How Numerical Integration Works

- Differential Equations can be solved with analytical solution or through numerical integration
- Numerical integration approximate the DE solution at successive small time intervals and then extrapolate the value of the derivative over each interval

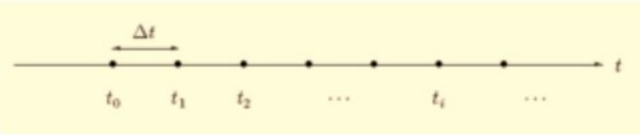
$$\frac{dx}{dt} = f(x) = x(1-x)$$

- with an initial value x_0=0.1 at an initial time t=0, the derivative is

$$f(0.1) = 0.1 \times (1 - 0.1) = 0.09$$

How Numerical Integration Works

$$\frac{dx}{dt} = f(x) = x(1-x)$$



Lets solve this further we need to choose a small interval step Δt = 0.5 with assumption that from t_0 to t = 0.5, the derivative is same. With this small interval, the x will increase by

$$dx/dt \times \Delta t = 0.09 \times 0.5 = 0.045$$

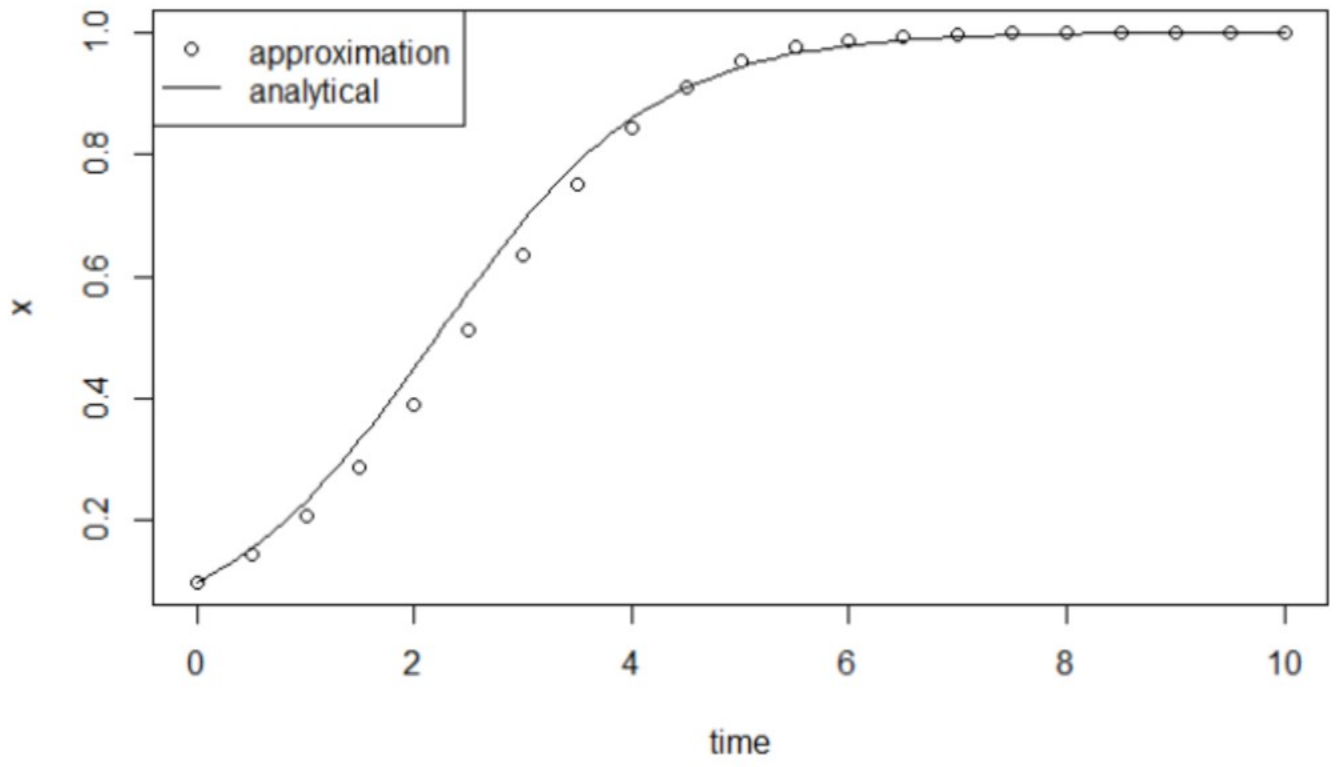
The approximate solution for x at t = 0.5 will be

$$x(0) + 0.045 = 0.1 + 0.045 = 0.145$$

We will then need to use x(0.5) to calculate the next point in time t=1. :

t	x	$\frac{dx}{dt}$
0	0.1	0.09
0.5	0.145	0.123975
1.0	0.206987	0.164144
1.5	0.289059	0.205504
2.0	0.391811	0.238295

