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Research article

Sub-optimal control of fuzzy linear dynamical systems under granular differentiability concept

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ABSTRACT

This paper deals with sub-optimal control of a fuzzy linear dynamical system. The aim is to keep the state variables of the fuzzy linear dynamical system close to zero in an optimal manner. In the fuzzy dynamical system, the fuzzy derivative is considered as the granular derivative; and all the coefficients and initial conditions can be uncertain. The criterion for assessing the optimality is regarded as a granular integral whose integrand is a quadratic function of the state variables and control inputs. Using the relative-distance-measure (RDM) fuzzy interval arithmetic and calculus of variations, the optimal control law is presented as the fuzzy state variables feedback. Since the optimal feedback gains are obtained as fuzzy functions, they need to be defuzzified. This will result in the sub-optimal control law. This paper also sheds light on the restrictions imposed by the approaches which are based on fuzzy standard interval arithmetic (FSIA), and use strongly generalized Hukuhara and generalized Hukuhara differentiability concepts for obtaining the optimal control law. The granular eigenvalues notion is also defined. Using an RLC circuit mathematical model, it is shown that, due to their unnatural behavior in the modeling phenomenon, the FSIA-based approaches may obtain some eigenvalues sets that might be different from the inherent eigenvalues set of the fuzzy dynamical system. This is, however, not the case with the approach proposed in this study. The notions of granular controllability and granular stabilizability of the fuzzy linear dynamical system are also presented in this paper. Moreover, a sub-optimal control for regulating a Boeing 747 in longitudinal direction with uncertain initial conditions and parameters is gained. In addition, an uncertain suspension system of one of the four wheels of a bus is regulated using the sub-optimal control introduced in this paper.

1. Introduction

Mathematical models play an important role in analyzing the behavior of dynamical phenomena. Linear dynamical systems are referred to as the dynamical phenomena whose mathematical model is considered as a system of linear differential equations. As a matter of fact, the linear dynamical system is a description of the main dynamical system - or dynamical phenomenon - whose model is approximated usually around an operating point. In control theory, linear dynamical systems are widely used for designing a controller as well as assessing some of dynamical system features such as stability, controllability, observability, etc. In almost all cases, the linear dynamical systems are considered with this assumption that all dynamical system parameters and conditions are certain. However, for some reasons such as error in measuring and identifying system parameters, dynamical system parameters and conditions are not always determined precisely and certainly. As a result, the uncertain linear dynamical system presents a more comprehensive description of the dynamical system than the crisp system does.

Fuzzy sets have been well known as effective tools for modeling uncertainty in many fields, including food science [1], mathematics [2–4], and medicine [5]. Thus, the uncertainty in the linear dynamical system can be taken into account as the fuzzy sets. Fuzzy linear dynamical systems are referred to as dynamical systems whose model is considered as a system of Fuzzy Differential Equations (FDEs). FDEs are those in which some parameters and/or boundary

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conditions are fuzzy sets. In recent years, FDEs have attracted more attention, and their applications in medicine [6], economy [7], fractional calculus [8–11], control theory [12], etc. show that they have been rapidly growing. One of the important branches in which FDEs can prove their advantages is optimal control theory. Nonetheless, compared to the enormous volume of the literature carried out for solving FDEs, little research has ever been performed in the field of optimal control theory based on FDEs.

Najariyan and Farahi studied fuzzy optimal control of a linear dynamical system with uncertain initial condition in Ref. [13] and with uncertain parameters in Ref. [14]. They used the notion of Strongly Generalized Hukuhara (SGH) differentiability in a fuzzy dynamical system and for designing the fuzzy optimal control. However, SGH derivative suffers from some limitations namely: 1. It does not always exist, 2. It necessitates the monotony of the uncertainty. To address the second limitation caused by SGH differentiability notion, they examined a class of fuzzy optimal control problems using generalized Hukuhara (gH) differentiability concept in Ref. [12]. Although the gH differentiability notion does not necessitate the monotony of the solution fuzziness, the existence of this kind of derivative cannot be guaranteed. Moreover, the approaches based on SGH differentiability and gH differentiability notions suffer from a main shortcoming called Unnatural Behavior in Modeling (UBM) phenomenon [15]. UBM phenomenon stems from this fact that, based on the fuzzy standard interval arithmetic (FSIA) the solutions of different forms of a fuzzy differential equation with the same structure may be different from each other. As a way of illustration, consider the fuzzy differential equations $\dot{\widetilde{x}}(t) = \widetilde{x}(t) + \widetilde{b}$, $\dot{\widetilde{x}}(t) - \widetilde{x}(t) = \widetilde{b}$, and $\dot{\widetilde{x}}(t) - \widetilde{b} = \widetilde{x}(t)$ where the initial conditions are the same and equal to a fuzzy number, $\widetilde{b}$ is a fuzzy number and the derivative of $\widetilde{x}(t)$, i.e. $\dot{\widetilde{x}}(t)$, is SGH or gH derivative. Then, the solutions obtained by solving the fuzzy differential equations may be different from each other, and this is the very UBM phenomenon. Moreover, there are some other drawbacks associated to the mentioned derivatives such as doubling property and multiplicity of solutions which were investigated in Ref. [15]. As a result, these limitations motivated us to examine fuzzy optimal control problem using an approach by which not only does a definition of fuzzy derivative exist, but also the mentioned shortcomings are handled.

With significant applications in clustering algorithms [16,17], measurement uncertainty [18], etc. granular computing has attracted more attention in recent years. Granularity can be considered as a concept which reflects detailed information. Piegat with his co-workers applied the granular computing in the crisp and fuzzy interval arithmetic. Piegat and Landowski [19–22] studied crisp interval arithmetic based on the notion of Relative-Distance-Measure (RDM) variables and introduced multidimensional RDM interval arithmetic. With successful results achieved, they showed how useful multidimensional RDM interval arithmetic is for the interval computations in comparision with standard interval arithmetic. Additionally, in the context of fuzzy mathematics, they introduced the notion of horizontal membership functions using the RDM variables [23,24]. With an approach called RDM fuzzy interval arithmetic introduced, they demonstrated that the result granule obtained by this approach is more fruitful than the result obtained by FSIA [25–28]. Afterwards, based on RDM interval arithmetic and the horizontal membership functions, Mazandarani et al. [15] proposed a new definition of the differentiability of fuzzy functions called granular differentiability (gr-differentiability). They proved that by considering FDEs under gr-differentiability and RDM interval arithmetic, the shortcomings, i.e. monotonic uncertainty, multiplicity of solutions, doubling property, and UBM phenomenon - that stem from existing definitions of fuzzy derivatives and FSIA - are successfully handled.

