write $U = (u_1 \cdots u_n)$ in column form and let $B := U^*TU = (b_{st})$. Our goal is to show that $B$ is symmetric. Note that $b_{st} = u_s^* T u_t$.

(1) When $n = 2m$,

$$u_k = \begin{cases} (e_k + e_{2m-k+1})/\sqrt{2} & k \leq m \\ (e_{2m-k+1} - e_k)i/\sqrt{2} & k > m. \end{cases}$$

By straightforward computation, we have

$$b_{st} = b_{ts} = \begin{cases} \frac{1}{2}(a_{t-s} + a_{s-t} + a_{s+t-2m-1} + a_{2m+1-s-t}), & s \leq m, t \leq m \\ \frac{1}{2}i(a_{t-s} - a_{s-t} + a_{s+t-2m-1} - a_{2m+1-s-t}), & s \leq m, t > m \\ \frac{1}{2}(a_{t-s} + a_{s-t} - a_{s+t-2m-1} - a_{2m+1-s-t}), & s > m, t > m. \end{cases}$$

(2) When $n = 2m + 1$,

$$u_k = \begin{cases} (e_k + e_{2m-k+2})/\sqrt{2} & k \le m \\ e_{m+1} & k = m+1 \\ (e_{2m-k+2} - e_k)i/\sqrt{2} & k > m+1. \end{cases}$$

Straightforward computation yields

$$b_{st} = b_{ts} = \begin{cases} \frac{1}{2}(a_{t-s} + a_{s-t} + a_{s+t-2m-2} + a_{2m+2-s-t}) & s \le m, t \le m \\ \frac{\sqrt{2}}{2}(a_{m+1-s} + a_{s-m-1}) & s \le m, t = m+1 \\ \frac{1}{2}i(a_{t-s} - a_{s-t} + a_{s+t-2m-2} - a_{2m+2-s-t}) & s \le m, t > m+1 \\ a_0 = \frac{1}{2}2a_0 & s = m+1, t = m+1 \\ \frac{\sqrt{2}}{2}i(a_{t-m-1} - a_{m+1-t}) & s = m+1, t > m+1 \\ \frac{1}{2}(a_{t-s} + a_{s-t} - a_{s+t-2m-2} - a_{2m+2-s-t}) & s > m+1, t > m+1. \end{cases}$$

It follows that $B = U^*TU$ is symmetric.

□

We remark that the corresponding symmetric matrices $B = U^*TU$ given by Theorem 3.1 and 3.3 are different.

**Example 3.4:** When $n = 4$, $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}$ and

$$U^*TU = \frac{1}{2}\begin{pmatrix} 2a_0 + a_3 + a_{-3} & a_1 + a_{-1} + a_2 + a_{-2} & i(a_2 - a_{-2} + a_{-1} - a_1) & i(a_3 - a_{-3}) \\ a_1 + a_{-1} + a_2 + a_{-2} & 2a_0 + a_1 + a_{-1} & i(a_1 - a_{-1}) & i(a_2 - a_{-2} + a_1 - a_{-1}) \\ i(a_2 - a_{-2} + a_{-1} - a_1) & i(a_1 - a_{-1}) & 2a_0 - a_1 - a_{-1} & a_1 + a_{-1} - a_2 - a_{-2} \\ i(a_3 - a_{-3}) & i(a_2 - a_{-2} + a_1 - a_{-1}) & a_1 + a_{-1} - a_2 - a_{-2} & 2a_0 - a_3 - a_{-3} \end{pmatrix}$$

Denote by $S_n$ the subspace of complex symmetric matrices in $\mathbb{C}_{n\times n}$ and $\mathbb{T}_n$ the set of Toeplitz matrices in $\mathbb{C}_{n\times n}$. Theorem 3.1 and 3.3 imply that $\{U^*TU : U \in \mathbb{U}(n)\}$ and $S_n$ have nonempty intersection for all $T \in \mathbb{T}_n$.

Given $S \in S_n$, can we find a unitary matrix $U$ such that $USU^*$ is Toeplitz? If the answer is affirmative, then it can be viewed as a (weak) converse to Theorem 3.1 and 3.3, that is, every symmetric $S \in \mathbb{C}_{n\times n}$ is unitarily similar to a Toeplitz matrix.