Solution



Euler's Method

- The method we just used is called the Euler method. It is a method to numerically approximate the solution to a differential equation

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0$$

$$f(y) = \frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{h}$$

$$\frac{dy}{dt} = f(y) \quad \Rightarrow \quad y(t+h) = y(t) + hf(y(t))$$

The solution at some point in time, y_n

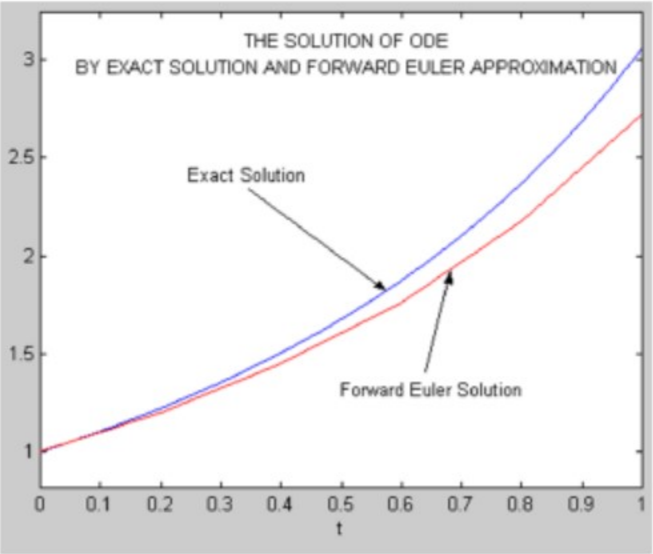
$$y_{n+1} = y_n + hf(y_n)$$

Euler's Method

- Euler's method is not exact. There is always an error between the solution and the error. There are many differential equations that can not be solved. We can still find an approximate solution.
- It works reasonably well for the differential equation as discussed previously but, in many cases it does not perform very well.
- There are other sophisticated methods with good algorithms that work well in most situations

Euler's Method

$$dy/dt = y' = ty + 1, \quad y_0 = y(0) = 1, \quad 0 \le t \le 1, \quad h = 0.25$$



$$\text{for} \quad t_0 = 0, \quad y_0 = y(0) = 1$$

$$\text{for} \quad t_1 = 0.25, \quad y_1 = y_0 + hy_0'$$
$$= y_0 + h(t_0 y_0 + 1)$$
$$= 1 + 0.25(0*1+1) = 1.25$$

$$\text{for} \quad t_2 = 0.5, \quad y_2 = y_1 + hy_1'$$
$$= y_1 + h(t_1 y_1 + 1)$$
$$= 1.25 + 0.25(0.25*1.25+1) = 1.5781$$

