At the displacement level, the closure of each kinematic loop can be expressed in the vector form as

$$\overrightarrow{AB} = \overrightarrow{AA_i} + \overrightarrow{A_iB_i} - \overrightarrow{BB_i} \quad \text{for } i = 1, 2, \ldots, n, \tag{3.1}$$

in which $\overrightarrow{AA_i}$ and $\overrightarrow{BB_i}$ can be easily obtained from the geometry of the attachment points in the base and in the moving platform. Let us define vector $\boldsymbol{a}_i = \overrightarrow{AA_i}$ in the fixed frame $\{\boldsymbol{A}\}$, and $\boldsymbol{b}_i = \overrightarrow{BB_i}$ in the moving frame $\{\boldsymbol{B}\}$. Furthermore, $\boldsymbol{q}_i = \overrightarrow{A_iB_i}$ is defined as the limb variable, which indicates the geometry of the limb, and generally includes the active and passive limb segments. Hence, the loop closure can be written as the unknown pose variables $\boldsymbol{p}$, and $\boldsymbol{R}$, the position vectors describing the known geometry of the base and the moving platform, $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$. Furthermore, the limb vector $\boldsymbol{q}_i$ is only known from the kinematic equation of the limb together with the readouts of the position sensors located at the different joints of that limb. Therefore, one may write the loop closure equations as follows:

$$\boldsymbol{p} = \boldsymbol{a}_i + \boldsymbol{q}_i - \boldsymbol{R}\,\boldsymbol{b}_i \quad \text{for } i = 1, 2, \ldots, n. \tag{3.2}$$

These equations are the main body of kinematic analysis. For an inverse kinematic problem, it is assumed that the moving platform position $\boldsymbol{p}$ and orientation $\boldsymbol{R}$ are given and the problem is to solve the active limb variables. Hence, from the above equations the passive limb variables must be eliminated to solve the inverse kinematic problem. This analysis is usually straightforward and results in a unique solution for the limb variables, even for redundant parallel manipulators. However, the inverse solution is not straightforward, and usually numerical methods are used for forward kinematic solution. In the proceeding sections, the inverse and forward kinematic analysis of three case studies are elaborated in detail.

3.3 Kinematic Analysis of a Planar Manipulator

In this section, the kinematics of a planar parallel manipulator is discussed in detail. In this analysis, the mechanism kinematic structure is described first and then the geometry of the manipulator is elaborated. Next, the inverse and forward kinematic analyses of the manipulator are worked out in detail. In order to verify the formulations, in Section 3.3.5, the simulation results for computation of inverse kinematics and forward kinematics of the manipulator are presented, and the accuracy of a numerical solution to forward kinematic problem is verified.

3.3.1 Mechanism Description

The architecture of a planar $4R\underline{P}R$ parallel manipulator considered here is shown in Figure 3.3. In this manipulator the moving platform is supported by four limbs of an identical kinematic structure. Each limb connects the fixed base to the manipulator moving platform by a revolute joint (R), followed by an actuated prismatic joint ($\underline{P}$), and another revolute joint (R). Hence, the total structure of the manipulator is $4R\underline{P}R$. The kinematic structure of a prismatic joint is used to model either a piston-cylinder actuator at each limb, or a cable-driven one. In order to avoid singularities at the central position of the

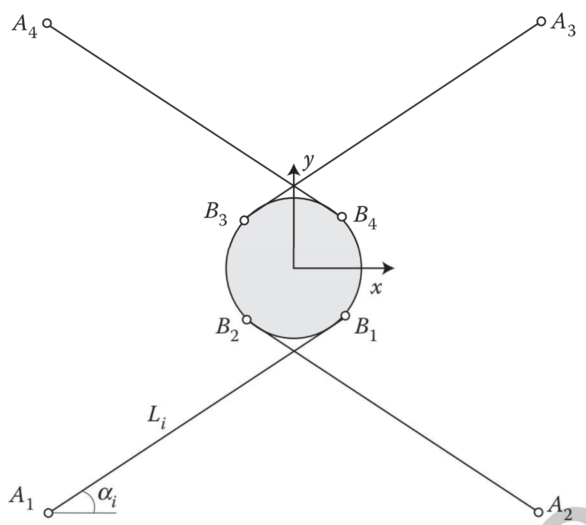

FIGURE 3.3
The schematics of a $4R\underline{P}R$ parallel manipulator.

manipulator, the limbs are considered to be crossed. Complete singularity analysis of the mechanism is presented in Section 4.6.2.