In this paper, the aim is to keep the state variables of a fuzzy linear dynamical system close to zero in an optimal manner. In the fuzzy dynamical system, the fuzzy derivative is considered as gr-derivative; and all the coefficients and initial conditions can be uncertain. The criteria for assessing the optimality is taken into account as a granular integral whose integrand is a quadratic function of the state variables and control inputs. Using the calculus of variations, the optimal control law is presented as the state variables feedback. Since the optimal feedback gains are obtained as fuzzy functions, they need to be defuzzified, and this results in the sub-optimal control law. The granular eigenvalues, controllability and stabilizability of the fuzzy linear dynamical system are also defined in this paper. Moreover, based on the obtained results, sub-optimal control for regulating a Boeing 747 in longitudinal direction with uncertain initial conditions and parameters is achieved.

2. Preliminaries

This section presents some necessary definitions, theorems, and propositions which will be used in this paper. Throughout this paper, the set of all real numbers is denoted by $\mathbb{R}$, the set of all positive real numbers is denoted by $\mathbb{R}^+$, and the set of all the fuzzy numbers on $\mathbb{R}$ by E_1. The left and right end-points of μ-level sets of the fuzzy set $\widetilde{A}$, $[\widetilde{A}]^\mu$, are denoted by $\underline{A}^\mu$ and $\overline{A}^\mu$, respectively. The transpose of a matrix $Y = [y_{ij}]_{n\times n}, i, j = 1, \ldots, n$ is denoted by Y^T.

Definition 1. [29]. The fuzzy set $\widetilde{u} : \mathbb{R} \to [0, 1]$ is called a fuzzy number if it is normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy subsets of the real numbers. The fuzzy number $\widetilde{u}$ can be represented in a parametric form by the ordered pair of functions $(\underline{u}^\mu, \overline{u}^\mu)$, $0 \le \mu \le 1$, satisfying the following properties:

1. $\underline{u}^\mu$ is a bounded non-decreasing left continuous function in $(0, 1]$, and it is right continuous at $\mu = 0$,
2. $\overline{u}^\mu$ is a bounded non-increasing left continuous function in $(0, 1]$, and it is right continuous at $\mu = 0$,
3. $\underline{u}^\mu \le \overline{u}^\mu$.

Based on fuzzy standard interval arithmetic; addition, subtraction, and multiplication of two fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$ are characterized as follows, respectively:

$$\mathit{Addition} : [\widetilde{u} + \widetilde{v}]^\mu = [\underline{u}^\mu + \underline{v}^\mu, \overline{u}^\mu + \overline{v}^\mu],$$

$$\mathit{Subtraction} : [\widetilde{u} - \widetilde{v}]^\mu = [\underline{u}^\mu - \overline{v}^\mu, \overline{u}^\mu - \underline{v}^\mu],$$

$$\mathit{Multiplication} : [\widetilde{u}\widetilde{v}]^\mu = [\min\{\underline{u}^\mu\overline{v}^\mu, \underline{u}^\mu\underline{v}^\mu, \overline{u}^\mu\overline{v}^\mu, \overline{u}^\mu\underline{v}^\mu\}, \max\{\underline{u}^\mu\overline{v}^\mu, \underline{u}^\mu\underline{v}^\mu, \overline{u}^\mu\overline{v}^\mu, \overline{u}^\mu\underline{v}^\mu\}];$$

Note 1. Let $\widetilde{u}, \widetilde{v}, \widetilde{w} \in E_1$. Based on fuzzy standard interval arithmetic, it can be proved that, as a whole, $(\widetilde{u} + \widetilde{v})\widetilde{w} \ne \widetilde{u}\widetilde{w} + \widetilde{v}\widetilde{w}$ and $\widetilde{u} - \widetilde{u} \ne 0$.

Definition 2. The function $\widetilde{f} : [a, b] \subseteq \mathbb{R} \to E_1$ is called a fuzzy function. Moreover, $\widetilde{f} : [a, b] \subseteq \mathbb{R} \to E_1^n = \underbrace{E_1 \times E_1 \times \cdots \times E_1}_{n}$ is called an n-dimensional vector of fuzzy functions.

Definition 3. [30].The fuzzy function $\widetilde{f} : (a, b) \subseteq \mathbb{R} \to E_1$ is said to be Strongly Generalized Hukuhara (SGH) differentiable in the first form at $t \in (a, b)$, if there exists a fuzzy number $\frac{d\widetilde{f}(t)}{dt} \in E_1$ such that for $h > 0$ sufficiently near zero, there are $\widetilde{f}(t + h) \ominus \widetilde{f}(t)$, $\widetilde{f}(t) \ominus \widetilde{f}(t - h)$ and the limits

$$\lim_{h\to 0}\frac{\widetilde{f}(t + h) \ominus \widetilde{f}(t)}{h} = \lim_{h\to 0}\frac{\widetilde{f}(t) \ominus \widetilde{f}(t - h)}{h} = \frac{d\widetilde{f}(t)}{dt}$$

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Additionally, the fuzzy function $\widetilde{f}$ is said to be SGH differentiable in the second form at $t \in (a,b)$, if there exists a fuzzy number $\frac{d\widetilde{f}(t)}{dt} \in E_1$ such that for $h > 0$ sufficiently near zero, there are $\widetilde{f}(t) \ominus \widetilde{f}(t+h)$, $\widetilde{f}(t-h) \ominus \widetilde{f}(t)$ and the limits

$$\lim_{h\to 0}\frac{\widetilde{f}(t)\ominus\widetilde{f}(t+h)}{-h} = \lim_{h\to 0}\frac{\widetilde{f}(t-h)\ominus\widetilde{f}(t)}{-h} = \frac{d\widetilde{f}(t)}{dt}$$

where "$\ominus$" stands for the Hukuhara difference and means $\widetilde{u} \ominus \widetilde{v} = \widetilde{z} \Leftrightarrow \widetilde{u} = \widetilde{v} + \widetilde{z}$.

Theorem 1. [31]. Let $\widetilde{f} : (a,b) \subseteq \mathbb{R} \to E_1$ be a fuzzy function such that $[\widetilde{f}(t)]^\mu = [\underline{f}^\mu(t), \overline{f}^\mu(t)]$. Then,

1. If $\widetilde{f}$ is SGH differentiable in the first form, then $\underline{f}^\mu(t)$ and $\overline{f}^\mu(t)$ are differentiable functions and $[\frac{d\widetilde{f}(t)}{dt}]^\mu = [\frac{d\underline{f}^\mu(t)}{dt}, \frac{d\overline{f}^\mu(t)}{dt}]$,
2. If $\widetilde{f}$ is SGH differentiable in the second form, then $\underline{f}^\mu(t)$ and $\overline{f}^\mu(t)$ are differentiable functions and $[\frac{d\widetilde{f}(t)}{dt}]^\mu = [\frac{d\overline{f}^\mu(t)}{dt}, \frac{d\underline{f}^\mu(t)}{dt}]$.