It is not hard to see that the claim is true when $n = 2$ since each $A \in \mathbb{C}_{n\times n}$ is unitarily similar to a matrix of equal diagonal entries [17, p.18]. How about the $3 \times 3$ case? The answer is affirmative and we are going to prove it. We first note that for any $3 \times 3$ complex matrix

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{pmatrix},$$

we have

$$USU^* = \frac{1}{2}\begin{pmatrix} s_{11}+s_{33} & \sqrt{2}(s_{12}+is_{23}) & s_{11}+2is_{13}-s_{33} \\ \sqrt{2}(s_{12}-is_{23}) & 2s_{22} & \sqrt{2}(s_{12}+is_{23}) \\ s_{11}-2is_{13}-s_{33} & \sqrt{2}(s_{12}-is_{23}) & s_{11}+s_{33} \end{pmatrix},$$

where $U$ is the unitary matrix given in Theorem 3.3. So, if $s_{11}+s_{33}=2s_{22}$, then the matrix $USU^*$ is Toeplitz. Let us return to the case that $S$ is symmetric. If we can find a rotation matrix $W$ such that $B=(b_{ij})=WSW^T$ satisfies

$$b_{11}+b_{33}=2b_{22}, \tag{3.1}$$

then we have the desired result by applying the unitary similarity via $U$ to $B$ for the $3\times 3$ case. We will show that such a rotation matrix exists.

Denote by $\mathrm{SO}(n)$ the $n\times n$ proper orthogonal group. Let $S\in\mathcal{S}_3$ and $\tilde{S}=S-\frac{1}{3}(\mathrm{tr}\,S)I_3$. Then $\mathrm{tr}\,\tilde{S}=0$. If we can show that $\tilde{S}$ is unitarily similar to some Toeplitz matrix $T$, then $S$ is unitarily similar to the Toeplitz matrix $T+\frac{1}{3}(\mathrm{tr}\,S)I_3$. Thus we may assume $\mathrm{tr}\,S=0$.

**Lemma 3.5:** *Suppose that $S\in\mathcal{S}_3$ satisfying $\mathrm{tr}\,S=0$. Then there is $W\in\mathrm{SO}(3)$ such that the (2, 2)-entry of $WSW^T$ is 0, and hence*

$$WSW^T = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}$$

*for some $b_{11},b_{12},b_{13},b_{23},b_{22},b_{33}\in\mathbb{C}$ with $b_{22}=0$, $b_{33}=-b_{11}$. It follows that $b_{11}+b_{33}-2b_{22}=0$.*

***Proof:*** Note that $\mathrm{tr}\,S=0$. By a result of Brickman [18], (also see [19,20]), the range

$$W(S):=\{(WSW^T)_{22} : W\in\mathrm{SO}(3)\}$$

is convex. Since $s_{11},s_{22},s_{33}\in W(S)$,

$$0=\frac{1}{3}\mathrm{tr}\,S=\frac{1}{3}(s_{11}+s_{22}+s_{33})\in W(S).$$

So there is $W\in\mathrm{SO}(3)$ such that $(WSW^T)_{22}=0$. □

**Theorem 3.6:** *Any $3\times 3$ complex symmetric matrix is unitarily similar to some $3\times 3$ Toeplitz matrix.*

***Proof:*** Let $S\in\mathcal{S}_3$. Let $\tilde{S}=S-\frac{1}{3}(\mathrm{tr}\,S)I_3$. By Lemma 3.5, there exists $W\in\mathrm{SO}(3)$ such that $\tilde{B}:=(\tilde{b}_{ij})=W\tilde{S}W^T$ and $\tilde{b}_{22}=0$; thus $\tilde{b}_{11}+\tilde{b}_{33}=2\tilde{b}_{22}$. Now $B:=(b_{ij})=WSW^T=W\tilde{S}W^T+\frac{1}{3}(\mathrm{tr}\,S)I_3$ is symmetric and $b_{11}+b_{33}=2b_{22}$ and $b_{22}=\frac{1}{3}\mathrm{tr}\,S$, that is, (3.1) is satisfied. By the previous discussion

$$UBU^* = UWSW^TU^* = UWS(UW)^*$$

is Toeplitz, where $U$ is the $3 \times 3$ unitary matrix given in Theorem 3.3. □

## 4. A standard form of complex symmetric matrices

To consider the class of complex symmetric matrices which are unitarily similar to Toeplitz matrices, we introduce a new standard form for complex symmetric matrices.

