Math 447/647

Probability Models

Test 2

November 21, 2022

Name:

The total number of points is 100 points. No calculators allowed.

Problem 1 (10 points) Let P be a transition probability matrix of a Markov chain and let i be a state. If $P_{ii}^n = 1/2^n$ what can you say about the state i, namely is it transient, recurrent, null recurrent, positive recurrent or ergodic and what is it period?

Problem 2 (10 points) The life expectancy of a patient that needs an organ transplant is exponentially distributed with rate μ. Organs for transplantation become available according to a Poisson process with rate λ. What is the probability that the patient obtains the transplant, if he is the first in line to receive it?

Problem 3 The Millers receive the newspaper every morning and place it on a pile after reading it. Each afternoon, with probability 1/3, someone takes all the papers in the pile and puts them in recycle bin. Also, Mr. Miller is very clutter-conscious, so if he sees there are at least five papers in the pile, Mr. Miller takes all the papers to the bin before going to bed. Consider the number of papers in the pile in the evening (after all Millers went to bed).

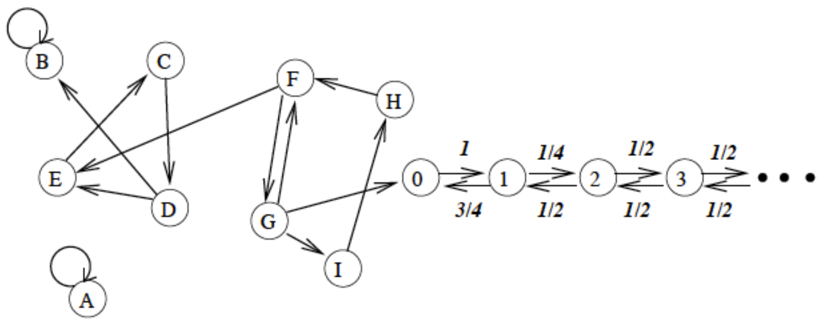
a) (10 points) Define the state space and find the transition matrix of the Markov chain.

b) (10 points) Find the stationary distribution.

c) (5 points) Assuming Millers have been doing this (buying and recycling newspapers) for a long time, what is the expected number of papers in the pile.

d) (5 points) Assume the pile is empty. What is the expected time until the pile will be empty again?

Problem 4 Consider the Markov chain X_n illustrated with the transition diagram below. The state space is the infinite set $S = \{A, B, B, C, D, E, F, G, H, I, 0, 1, 2, 3, \dots\}$. For $i \geq 2$, probabilities $P_{i,i+1} = P_{i,i-1} = 1/2$. Positive transition probabilities between states denoted with letters are represented by an arrow, but their values are not specified.



a) (5 points) List all the classes.

b) (5 points) For each class determine its period.

c) (5 points) For **letter classes only** determine if they are transient or recurrent. If recurrent, are they positive or null recurrent? (If you have the time, you can do this for the number class, but that is not mandatory.)

d) (5 points) What is the probability that starting in state C, the chain will ever visit state 3?

e) (5 points) Find $\lim_{n\to\infty} P(X_n \in \{C, G, H\})$.

Problem 5 Let $N(t)$ be a Poisson process with rate λ.

a) (5 points) What is the probability that there are exactly two events in the interval (1,3]?

b) (10 points) What is the probability that there is exactly one event in the interval (1,3], exactly two in (3,7] and exactly three in (7,10]?

c) (10 points) What is the probability that there are exactly two events in (1,7] and exactly three in (3,10]? (*Hint.* Condition on the number of events in the intersection of the two intervals.)