Definition 4. [32]. The fuzzy function $\widetilde{f} : (a,b) \subseteq \mathbb{R} \to E_1$ is said to be generalized Hukuhara (gH) differentiable at $t \in (a,b)$, if there exists a fuzzy number $\frac{d\widetilde{f}(t)}{dt} \in E_1$ such that the following limit exists

$$\lim_{h\to 0}\frac{\widetilde{f}(t+h)\ominus_{gh}\widetilde{f}(t)}{h} = \frac{d\widetilde{f}(t)}{dt}$$

where "$\ominus_{gh}$" stands for the generalized Hukuhara difference and means $\widetilde{u}\ominus_{gh}\widetilde{v} = \widetilde{z} \Leftrightarrow \widetilde{u} = \widetilde{v} + \widetilde{z}$ or $\widetilde{u} - \widetilde{z} = \widetilde{v}$.

Theorem 2. [32]. Let $\widetilde{f} : (a,b) \subseteq \mathbb{R} \to E_1$ be a fuzzy function such that $[\widetilde{f}(t)]^\mu = [\underline{f}^\mu(t), \overline{f}^\mu(t)]$, and the functions $\underline{f}^\mu(t), \overline{f}^\mu(t)$ be differentiable with respect to t, uniformly in $\mu \in [0,1]$. Then, the fuzzy function $\widetilde{f}$ is gH differentiable at $t \in (a,b)$ if and only if one of the following conditions occurs:

1. $\frac{d\underline{f}^\mu(t)}{dt}$ and $\frac{d\overline{f}^\mu(t)}{dt}$ are increasing and decreasing functions of μ, respectively, and $\frac{d\underline{f}^{\mu=1}(t)}{dt} \le \frac{d\overline{f}^{\mu=1}(t)}{dt}$. This case is called gH differentiability in the first form.
2. $\frac{d\underline{f}^\mu(t)}{dt}$ and $\frac{d\overline{f}^\mu(t)}{dt}$ are decreasing and increasing functions of μ, respectively, and $\frac{d\underline{f}^{\mu=1}(t)}{dt} \ge \frac{d\overline{f}^{\mu=1}(t)}{dt}$. This case is called gH differentiability in the second form.

Moreover,

$$[\frac{d\widetilde{f}(t)}{dt}]^\mu = [\min\{\frac{d\overline{f}^\mu(t)}{dt}, \frac{d\underline{f}^\mu(t)}{dt}\}, \max\{\frac{d\overline{f}^\mu(t)}{dt}, \frac{d\underline{f}^\mu(t)}{dt}\}].$$

Definition 5. [15,23]. Let $\widetilde{u} : [a,b] \subseteq \mathbb{R} \to [0,1]$ be a fuzzy number. The horizontal membership function $u^{gr} : [0,1]\times[0,1] \to [a,b]$ is a representation of $\widetilde{u}(x)$ as $u^{gr}(\mu,\alpha_u) = x$ in which "gr" stands for the granule of information included in $x \in [a,b]$, $\mu \in [0,1]$ is the membership degree of x in $\widetilde{u}(x)$, $\alpha_u \in [0,1]$ is called relative-distance-measure (RDM) variable, and $u^{gr}(\mu,\alpha_u) = \underline{u}^\mu + (\overline{u}^\mu - \underline{u}^\mu)\alpha_u$.

Note 2. The horizontal membership function of $\widetilde{u}(x) \in E_1$ is also denoted by $\mathcal{H}(\widetilde{u}) \triangleq u^{gr}(\mu,\alpha_u)$. Furthermore, if the triangular fuzzy number $\widetilde{u} \in E_1$ is denoted by the triple $(a,b,c), a \le b \le c$, then the horizontal membership function of $\widetilde{u} = (a,b,c)$ can be characterized as $\mathcal{H}(\widetilde{u}) = [a + (b-a)\mu] + [(1-\mu)(c-a)]\alpha_u$. For more illustration, Fig. 1 shows the triangular fuzzy number $\widetilde{u} = (3,6,10)$ and its horizontal membership function.

Definition 6. [15]. Two fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$ are said to be equal if and only if $\mathcal{H}(\widetilde{u}) = \mathcal{H}(\widetilde{v})$ for all $\alpha_u = \alpha_v \in [0,1]$.

Note 3. The μ-level sets of $\widetilde{u} \in E_1$ which are in fact the span of the information granule can be obtained using

$$\mathcal{H}^{-1}(u^{gr}(\mu,\alpha_u)) = [\widetilde{u}]^\mu = \left[\inf_{\beta\ge\mu}\min_{\alpha_u} u^{gr}(\beta,\alpha_u), \sup_{\beta\ge\mu}\max_{\alpha_u} u^{gr}(\beta,\alpha_u)\right]$$

What follows presents the four basic operations defined in RDM fuzzy interval arithmetic.

Definition 7. [15]. Let $\widetilde{u}$ and $\widetilde{v}$ be two fuzzy numbers whose horizontal membership functions are $u^{gr}(\mu,\alpha_u)$ and $v^{gr}(\mu,\alpha_v)$, respectively, and "$\odot$" denote one of the four basic operations, i.e. addition, subtraction, multiplication, and division. Then, $\widetilde{u}\odot\widetilde{v}$ is a fuzzy number $\widetilde{m}$ such that $\mathcal{H}(\widetilde{m}) \triangleq u^{gr}(\mu,\alpha_u) \odot v^{gr}(\mu,\alpha_v)$. It should be noted that $0 \notin v^{gr}(\mu,\alpha_v)$ when "$\odot$" denotes the division operator.

The difference between two fuzzy numbers defined in Definition 7 is called granular difference (gr-difference).

Definition 8. Let $\widetilde{u}$ and $\widetilde{v}$ be two fuzzy numbers. We say that $\widetilde{u} \ge \widetilde{v}$ if and only if $\mathcal{H}(\widetilde{u}) \ge \mathcal{H}(\widetilde{v})$ for all $\alpha_u = \alpha_v \in [0,1]$, $\mu \in [0,1]$.

Note 4. [15]. Based on RDM fuzzy interval arithmetic, the following relations hold for $\widetilde{u}, \widetilde{v}, \widetilde{w} \in E_1$:

1. $\widetilde{u} - \widetilde{v} = -(\widetilde{v} - \widetilde{u})$,
2. $\widetilde{u} - \widetilde{u} = 0$,
3. $\widetilde{u} \div \widetilde{u} = 1$,
4. $(\widetilde{u} + \widetilde{v})\widetilde{w} = \widetilde{u}\widetilde{w} + \widetilde{v}\widetilde{w}$.

Definition 9. [15]. Let $\widetilde{u}, \widetilde{v} \in E_1$. The function $D_{gr} : E_1 \times E_1 \to \mathbb{R}^+ \cup \{0\}$, $D_{gr}(\widetilde{u},\widetilde{v}) = \sup_\mu \max_{\alpha_u,\alpha_v} |u^{gr}(\mu,\alpha_u) - v^{gr}(\mu,\alpha_v)|$ is a distance between two fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$.

Note 5. The function D_{gr} is a metric on the space of fuzzy numbers and is called granular metric. Moreover, the metric space (E_1, D_{gr}) is a complete metric space - see Ref. [15] for more details.