Lemma 3.5 can be extended in the following theorem.

**Theorem 4.1:** Let $S \in \mathcal{S}_n$. There exists $W \in \mathrm{SO}(n)$ for which the diagonal entries $(d_1, d_2, \dots, d_n)$ of the complex symmetric matrix $W^TSW$ satisfy

$$d_j = \frac{2}{n}\mathrm{tr}\, S - d_{n+1-j}$$

for $j = 1, 2, \dots, n$. In particular, $d_j = -d_{n+1-j}$ when $\mathrm{tr}\, S = 0$ for $j = 1, 2, \dots, n$.

***Proof*:** Without loss of generality, we may assume that $\mathrm{tr}\, S = 0$. It suffices to prove that there exists $W \in \mathrm{SO}(n)$ for which the diagonal entries $(d_1, d_2, \dots, d_n)$ of $W^TSW$ satisfy $d_j = -d_{n+1-j}$ for $j = 1, 2, \dots, n$.

It is trivial when $n = 2$ and Lemma 3.5 handles the $n = 3$ case.

Now we let $n \geq 4$. When $n = 2m$, let $C = \mathrm{diag}(1, 0, \dots, 0, 1)$. By a result of Au-Yeung and Tsing [19] (also see [20, Theorem 11.7]), the range

$$W_C(S) = \{(WSW^T)_{11} + (WSW^T)_{nn} : W \in \mathrm{SO}(n)\}$$

is convex. If $0 \notin W_C(S)$, we can separate $W_C(S)$ from 0 by the line $x = a$ for some $a > 0$ by rotating the range. We may assume $W_C(S) \subset \{z \in \mathbb{C} : \Re(z) \geq a\}$. This relation implies that

$$\begin{aligned}
\Re((WSW^T)_{11} + (WSW^T)_{nn}) &\geq a, \\
\Re((WSW^T)_{22} + (WSW^T)_{(n-1)(n-1)}) &\geq a, \\
&\cdots \\
\Re((W^T)_{mm} + (WSW^T)_{(m+1)(m+1)}) &\geq a,
\end{aligned}$$

and hence $\Re(\mathrm{tr}\,(WSW^T)) = \Re(\mathrm{tr}\, S) \geq ma > 0$, which contradicts $\mathrm{tr}\, S = 0$. So we have $0 \in W_C(S)$, and thus there exists $W \in \mathrm{SO}(n)$ such that the diagonal entries $(d_1, d_2, \dots, d_n)$ of $W^TSW$ satisfy $d_n = -d_1$. The argument can be used to prove $d_j = -d_{n-j+1}$ by taking $C = \mathrm{diag}\,(0_{j-1}, 1, 0_{n-2j}, 1, 0_{j-1})$. Hence, we have $d_j = -d_{n+1-j}$, where $j = 1, \cdots, m$.

When $n = 2m + 1$, let $C = \mathrm{diag}(0_m, 1, 0_m)$. Using the idea in Lemma 3.5 we can prove that there exists $W \in \mathrm{SO}(n)$ such that the diagonal entries $(d_1, \dots, d_{m+1}, \dots, d_n)$ of $W^TSW$ satisfy $d_{m+1} = 0$. Then we consider the $(2m) \times (2m)$ matrix $\tilde{S}$ obtained by deleting the $(m + 1)$-st row and column from $S$. We then apply the inductive hypothesis to $\tilde{S}$ and complete the proof of our assertion. □

We consider the class of $4 \times 4$ standard form of complex symmetric matrices $S$ with $\mathrm{tr}\, S = 0$:

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{12} & s_{22} & s_{23} & s_{24} \\ s_{13} & s_{23} & -s_{22} & s_{34} \\ s_{14} & s_{24} & s_{34} & -s_{11} \end{pmatrix},$$

which is parametrized by the 8 complex numbers $s_{11}, s_{22}, s_{12}, \ldots, s_{34}$. In order to have $S = U^*TU$ for some $T \in \mathbb{T}_4$ and the unitary matrix $U$ given in Section 3, the following two equations are necessary and sufficient conditions:

$$s_{12} + s_{34} - 2s_{22} = 0, s_{13} - s_{24} + 2s_{23} = 0.$$

A general form of $\tilde{T} = USU^*$ for the above $S$ is given by

$$\tilde{T} = \begin{pmatrix} 0 & t_{12} & t_{13} & t_{14} \\ t_{21} & 0 & t_{23} & t_{13} \\ t_{31} & t_{32} & 0 & t_{12} \\ t_{41} & t_{31} & t_{21} & 0 \end{pmatrix},$$

which is parametrized by 8 complex numbers $t_{12}, t_{13}, t_{14}, t_{23}$ and $t_{21}, t_{31}, t_{41}, t_{32}$. We denote by $\tilde{\mathbb{T}}_4$ the complex vector space of matrices of the form

$$\tilde{T} = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{11} & t_{23} & t_{13} \\ t_{31} & t_{32} & t_{11} & t_{12} \\ t_{41} & t_{31} & t_{21} & t_{11} \end{pmatrix}.$$

The matrix $\tilde{T}$ is Toeplitz if $t_{23} = t_{12}, t_{32} = t_{21}$.

## 5. Comparision of the two dimensions, 27 vs. 26

In this section, we shall prove that the following inclusion is proper:

$$\{UTU^* : T \in \mathbb{T}_4, U \in \mathbb{U}(4)\} \subset \{USU^* : S \in S_4, U \in \mathbb{U}(4)\}.$$

We first establish that the following theorem.

**Theorem 5.1:** *The set*

$$\{UTU^* : T \in \mathbb{T}_4, \operatorname{tr} T = 0, U \in \mathbb{U}(4)\}, \tag{5.1}$$

*is parametrized by real 26-variables. The following set*

$$\{USU^* : S \in \tilde{\mathbb{T}}_4, \operatorname{tr} S = 0, U \in \mathbb{U}(4)\} \tag{5.2}$$

*contains the above set (5.1) and its dimension is 27. Hence there is a set $S \in \tilde{\mathbb{T}}_4$ with $\operatorname{tr} S = 0$ for which $USU^*$ does not belong to (5.1) for any $U \in \mathbb{U}(4)$.*

***Proof*:** We first examine the set (5.1). Note that the dimension of the real vector space

$$\{T \in \mathbb{T}_4, \operatorname{tr} T = 0\}$$

is 12. The Lie group SU(4) is a 15-dimensional real analytic manifold with the real tangent space at the identity composed of the $4 \times 4$ skew-Hermitian matrices of zero trace:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}$$

with $x_{11} = ip_1, x_{22} = ip_2, x_{33} = ip_3, x_{44} = -i(p_1+p_2+p_3), x_{12} = r_1+is_1, x_{21} = -r_1+is_1$, $x_{13} = r_2+is_2, x_{31} = -r_2+is_2, x_{14} = r_3+is_3, x_{41} = -r_3+is_3, x_{23} = r_4+is_4, x_{32} = -r_4+is_4$, $x_{24} = r_5 + is_5, x_{42} = -r_5 + is_5, x_{34} = r_6 + is_6, x_{43} = -r_6 + is_6$, where $p_1, p_2, p_3, r_1, \ldots, r_6$, $s_1, \ldots, s_6$ are 15 real parameters. For a general point $U_g T_0 U_g^*$ of the set

$$\{UTU^* : T \in \mathbb{T}_4, \operatorname{tr} T = 0, U \in \mathrm{SU}(4)\},$$

we shall estimate the dimension of its tangent space. By using the operation

$$Z \mapsto U_g^* Z U_g,$$

we may assume that $U_g = I$ and restrict ourselves to consider the dimension of the tangent space at $T_0 \in \mathbb{T}_4$ with $\operatorname{tr} T_0 = 0$. By the Taylor expansion of an element of a neighborhood of $T_0$, we obtain

$$\exp(tX)(T_0 + tT_1)\exp(-tX) = T_0 + tT_1 + tXT_0 - tTX_0 + O(t^2)$$

for $X \in \mathbb{C}_{4\times4}, X^* = -X, \operatorname{tr} X = 0$ and $T_1 \in \mathbb{T}_4, \operatorname{tr} T_1 = 0$. These $T_1$'s form a 12 dimensional real vector space. We compute the dimension of the derivation range

$$\{T_0X - XT_0 : X \in \mathbb{C}_{4\times4}, X^* = -X, \operatorname{tr} X = 0\}$$