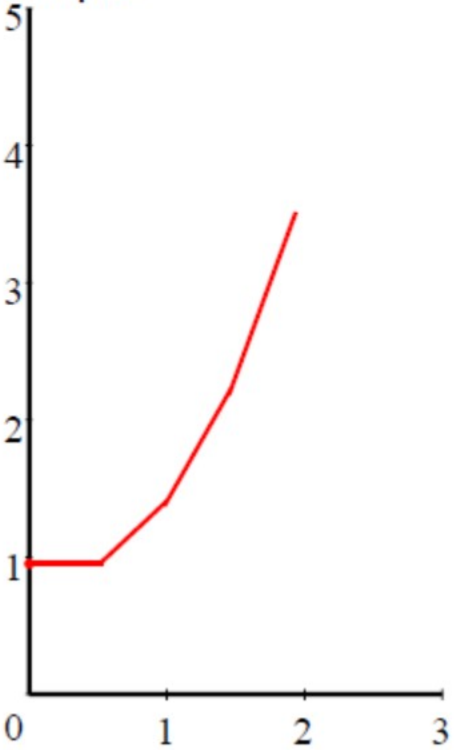
Euler's Method

$$\frac{dy}{dx} = 2x \qquad y_0 = 1 \qquad dx = 0.5$$

Lets practice another example

$$\frac{dy}{dx} = 2x$$

Initial value: $y_0 = 1$



(x_n, y_n)	$\frac{dy}{dx}$	dy	(x_{n+1}, y_{n+1})
$(0,1)$	0	0	$(.5,1)$
$(.5,1)$	1	.5	$(1,1.5)$
$(1,1.5)$	2	1	$(1.5,2.5)$

$$dy = \frac{dy}{dx} \cdot dx \qquad y_n + dy = y_{n+1}$$

Exact Solution

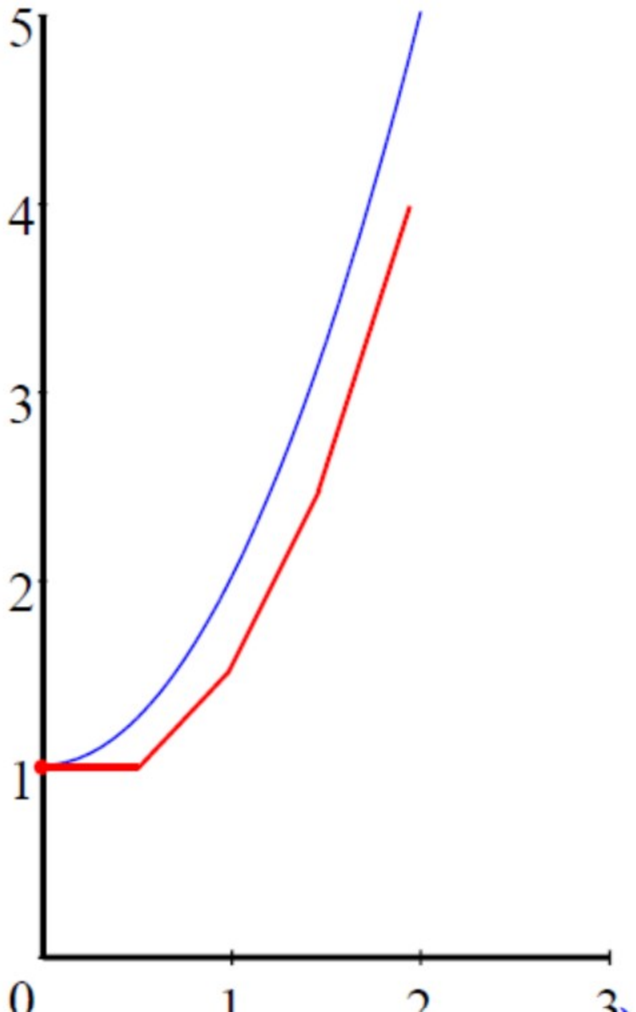
$$\frac{dy}{dx} = 2x$$

$$dy = 2x\ dx$$

$$y = x^2 + C$$

$$1 = 0 + C$$

$$y = x^2 + 1$$



Marked Labwork 1 (Week 9)

Use Euler's Method with increments of $\Delta x = .1$ to approximate the value of y when x = 1.3 and y = 3 when x = 1.

(x,y)	$\frac{dy}{dx} = y - 1$	Δx	$\left(\frac{dy}{dx}\right) \cdot \Delta x$	$(x + \Delta x, y + \Delta y)$

Marked Labwork 1 (Week 9)

Reproduce the values in table using R/Python and submit your notebook before you leave the lab. No sharing please ☺

Plot the values of x for t = 0, 0.5 …. 2.5

t	x	$\frac{dx}{dt}$
0	0.1	0.09
0.5	0.145	0.123975
1.0	0.206987	0.164144
1.5	0.289059	0.205504
2.0	0.391811	0.238295
2.5	??	??

Marked Labwork 2 (Week 9)

Use Euler method to solve the differential equation. Find the value of y_1 which should be the approximate solution at the first time step.

$$\frac{dy}{dt} = y(4 - y), \quad y(0) = 2,$$

$$\Delta t = h = 0.1$$