Definition 10. [15]. Let $\widetilde{f} : [a,b] \subseteq \mathbb{R} \to E_1$ include $n \in \mathbb{N}$ distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n$. The horizontal membership function of $\widetilde{f}(t)$ at the point $t \in [a,b]$ is denoted by $\mathcal{H}(\widetilde{f}(t)) \triangleq f^{gr}(t,\mu,\alpha_f)$, and defined as $f^{gr} : [a,b]\times[0,1]\times\underbrace{[0,1]\times\cdots\times[0,1]}_{n} \to [c,d] \subseteq \mathbb{R}$ in which $\alpha_f \triangleq (\alpha_{u_1}, \alpha_{u_2}, \ldots, \alpha_{u_n})$ are the RDM variables corresponding to the fuzzy numbers.

Definition 11. [15]. The fuzzy function $\widetilde{f} : [a,b] \subseteq \mathbb{R} \to E_1$ mapping $t \mapsto \widetilde{f}(t)$ is said to be continuous if the following conditions hold:

1. $\forall t_0 \in (a,b), \forall \epsilon > 0, \exists \delta > 0 \ni |t - t_0| < \delta$ $\to D_{gr}\left(\widetilde{f}(t), \widetilde{f}(t_0)\right) < \epsilon$, and
2. $\forall \epsilon > 0, \exists \delta > 0 \ni 0 < t - a < \delta \to D_{gr}\left(\widetilde{f}(t), \widetilde{f}(a)\right) < \epsilon$, and
3. $\forall \epsilon > 0, \exists \delta > 0 \ni -\delta < t - b < 0 \to D_{gr}\left(\widetilde{f}(t), \widetilde{f}(b)\right) < \epsilon$.

Definition 12. [15]. The fuzzy function $\widetilde{f} : [a,b] \subseteq \mathbb{R} \to E_1$ is said to be granular differentiable (gr-differentiable) at $t \in [a,b]$ if there exists a fuzzy number $\frac{d\widetilde{f}(t)}{dt} \in E_1$ such that the following limit exists:

$$\lim_{h\to 0}\frac{\widetilde{f}(t+h) - \widetilde{f}(t)}{h} = \frac{d\widetilde{f}(t)}{dt} \quad (1)$$

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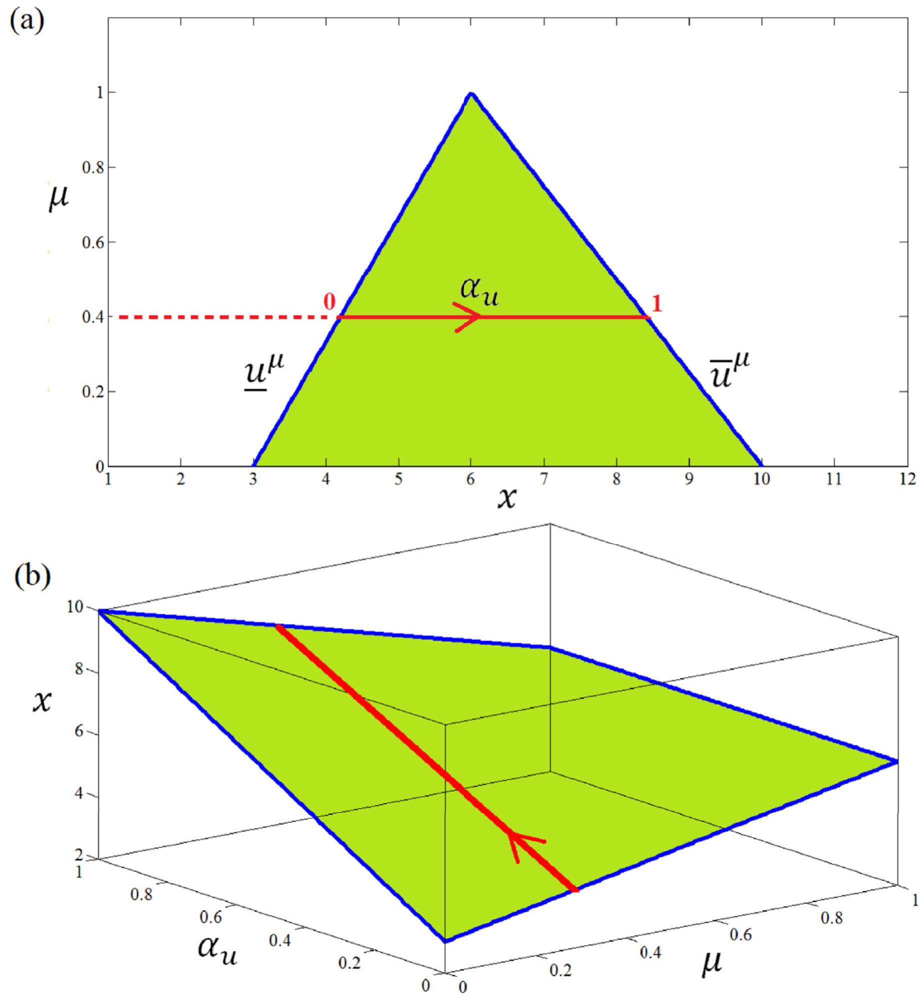

Fig. 1. The triangular fuzzy number $\widetilde{u} = (3, 6, 10)$ (a), and its horizontal membership function (b).

The limit is taken in the metric space (E_1, D_{gr}).

Theorem 3. [15]. The fuzzy function $\widetilde{f} : [a, b] \subseteq \mathbb{R} \to E_1$ is granular differentiable at the point $t \in [a, b]$ if and only if its horizontal membership function is differentiable with respect to t at that point. Moreover, $\mathcal{H}\left(\frac{d\widetilde{f}(t)}{dt}\right) = \frac{\partial f^{gr}(t,\mu,\alpha_f)}{\partial t}$.

Definition 13. [15]. Let $\widetilde{f} : [a, b] \subseteq \mathbb{R} \to E_1$ mapping $t \mapsto \widetilde{f}(t)$ be a continuous fuzzy function whose horizontal membership function, i.e. $f^{gr}(t, \mu, \alpha_f)$, is integrable on $t \in [a, b]$. Let $\int_a^b \widetilde{f}(t)dt$ denote the integral of $\widetilde{f}$ on $[a, b]$. Then, the fuzzy function $\widetilde{f}$ is said to be granular integrable (gr-integrable) on $[a, b]$ if there exists a fuzzy number $\widetilde{m} = \int_a^b \widetilde{f}(t)dt$ such that $\mathcal{H}(\widetilde{m}) = \int_a^b \mathcal{H}(\widetilde{f}(t))dt$.

Proposition 1. [15]. Suppose the fuzzy function $\widetilde{F} : [a, b] \subseteq \mathbb{R} \to E_1$ mapping $t \mapsto \widetilde{F}(t)$ is gr-differentiable, and $\widetilde{f}(t) = \frac{d\widetilde{F}(t)}{dt}$ is continuous on $[a, b]$. Then, $\int_a^b \widetilde{f}(t)dt = \widetilde{F}(b) - \widetilde{F}(a)$.