modulo the vector space

$$\{\tilde{T} \in \mathbb{T}_4 : \operatorname{tr} \tilde{T} = 0\}.$$

We denote by $W_{ij}$ the $(i,j)$-entry of $W = TX - XT$. Let

$$\begin{aligned} W_1 &= W_{1,1}, W_2 = W_{2,2}, W_3 = W_{3,3}, W_4 = W_{1,2} - W_{3,4}, W_5 = W_{2,1} - W_{4,3}, \\ W_6 &= W_{1,3} - W_{2,4}, W_7 = W_{3,1} - W_{4,2}, W_8 = W_{2,3} - W_{3,4}, W_9 = W_{3,2} - W_{4,3}. \end{aligned}$$

Each $W_j$ can be expressed as

$$W_j = t_{j,1}p_1 + t_{j,2}p_2 + t_{j,3}p_3 + \sum_{k=4}^{9} t_{j,k-3}r_k + \sum_{10}^{15} t_{j,k-9}s_k$$

for some complex coefficients $t_{j,k}$. The coefficients $t_{j,k}$ satisfy

$$t_{j,k} = 0,$$

for $j, k = 1, 2, 3$. We are going to give the coefficient vectors

$$P_k = (t_{4,k}, t_{5,k}, t_{6,k}, t_{7,k}, t_{8,k}, t_{9,k}), \quad k = 1, 2, 3.$$

They are

$$\begin{aligned} P_1 &= (0, 0, 0, 0, ia_1 - b_1, -ic_1 + d_1), \\ P_2 &= 2(ia_1 - b_1, -ic_1 + d_1, ia_2 - b_2, -ic_2 + d_2, 0, 0), \\ P_3 &= (2ia_1 - 2b_1, -2ic_1 + 2d_1, 2ia_2 - 2b_2, -2ic_2 + 2d_2, 3ia_1 - 3b_1, -3ic_1 + 3d_1), \end{aligned}$$

and hence the vectors $P_j's$ satisfy the linear equation

$$P_3 - P_2 - 3P_1 = 0.$$

Hence, the rank of the $18 \times 15$ matrix $(\Re(t_{j,k}), \Im(t_{j,k}))^T$ is necessarily less than or equal to 14. By taking a rather general coefficient $a_j, b_j, c_j, d_j$, the rank of such a matrix is just 14. Thus, we conclude that the dimension of the set (5.1) is 26.

We then examine the set (5.2) and show that it has dimension $27 = 16 + 11$. We take a generic matrix $\tilde{T}$ in $\tilde{\mathbb{T}}_4$ with $\operatorname{tr} \tilde{T} = 0$ as follows:

$$\tilde{T} = \begin{pmatrix} 0 & 2-2i & 3+7i & 7+8i \\ 1+4i & 0 & 23+3i & 3+7i \\ 23+7i & 13-3i & 0 & 2-2i \\ -11+11i & 23+7i & 1+4i & 0 \end{pmatrix},$$

at which we consider the tangent space of the set (5.2). By the Taylor expansion of an element of a neighborhood of $\tilde{T}$, we obtain

$$\exp(tX)(\tilde{T} + tT_1)\exp(-tX) = \tilde{T} + tT_1 + X\tilde{T} - \tilde{T}X + O(t^2)$$

for $X \in \mathbb{C}_{4\times 4}, X^* = -X, \operatorname{tr} X = 0$ and $T_1 \in \tilde{\mathbb{T}}_4, \operatorname{tr} T_1 = 0$. These $T_1$'s form a 16 dimensional real vector space. We compute the dimension of the derivation range

$$\{\tilde{T}X - X\tilde{T} : X \in \mathbb{C}_{4\times 4}, X^* = -X, \operatorname{tr} X = 0\}$$