As shown in Figure 3.3, A_i denotes the fixed base points of the limbs, B_i denotes point of connection of the limbs to the moving platform, L_i denotes the limbs' lengths, and α_i denotes the limbs' angles. The position of the center of the moving platform G is denoted by $G = [x_G, y_G]$, and orientation of the manipulator moving platform is denoted by ϕ with respect to the fixed coordinate frame. Hence, the manipulator possesses three-degrees-of-freedom $\boldsymbol{\mathcal{X}} = [x_G, y_G, \phi]$, with one-degree-of-redundancy in the actuators.

3.3.2 Geometry of the Manipulator

For the purpose of analysis and as illustrated in Figure 3.4, a fixed frame O: xy is attached to the fixed base at point O, the center of the base point circle which passes through A_i's. Moreover, another moving coordinate frame G: UV is attached to the manipulator moving platform at point G. Furthermore, assume that point A_i lies at a radial distance of R_A from point O, and point B_i lies at a radial distance of R_B from point G in the xy plane, when the manipulator is at a central location.

In order to specify the geometry of the manipulator, let us define θ_{Ai} and θ_{Bi} as the absolute angles of points A_i and B_i at the central configuration of the manipulator with respect to the fixed frame O. The instantaneous orientation angle of B_i's is defined as

$$\phi_i = \phi + \theta_{B_i}. \tag{3.3}$$

Therefore, for each limb, $i = 1, 2, 3, 4$, the positions of the base points A_i are given by

$$A_i = [R_A \cos(\theta_{A_i}),\, R_A \sin(\theta_{A_i})]^T. \tag{3.4}$$

3.3.3 Inverse Kinematics

For inverse kinematic analysis, it is assumed that the position and orientation of the moving platform $\boldsymbol{\mathcal{X}} = [x_G, y_G, \phi]^T$ is given and the problem is to find the joint variables of the

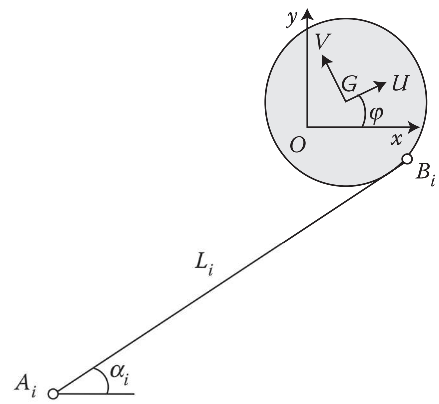

FIGURE 3.4
Kinematic configuration of the $4R\underline{P}R$ planar manipulator.

manipulator, $\boldsymbol{L} = [L_1, L_2, L_3, L_4]^T$. From the geometry of the manipulator, as illustrated in Figure 3.4, the loop closure equation for each limb, $i = 1, 2, 3, 4$, may be written as,

$$\overrightarrow{A_i G} = \overrightarrow{A_i B_i} - \overrightarrow{G B_i}. \tag{3.5}$$

Rewriting the vector-loop closure componentwise,

$$x_G - x_{Ai} = L_i \cos(\alpha_i) - R_B \cos(\phi_i), \tag{3.6}$$

$$y_G - y_{Ai} = L_i \sin(\alpha_i) - R_B \sin(\phi_i), \tag{3.7}$$

in which α_i's are the absolute limb angles. To solve the inverse kinematic problem, it is required to eliminate α_i's from the above equations and solve L_i's. This can be accomplished by reordering the above equations as,

$$L_i \cos(\alpha_i) = x_G - x_{Ai} + R_B \cos(\phi_i) \tag{3.8}$$

$$L_i \sin(\alpha_i) = y_G - y_{Ai} + R_B \sin(\phi_i). \tag{3.9}$$

By adding the square of both the sides of Equations 3.8 and 3.9, the limb lengths are uniquely determined as follows:

$$L_i = \Big[(x_G - x_{Ai} + R_B \cos(\phi_i))^2 + (y_G - y_{Ai} + R_B \sin(\phi_i))^2\Big]^{1/2}. \tag{3.10}$$

Furthermore, the limb angles α_i's can be determined from the following equation:

$$\alpha_i = \text{Atan2}\left[(y_G - y_{Ai} + R_B \sin(\phi_i)), (x_G - x_{Ai} + R_B \cos(\phi_i))\right]. \tag{3.11}$$

Hence, corresponding to each given manipulator location $\boldsymbol{\mathcal{X}} = [x_G, y_G, \phi]^T$, there is a unique solution for the limb lengths L_i's, and limb angles α_i's.