Proposition 2. Let the fuzzy functions $\widetilde{f} : [a, b] \subseteq \mathbb{R} \to E_1$ mapping $t \mapsto \widetilde{f}(t)$ and $\widetilde{g} : [a, b] \subseteq \mathbb{R} \to E_1$ mapping $t \mapsto \widetilde{g}(t)$ satisfy the conditions mentioned in Definition 13. Then, the relation $\int_a^b \left(\widetilde{f}(t) + \widetilde{g}(t)\right) dt = \int_a^b \widetilde{f}(t)dt + \int_a^b \widetilde{g}(t)dt$ holds.

3. Sub-optimal control of fuzzy dynamical linear systems

This section aims at finding a fuzzy optimal control so as to regulate a fuzzy dynamical linear system in an optimal way. For this purpose, the optimal control of fuzzy dynamical linear systems problem is introduced at first, then using the calculus of variations the problem is solved.

Consider the following fuzzy linear dynamical system:

$$\begin{cases} \dot{\widetilde{X}}(t) = \widetilde{A}\widetilde{X}(t) + \widetilde{B}\widetilde{U}(t) \\ \widetilde{X}(t_0) = \widetilde{X}_0 \\ t \in [t_0, t_f] \end{cases} \tag{2}$$

where $\widetilde{X}(t) \triangleq [\widetilde{x}_1(t), \widetilde{x}_2(t), \ldots, \widetilde{x}_n(t)]^T$, $\widetilde{X} : [t_0, t_f] \subseteq \mathbb{R}^+ \bigcup \{0\} \to E_1^n$, is the states vector of the system, $\widetilde{U}(t) \triangleq [\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_m(t)]^T$, $\widetilde{U} : [t_0, t_f] \subseteq \mathbb{R}^+ \bigcup \{0\} \to E_1^m$ is the vector of control functions, and $\dot{\widetilde{X}}(t) \triangleq [\dot{\widetilde{x}}_1(t), \dot{\widetilde{x}}_2(t), \ldots, \dot{\widetilde{x}}_n(t)]^T$ in which the derivatives are in the sense of gr-derivative. The matrices $\widetilde{A} = [\widetilde{a}_{ij}]_{n\times n}, i, j = 1, \ldots, n$, and $\widetilde{B} = [\widetilde{b}_{ik}]_{n\times m}, i = 1, \ldots, n, k = 1, \ldots, m$, are fuzzy matrices meaning $\widetilde{a}_{ij}, \widetilde{b}_{ik} \in E_1$. The initial condition is known and equal to the vector of fuzzy numbers $\widetilde{X}_0 \in E_1^n$, and the final condition, i.e. $\widetilde{X}(t_f) \in E_1^n$, is free. In the fuzzy optimal control problem considered here, the fuzzy control functions are meant to be found such that the states of uncertain system (2) are maintained close to the origin and the following performance measure

$$\widetilde{J}(\widetilde{U})=\frac{1}{2}\widetilde{X}^T(t_f)P\widetilde{X}(t_f)+\frac{1}{2}\int_{t_0}^{t_f}\left(\widetilde{X}^T(t)Q\widetilde{X}(t)+\widetilde{U}^T(t)R\widetilde{U}(t)\right)dt \tag{3}$$

is minimized. In the performance measure (3), the integral operator is in the sense of the granular integral defined in Definition 13; P, Q are positive semi-definite symmetric matrices, and the matrix R is a positive definite symmetric matrix. It should be noted that $\widetilde{J}$ assigns to each $\widetilde{U}$ a unique fuzzy number. Some definitions and a proposition are needed to be introduced before we proceed to solve the fuzzy optimal control problem.

Definition 14. The fuzzy eigenvalues of the fuzzy matrix $\widetilde{A}=[\widetilde{a}_{ij}]_{n\times n}, i,j=1,\dots,n$, denoted by $\widetilde{\lambda}_i$ are the roots of the fuzzy characteristic polynomial $det\left(\widetilde{\lambda}_i I_{n\times n}-\widetilde{A}\right)$, where $det(\bullet)$ means determinant, and $I_{n\times n}$ is the $n\times n$ identity matrix.

As we know, the poles of the open-loop linear dynamical system $\dot{X}(t)=AX(t)+BU(t)$ just depends on the eigenvalues of matrix A. However, as a whole, based on each of the concepts of SGH differentiability and gH differentiability the sets of eigenvalues corresponding to the following open-loop fuzzy dynamical systems:

$$\begin{cases}\dot{\widetilde{X}}(t)=A\widetilde{X}(t)+B\widetilde{U}(t)\\ \widetilde{X}(t_0)=\widetilde{X}_0\\ t\in[t_0,t_f]\end{cases} \tag{4}$$

$$\begin{cases}\dot{\widetilde{X}}(t)-A\widetilde{X}(t)=B\widetilde{U}(t)\\ \widetilde{X}(t_0)=\widetilde{X}_0\\ t\in[t_0,t_f]\end{cases} \tag{5}$$

may be not only different from each other, but also be different from the set of eigenvalues of the matrix A. This is while, based on Definition 14, the sets of eigenvalues of fuzzy dynamical systems (4) and (5) are always the same, and they are equal to the set of eigenvalues of the matrix A. For more illustration, see Example 1. In the following the concepts of granular controllability and stabilizability of fuzzy dynamical system (2) are presented. These concepts are similar to those introduced for the crisp dynamical system in Ref. [33].

Definition 15. A fuzzy linear dynamical system is said to be granular controllable if there exists a fuzzy control input $\widetilde{U}(t)$ which can transfer the system states from any fuzzy initial condition $\widetilde{X}(t_0)$ to any fuzzy final condition $\widetilde{X}(t_f)$ on the finite time interval $[t_0,t_f]$.

Proposition 3. Consider fuzzy dynamical system (2) and let $A^{gr}(\mu,\alpha_A)$ and $B^{gr}(\mu,\alpha_B)$ be the horizontal membership functions of the matrices $\widetilde{A}$ and $\widetilde{B}$, respectively. Then, fuzzy dynamical system (2) is granular controllable if and only if the granular matrix

$$[B^{gr}(\mu,\alpha_B)\quad A^{gr}(\mu,\alpha_A)B^{gr}(\mu,\alpha_B)\quad (A^{gr}(\mu,\alpha_A))^2B^{gr}(\mu,\alpha_B)\dots$$
$$(A^{gr}(\mu,\alpha_A))^{n-1}B^{gr}(\mu,\alpha_B)] \tag{6}$$

has full row rank $\forall\mu,\alpha_A,\alpha_B\in[0,1]$.