modulo the vector space

$$\{T_1 \in \tilde{\mathbb{T}}_4 : \operatorname{tr} T_1 = 0\}.$$

We denote by $W_{ij}$ the $(i,j)$-entry of $W = TX - XT$. Let

$$W_1 = W_{1,1}, W_2 = W_{2,2}, W_3 = W_{3,3}, W_4 = W_{1,2} - W_{3,4},$$

$$W_5 = W_{2,1} - W_{4,3}, W_6 = W_{1,3} - W_{2,4}, W_7 = W_{3,1} - W_{4,2}.$$

Let

$$\begin{aligned} \Re(W_j) &= c_{2j-1,1}p_1 + c_{2j-1,2}p_2 + c_{2j-1,3}p_3 + c_{2j-1,4}r_1 + \cdots + c_{2j-1,9}r_6 + c_{2j-1,10}s_1 \\ &\quad + \ldots + c_{2j-1,15}s_6, \\ \Im(W_j) &= c_{2j,1}p_1 + c_{2j,2}p_2 + c_{2j,3}p_3 + c_{2j,4}r_1 + \cdots + c_{2j,9}r_6 + c_{2j,10}s_1 + \ldots + c_{2j,15}s_6, \end{aligned}$$

$j = 1, \ldots, 7$. We consider the $14 \times 15$ matrix $\tilde{T} = (c_{ij}) = [A|B]$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & -26 & 4 & 0 & 0 \\ 0 & 0 & 0 & -2 & -14 & -19 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & -36 & -26 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & -14 \\ 0 & 0 & 0 & 0 & 26 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 & 14 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & -20 & -46 & -6 & -20 \\ 0 & 4 & 4 & 0 & -5 & -14 & -14 & -5 \\ 0 & 8 & 8 & 0 & -12 & -6 & -46 & -12 \\ 0 & -2 & -2 & 0 & -14 & -14 & -14 & -14 \\ 0 & -14 & -14 & -30 & 0 & -2 & 4 & 0 \\ 0 & 6 & 6 & -11 & 0 & -8 & -4 & 0 \\ 0 & 14 & 14 & -2 & 0 & -4 & 2 & 0 \\ 0 & -46 & -46 & -8 & 0 & 4 & 8 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 6 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & -20 & 18 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & -6 & 0 & 0 \\ 0 & -1 & 0 & 0 & 10 & -20 & -20 \\ -3 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & -2 & 20 & 0 & -10 & 0 & 0 \\ 0 & 0 & -11 & 14 & -14 & -11 & -11 \\ 0 & 0 & -6 & -46 & 6 & -6 & -6 \\ 0 & 0 & 8 & -14 & 14 & 8 & 8 \\ 0 & 0 & 34 & 6 & -46 & 34 & 34 \\ -30 & -5 & 0 & 8 & 4 & 0 & 0 \\ -11 & -16 & 0 & -2 & 4 & 0 & 0 \\ -2 & 14 & 0 & 4 & 8 & 0 & 0 \\ -8 & 24 & 0 & 4 & -2 & 0 & 0 \end{pmatrix}.$$

We are going to show that the rank of $\tilde{T}$ is at least 11. For this purpose we delete the first two columns and the last two columns from $\tilde{T}$ without increasing the rank. These columns correspond to the variables $p_1, p_2, s_5, s_6$. We also delete the first two rows and the last row from $\tilde{T}$ without increasing the rank. These two rows correspond to the $(1,1)$-entry of $W = TX - XT$ and the imaginary part of $W_7 = W_{3,1} - W_{4,2}$. Then we obtain an $11 \times 11$ real invertible matrix. This shows that the set (5.2) has a tangent space of dimension at least 27 at the point $\tilde{T}$. In fact, with some additional computations we can show that the rank is just 11 but it is unnecessary for the proof of the assertion in the theorem, so the set (5.1) cannot cover the set (5.2). Thus, we just proved that there exists a $4 \times 4$ complex symmetric matrix which is not unitarily similar to any $4 \times 4$ Toeplitz matrix. □

## 6. 5 × 5 matrices

We shall prove the following theorem.

**Theorem 6.1:** *There exists a $5 \times 5$ complex symmetric matrix $A$ with* $\operatorname{tr} A = 0$ *for which $F_A$ is not realized by $F_T$ for any $T \in \mathbb{T}_5$ with* $\operatorname{tr} T = 0$.

