Math 5300, Fall 2022 (Roos) – Homework 4.
Due Wednesday, Nov 2.

Important: Please submit your homework as a single compressed pdf file (<2 MB if possible, scans of handwritten work are okay, use an appropriate app) online via Blackboard. Problem 1 must appear on top of the first page.

Only Problem 1 will be graded. The other problems are strongly recommended, but will not be graded. Problems marked with an asterisk (*) may be more challenging.

1 (Graded). (i) Determine the exact solution to the IVP

$$\begin{cases} y' = \sin(x)y \\ y(1) = 2 \end{cases}$$

(ii) Implement the 4-stage Runge-Kutta method given by the Butcher tableau

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{3} & \frac{1}{3} & & & \\ \frac{2}{3} & -\frac{1}{3} & 1 & & \\ 1 & 1 & -1 & 1 & \\ \hline & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

Test it on the IVP from (i) on the interval $I = [1, 6]$ (so $a = 5$): Generate a scatter plot of the approximate solution for step size $h = \frac{1}{5}$ (plot each point of the approximation, do not connect the points) and also plot the curve of the exact solution (in the same plot). Output the decimal value of the global approximation error

$$|E_N| = |y(6) - y_N|$$

where $y = y(x)$ is the exact solution from (i) and y_N is the approximation that your program produced (note $N = a/h = 25$).

(iii) Confirm that the order of the method is 4 by computing the approximation error for various small step sizes, say $h = 2^{-i}$ for $i = 2, \dots, 9$, then plotting $\log |E_N|$ against $\log(h)$ and estimating the slope of the line (as shown in class or differently).

Note: To receive full credit for (ii), (iii) it suffices to include the source code and its output including plots. Include brief comments in your source code to explain what you are doing.

2. Consider the second-order linear ODE

$$y'' = -y + 2y'$$

(i) Determine the general solution.

(ii) Determine the solution with initial conditions $y(0) = 1, y'(2) = -1$.

(iii) Determine all solutions (that is, the general solution) of the inhomogeneous equation

$$y'' = -y + 2y' + 3.$$

3. (Extra Credit!) Extend your code from the in-class programming session in such a way that it covers also systems of first-order ODEs. Test your code on the system

$$y_1' = y_2, y_2' = -4y_1$$

with initial conditions $y_1(0) = 1$, $y_2(0) = 2$ (first determine the exact solution).

Produce graphs that show the exact solution beside the numerical solutions using methods A,B,C,D from Exercise 3.52. Produce graphs that demonstrate the different orders of methods A,B,C,D empirically using the given system (double-logarithmic error plots).

Comments:

- Submissions without pictures will not receive any credit.
- The problems asks for an implementation that works for systems of arbitrary size (i.e. arbitrarily many equations), but if you only implement it for systems of two equations to cover the given example, you may still receive some credit.
- To determine the exact solution of the system, you may want to recognize it as a linear second-order equation in disguise.

4. Recall the definition of linear multistep methods from class.

(i) Define reasonable notions of truncation error, consistency and order for linear multistep methods (review the corresponding notions for one-step methods). (You can compare with the actual definitions in Süli-Mayers, Ch. 12.6).

(ii) Prove that the truncation error converges to zero for the linear two-step Adams-Bashforth method

$$y_{n+2} - y_{n+1} = h(\tfrac{3}{2}F_{n+1} - \tfrac{1}{2}F_n)$$

and show that it has order 2.