Proof. Based on Definition 6 and Theorem 3, fuzzy dynamical system (2) can be equivalently rewritten in the granular form as follows

$$\frac{\partial X^{gr}(t,\mu,\alpha_X)}{\partial t}=A^{gr}(\mu,\alpha_A)X^{gr}(t,\mu,\alpha_X)+B^{gr}(\mu,\alpha_B)U^{gr}(t,\mu,\alpha_U) \tag{7}$$

with the initial condition $X^{gr}(t_0,\mu,\alpha_X)=X_0^{gr}(\mu,\alpha_{X_0})$. Let $\alpha_X=\alpha_{X_c}, \alpha_A=\alpha_{A_c}, \alpha_B=\alpha_{B_c}, \alpha_U=\alpha_{U_c}$ and $\mu=\mu_c$ where $\alpha_{X_c},\alpha_{A_c},\alpha_{B_c},\alpha_{U_c}$ and μ_c are constant values belonging to $[0,1]$. Thus, dynamical system (7) can be considered as

$$\dot{X}^{gr}_{\mu_c,\alpha_{X_c}}(t)=A^{gr}_{\mu_c,\alpha_{A_c}}X^{gr}_{\mu_c,\alpha_{X_c}}(t)+B^{gr}_{\mu_c,\alpha_{B_c}}U^{gr}_{\mu_c,\alpha_{U_c}}(t) \tag{8}$$

where $A^{gr}_{\mu_c,\alpha_{A_c}}$ and $B^{gr}_{\mu_c,\alpha_{B_c}}$ are crisp constant matrices. It is well-known that the crisp dynamical system (8) is controllable if and only if the matrix

$$[B^{gr}_{\mu_c,\alpha_{B_c}}\quad A^{gr}_{\mu_c,\alpha_{A_c}}B^{gr}_{\mu_c,\alpha_{B_c}}\quad (A^{gr}_{\mu_c,\alpha_{A_c}})^2B^{gr}_{\mu_c,\alpha_{B_c}}\quad\cdots\quad (A^{gr}_{\mu_c,\alpha_{A_c}})^{n-1}B^{gr}_{\mu_c,\alpha_{B_c}}]$$

has full row rank. Therefore, it can be concluded that, granular dynamical system (7) is controllable if and only if the granular matrix (6) has full row rank $\forall\mu,\alpha_A,\alpha_B\in[0,1]$. As a result, since dynamical system (7), based on horizontal membership functions, is a representation of fuzzy dynamical system (2) in the granular form, then correspondingly the fuzzy dynamical system is granular controllable if and only if dynamical system (7) is controllable. □

Definition 16. Fuzzy dynamical system (2) is said to be granular stabilizable if and only if granular dynamical system (7) is stabilizable $\forall\mu,\alpha_A,\alpha_B,\alpha_X,\alpha_U\in[0,1]$.

Hereafter, we suppose fuzzy dynamical system (2) is granular stabilizable. What follows shows that how the mentioned optimal fuzzy control problem is converted to a problem which is similar to the well-known crisp linear quadratic regulator problem; and also how the system states and fuzzy control functions are determined. For the sake of simplicity, suppose

$$\widetilde{\phi}\left(\widetilde{X}(t)\right)\triangleq\frac{1}{2}\widetilde{X}^T(t)P\widetilde{X}(t)$$

$$\widetilde{L}\left(\widetilde{X}(t),\widetilde{U}(t)\right)\triangleq\frac{1}{2}\left(\widetilde{X}^T(t)Q\widetilde{X}(t)+\widetilde{U}^T(t)R\widetilde{U}(t)\right)$$

Since $\widetilde{\phi}$ is gr-differentiable, then based on Proposition 1 the following relation can be written

$$\widetilde{\phi}\left(\widetilde{X}(t_f)\right)=\int_{t_0}^{t_f}\left[\frac{d\widetilde{\phi}\left(\widetilde{X}(t)\right)}{dt}\right]dt+\widetilde{\phi}\left(\widetilde{X}(t_0)\right)$$

which helps us to rewrite the performance measure as

$$\widetilde{J}(\widetilde{U})=\widetilde{\phi}\left(\widetilde{X}(t_0)\right)+\int_{t_0}^{t_f}\left[\widetilde{L}\left(\widetilde{X}(t),\widetilde{U}(t)\right)+\frac{d\widetilde{\phi}\left(\widetilde{X}(t)\right)}{dt}\right]dt \tag{9}$$

Since $\widetilde{\phi}\left(\widetilde{X}(t_0)\right)$ is a vector of known fuzzy numbers, then it does not affect minimizing $\widetilde{J}(\widetilde{U})$, and therefore it can be neglected. Using Note 4, and the fuzzy Lagrange multipliers vector defined as $\widetilde{\Lambda}:[t_0,t_f]\to E_1^n$ the performance measure is formed as

$$\widetilde{J}(\widetilde{U})=\int_{t_0}^{t_f}\left[\widetilde{L}\left(\widetilde{X}(t),\widetilde{U}(t)\right)+\frac{d\widetilde{\phi}\left(\widetilde{X}(t)\right)}{dt}\right.$$
$$\left.+\widetilde{\Lambda}^T(t)\left(\widetilde{A}\widetilde{X}(t)+\widetilde{B}\widetilde{U}(t)-\dot{\widetilde{X}}(t)\right)\right]dt \tag{10}$$

Now, suppose $\widetilde{X}^*(t)$ and $\widetilde{U}^*(t)$ are optimal fuzzy states vector and control functions vector, respectively, and $\widetilde{\Lambda}^*(t)$ corresponds to $\widetilde{X}^*(t)$, $\widetilde{U}^*(t)$. Consider the deformations $\widetilde{X}(t)=\widetilde{X}^*(t)+\delta\widetilde{X}(t)$, $\widetilde{U}(t)=\widetilde{U}^*(t)+\delta\widetilde{U}(t)$, $\widetilde{\Lambda}(t)=\widetilde{\Lambda}^*(t)+\delta\widetilde{\Lambda}(t)$, and $\dot{\widetilde{X}}(t)=\dot{\widetilde{X}}^*(t)+\delta\dot{\widetilde{X}}(t)$ obtained using the small variations $\delta\widetilde{X}(t)$, $\delta\widetilde{U}(t)$, $\delta\widetilde{\Lambda}(t)$, and $\delta\dot{\widetilde{X}}(t)$. Due to the fact that, $\widetilde{X}^*(t)$ and $\widetilde{U}^*(t)$ satisfy system (2) and minimize the performance measure, then the increment of $\widetilde{J}(\widetilde{U}^*)$ defined as $\triangle\widetilde{J}\triangleq\widetilde{J}(\widetilde{U})-\widetilde{J}(\widetilde{U}^*)$ must be non-negative, i.e. $\triangle\widetilde{J}\triangleq\widetilde{J}(\widetilde{U})-\widetilde{J}(\widetilde{U}^*)\geq 0$. Using Proposition 2, $\triangle\widetilde{J}$ can be written as:

$$\triangle\widetilde{J}=\int_{t_0}^{t_f}\left[\widetilde{L}\left(\widetilde{X}(t),\widetilde{U}(t)\right)-\widetilde{L}\left(\widetilde{X}^*(t),\widetilde{U}^*(t)\right)+\frac{d\widetilde{\phi}\left(\widetilde{X}(t)\right)}{dt}-\frac{d\widetilde{\phi}\left(\widetilde{X}^*(t)\right)}{dt}+\widetilde{\Lambda}^T(t)\left(\widetilde{A}\widetilde{X}(t)+\widetilde{B}\widetilde{U}(t)-\dot{\widetilde{X}}(t)\right)-\widetilde{\Lambda}^{*T}(t)\left(\widetilde{A}\widetilde{X}^*(t)+\widetilde{B}\widetilde{U}^*(t)-\dot{\widetilde{X}}^*(t)\right)\right]dt$$

and according to Definitions 6 and 8, we have $\mathcal{H}(\triangle\widetilde{J})=\mathcal{H}\left(\widetilde{J}(\widetilde{U})-\widetilde{J}(\widetilde{U}^*)\right)$ meaning

$$\mathcal{H}(\triangle\widetilde{J})=\mathcal{H}\left(\int_{t_0}^{t_f}\left[\widetilde{L}\left(\widetilde{X}(t),\widetilde{U}(t)\right)-\widetilde{L}\left(\widetilde{X}^*(t),\widetilde{U}^*(t)\right)+\frac{d\widetilde{\phi}\left(\widetilde{X}(t)\right)}{dt}-\frac{d\widetilde{\phi}\left(\widetilde{X}^*(t)\right)}{dt}+\widetilde{\Lambda}^T(t)\left(\widetilde{A}\widetilde{X}(t)+\widetilde{B}\widetilde{U}(t)-\dot{\widetilde{X}}(t)\right)-\widetilde{\Lambda}^{*T}(t)\left(\widetilde{A}\widetilde{X}^*(t)+\widetilde{B}\widetilde{U}^*(t)-\dot{\widetilde{X}}^*(t)\right)\right]dt\right)$$

Based on Theorem 3 and Definition 13, the following relation is obtained:

$$\begin{aligned}\mathcal{H}(\triangle\widetilde{J})=\int_{t_0}^{t_f}&\left[L^{gr}\left(X^{gr},U^{gr},\mu,\alpha_L\right)-L^{gr}\left(X^{*gr},U^{*gr},\mu,\alpha_{L^*}\right)\right.\\&+\frac{\partial\phi^{gr}\left(X^{gr},\mu,\alpha_\phi\right)}{\partial t}-\frac{\partial\phi^{gr}\left(X^{*gr},\mu,\alpha_{\phi^*}\right)}{\partial t}\\&+\Lambda^{grT}\left(A^{gr}(\mu,\alpha_A)X^{gr}+B^{gr}(\mu,\alpha_B)U^{gr}-\frac{\partial X^{gr}}{\partial t}\right)\\&\left.-\Lambda^{*grT}\left(A^{gr}(\mu,\alpha_A)X^{*gr}+B^{gr}(\mu,\alpha_B)U^{*gr}-\frac{\partial X^{*gr}}{\partial t}\right)\right]dt\end{aligned}\tag{11}$$

where $X^{gr}\triangleq X^{*gr}+\delta X^{gr}$,

$X^{*gr}\triangleq X^{*gr}(t,\mu,\alpha_{X^*}),\quad \delta X^{gr}\triangleq\delta X^{gr}(t,\mu,\alpha_{\delta X}),$

$U^{gr}\triangleq U^{*gr}+\delta U^{gr}, U^{*gr}\triangleq U^{*gr}(t,\mu,\alpha_{U^*}),$

$\delta U^{gr}\triangleq\delta U^{gr}(t,\mu,\alpha_{\delta U}), \Lambda^{gr}\triangleq\Lambda^{*gr}+\delta\Lambda^{gr},$

$\Lambda^{*gr}\triangleq\Lambda^{*gr}(t,\mu,\alpha_{\Lambda^*}),\quad \delta\Lambda^{gr}\triangleq\delta\Lambda^{gr}(t,\mu,\alpha_{\delta\Lambda}).$

Relation (11) is similar to that could be obtained if the dynamical system was a crisp one. The difference between (11) and its corresponded relation for a crisp linear dynamical system is that in (11) we deal with multivariable functions $X^{*gr}(t,\mu,\alpha_{X^*})$, $U^{*gr}(t,\mu,\alpha_{U^*})$ and $\Lambda^{*gr}(t,\mu,\alpha_{\Lambda^*})$, etc. instead $X^*(t)$, $U^*(t)$, and $\Lambda^*(t)$, etc. Then, for minimizing, the first variations of $\mathcal{H}(\triangle\widetilde{J})$ with respect to X^{*gr}, $\frac{\partial X^{*gr}}{\partial t}$, U^{*gr}, and Λ^{*gr} must be zero. Suppose,

$$\begin{aligned}g(X^{gr},\dot{X}^{gr},U^{gr},\Lambda^{gr})=L^{gr}\left(X^{gr},U^{gr},\mu,\alpha_L\right)+\frac{\partial\phi^{gr}\left(X^{gr},\mu,\alpha_\phi\right)}{\partial t}\\+\Lambda^{grT}\left(A^{gr}(\mu,\alpha_A)X^{gr}+B^{gr}(\mu,\alpha_B)U^{gr}-\dot{X}^{gr}\right)\end{aligned}$$

where

$$\dot{X}^{gr}\triangleq\dot{X}^{*gr}+\delta\dot{X}^{gr},\dot{X}^{*gr}\triangleq\frac{\partial X^{*gr}}{\partial t},\quad \delta\dot{X}^{gr}\triangleq\frac{\partial(\delta X^{gr})}{\partial t}$$

Thus, relation (11) can be rewritten as

$$\begin{aligned}\mathcal{H}(\triangle\widetilde{J})=\int_{t_0}^{t_f}&\left[g(X^{*gr}+\delta X^{gr},\dot{X}^{*gr}+\delta\dot{X}^{gr},U^{*gr}+\delta U^{gr},\Lambda^{*gr}+\delta\Lambda^{gr})\right.\\&\left.-g(X^{*gr},\dot{X}^{*gr},U^{*gr},\Lambda^{*gr})\right]dt\end{aligned}\tag{12}$$