***Proof*:** The real vector space of all $5 \times 5$ Toeplitz matrices $T$ with $\operatorname{tr} T = 0$ is parametrized as

$$\begin{pmatrix} 0 & p_1 + iq_1 & p_2 + iq_2 & p_3 + iq_3 & p_4 + iq_4 \\ p_5 + iq_5 & 0 & p_1 + iq_1 & p_2 + iq_2 & p_3 + iq_3 \\ p_6 + iq_6 & p_5 + iq_5 & 0 & p_1 + iq_1 & p_2 + iq_2 \\ p_7 + iq_7 & p_6 + iq_6 & p_5 + iq_5 & 0 & p_1 + iq_1 \\ p_8 + iq_8 & p_7 + iq_7 & p_6 + iq_6 & p_5 + iq_5 & 0 \end{pmatrix}$$

by 16 real parameters $p_1, \dots, p_8, q_1, \dots, q_8$. The Kippenhahn polynomial of a general $5 \times 5$ unitarily symmetrizable matrix $\tilde{S}$ with $\operatorname{tr} \tilde{S} = 0$ is expressed by

$$\begin{aligned} F_{\tilde{S}}(t,x,y) &= t^5 + c_1 t^3 x^2 + c_2 t^3 xy + c_3 t^3 y^2 + c_4 t^2 x^3 + c_5 t^2 x^2 y + c_6 t^2 xy^2 \\ &\quad + c_7 t^2 y^3 + c_8 tx^4 + c_9 tx^3 y + c_{10} tx^2 y^2 + c_{11} txy^3 \\ &\quad + c_{12} ty^4 + c_{13} x^5 + c_{14} x^4 y + c_{15} x^3 y^2 + c_{16} x^2 y^3 + c_{17} xy^4 + c_{18} y^5, \end{aligned}$$

where $c_1, \dots, c_{18}$ are real coefficients. We will identify $F_{\tilde{S}}(t,x,y)$ by its coefficient vector $\bar{F}_{\tilde{S}} := (c_1, c_2, \dots, c_{18})$. We consider the subspace

$$\{\bar{F}_T = (c_1, c_2, \dots, c_{18}) : T \in \mathbb{T}_5, \operatorname{tr} T = 0\}.$$

The map

$$(p_1, q_1, \dots, p_8, q_8) \mapsto (c_1, \dots, c_{18})$$

is a polynomial map and hence it is infinitely differentiable. So the set of points $(c_1, \dots, c_{18})$ for $5 \times 5$ Toeplitz matrices $T$ with $\operatorname{tr} T = 0$ has dimension $\leq 16$. We show that there exists a $5 \times 5$ complex symmetric matrix $S$ for which the linear perturbation $\tilde{S} = 2S + H + iK$ of $2S$ by the matrices $H, K$:

$$H = \begin{pmatrix} a_1 & a_5 & a_9 & a_{12} & a_{14} \\ a_5 & a_2 & a_6 & a_{10} & a_{13} \\ a_9 & a_6 & a_3 & a_7 & a_{11} \\ a_{12} & a_{10} & a_7 & a_4 & a_8 \\ a_{14} & a_{13} & a_{11} & a_8 & a_{12} \end{pmatrix}, \quad K = \begin{pmatrix} b_1 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & 0 & b_5 \end{pmatrix},$$

with $a_{12} = -(a_1 + a_2 + a_3 + a_4)$, $b_5 = -(b_1 + b_2 + b_3 + b_4)$ for real coefficients $b_1, \dots, b_4, a_1, \dots, a_{14}$. Then the matrix $\tilde{S} = 2S + H + iK$ has non-vanishing Jacobian

$$\frac{\partial(c_1, \dots, c_4, c_5, \dots, c_{18})}{\partial(b_1, \dots, b_4, a_1, \dots, a_{14})}$$

at $(b_1, \dots, b_4, a_1, \dots, a_{18}) = (0, \dots, 0, 0, \dots, 0)$ for some symmetric matrix $S$. In fact, let

$$S = \begin{pmatrix} 0 & 2+3i & 1-3i & 4+2i & -3+5i \\ 2+3i & 4+2i & 3-2i & 3+5i & 4-6i \\ 1-3i & 3-2i & 2+i & 2+2i & 1-i \\ 4+2i & 3+5i & 2+2i & -2-i & 1+i \\ -3+5i & 4-6i & 1-i & 1+i & -4-2i \end{pmatrix}.$$

Then we compute the above Jacobian by using computer software. This value does not vanish and is about $3.81737 \times 10^{57}$. Hence, there exists a $5 \times 5$ symmetric matrix $A$ with $\operatorname{tr} A = 0$ for which $F_A$ is not realized by $F_T$ for any $T \in \mathbb{T}_5$ with $\operatorname{tr} T = 0$. □

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