Expanding the integrand of (12) in a Taylor series about X^{*gr}, $\dot{X}^{*gr}$, U^{*gr}, and Λ^{*gr}; and considering the terms in the expansion which are linear in δX^{gr}, $\delta\dot{X}^{gr}$, δU^{gr}, and $\delta\Lambda^{gr}$ result in

$$\begin{aligned}\delta\mathcal{H}(\triangle\widetilde{J})=&\left[\frac{\partial g}{\partial\dot{X}^{gr}}(X^{*gr},\dot{X}^{*gr},U^{*gr},\Lambda^{*gr})\right]^T\Bigg|_{t=t_f}\delta X^{gr}\big|_{t=t_f}\\&+\int_{t_0}^{t_f}\left\{\left[\left[\frac{\partial g}{\partial X^{gr}}(X^{*gr},\dot{X}^{*gr},U^{*gr},\Lambda^{*gr})\right]^T-\frac{d}{dt}\left[\frac{\partial g}{\partial\dot{X}^{gr}}(X^{*gr},\dot{X}^{*gr},U^{*gr},\Lambda^{*gr})\right]^T\right]\delta X^{gr}\right.\\&+\left[\frac{\partial g}{\partial U^{gr}}(X^{*gr},\dot{X}^{*gr},U^{*gr},\Lambda^{*gr})\right]^T\delta U^{gr}\\&\left.+\left[\frac{\partial g}{\partial\Lambda^{gr}}(X^{*gr},\dot{X}^{*gr},U^{*gr},\Lambda^{*gr})\right]^T\delta\Lambda^{gr}\right\}dt\end{aligned}$$

which is the first variation of $\mathcal{H}(\triangle\widetilde{J})$. Then, by $\delta\mathcal{H}(\triangle\widetilde{J})=0$ the following results can be gained

$$\frac{\partial X^{*gr}(t,\mu,\alpha_{X^*})}{\partial t}=A^{gr}(\mu,\alpha_A)X^{*gr}(t,\mu,\alpha_{X^*})+B^{gr}(\mu,\alpha_B)U^{*gr}(t,\mu,\alpha_{U^*})\tag{13}$$

$$\frac{\partial\Lambda^{*gr}(t,\mu,\alpha_\Lambda)}{\partial t}=-QX^{*gr}(t,\mu,\alpha_{X^*})-A^{gr^T}(\mu,\alpha_A)\Lambda^{*gr}(t,\mu,\alpha_{\Lambda^*})\tag{14}$$

$$U^{*gr}(t,\mu,\alpha_U)=-R^{-1}B^{gr^T}(\mu,\alpha_B)\Lambda^{*gr}(t,\mu,\alpha_{\Lambda^*})\tag{15}$$

$$\Lambda^{*gr}(t_f,\mu,\alpha_{\Lambda^*})=PX^{*gr}(t_f,\mu,\alpha_{X^*})\tag{16}$$

Now, assume the relation between $X^{*gr}(t,\mu,\alpha_{X^*})$ and $\Lambda^{*gr}(t,\mu,\alpha_{\Lambda^*})$ is as

$$\Lambda^{*gr}(t,\mu,\alpha_{\Lambda^*})=S^{gr}(t,\mu,\alpha_S)X^{*gr}(t,\mu,\alpha_{X^*})\tag{17}$$

in which $S^{gr}(t,\mu,\alpha_S)$ is an unknown matrix function such that $S^{gr}(t_f,\mu,\alpha_S)=P$. If such a matrix function can be found, then the assumption is valid. By taking derivative from both side of (17) with respect to t, and using relations (13) and (15) we have

$$\begin{aligned}&\frac{\partial\Lambda^{*gr}(t,\mu,\alpha_{\Lambda^*})}{\partial t}\\&=\frac{\partial S^{gr}(t,\mu,\alpha_S)}{\partial t}X^{*gr}(t,\mu,\alpha_{X^*})+S^{gr}(t,\mu,\alpha_S)\frac{\partial X^{*gr}(t,\mu,\alpha_{X^*})}{\partial t}\\&=\frac{\partial S^{gr}(t,\mu,\alpha_S)}{\partial t}X^{*gr}(t,\mu,\alpha_{X^*})+S^{gr}(t,\mu,\alpha_S)\\&\quad\times\left(A^{gr}(\mu,\alpha_A)X^{*gr}(t,\mu,\alpha_{X^*})-B^{gr}(\mu,\alpha_B)R^{-1}B^{gr^T}(\mu,\alpha_B)S^{gr}(t,\mu,\alpha_S)X^{*gr}(t,\mu,\alpha_{X^*})\right)\end{aligned}$$

Then, by the aid of relations (14) and (17), we have

$$\begin{aligned}&-\frac{\partial S^{gr}(t,\mu,\alpha_S)}{\partial t}X^{*gr}(t,\mu,\alpha_{X^*})\\&=\left(A^{gr^T}(\mu,\alpha_A)S^{gr}(t,\mu,\alpha_S)+S^{gr}(t,\mu,\alpha_S)A^{gr}(\mu,\alpha_A)\right.\\&\quad\left.-S^{gr}(t,\mu,\alpha_S)B^{gr}(\mu,\alpha_B)R^{-1}B^{gr^T}(\mu,\alpha_B)S^{gr}(t,\mu,\alpha_S)+Q\right)\\&\quad\times X^{*gr}(t,\mu,\alpha_{X^*})\end{aligned}$$

Due to the fact that the relation above must hold for all the system states trajectories on $[t_0,t_f]$, therefore the following relation hold

$$-\frac{\partial S^{gr}(t,\mu,\alpha_S)}{\partial t} = A^{gr^T}(\mu,\alpha_A)S^{gr}(t,\mu,\alpha_S) + S^{gr}(t,\mu,\alpha_S)A^{gr}(\mu,\alpha_A) - S^{gr}(t,\mu,\alpha_S)B^{gr}(\mu,\alpha_B)R^{-1}B^{gr^T}(\mu,\alpha_B)S^{gr}(t,\mu,\alpha_S) + Q \tag{18}$$

whose boundary condition is $S^{gr}(t_f,\mu,\alpha_S) = P$. Moreover, it is easy to see that $S^{gr}(t,\mu,\alpha_S)$ is a symmetric matrix function and can be obtained by solving granular matrix differential equation (18). Suppose (18) was solved and the matrix function $S^{gr}(t,\mu,\alpha_S)$ is at disposal. Then, by defining the granular feedback gain $K(t,\mu,\alpha_k) = [k^{gr}_{ri}(t,\mu,\alpha_{k_{ri}})]_{m\times n}, r = 1,\dots,m, i = 1,\dots,n$ as

$$K(t,\mu,\alpha_k) = R^{-1}B^{gr^T}(\mu,\alpha_B)S^{gr}(t,\mu,\alpha_S) \tag{19}$$

the granular optimal control functions vector is as follows

$$U^{*gr}(t,\mu,\alpha_U) = -K(t,\mu,\alpha_k)X^{*gr}(t,\mu,\alpha_{X^*}) \tag{20}$$

Eventually, relations (13) and (18) to (20) - using Theorem 3, Definitions 4 and 6 - can be written as

$$\begin{cases} \dot{\widetilde{X}}^*(t) = \widetilde{A}\widetilde{X}^*(t) + \widetilde{B}\widetilde{U}^*(t) \\ \widetilde{U}^*(t) = -\widetilde{K}(t)\widetilde{X}^*(t) \\ \widetilde{K}(t) = R^{-1}\widetilde{B}^T\widetilde{S}(t) \\ -\dot{\widetilde{S}}(t) = \widetilde{A}^T\widetilde{S}(t) + \widetilde{S}(t)\widetilde{A} - \widetilde{S}(t)\widetilde{B}R^{-1}\widetilde{B}^T\widetilde{S}(t) + Q \\ \widetilde{X}^*(t_0) = \widetilde{X}^*_0 \\ \widetilde{S}(t_f) = P \\ t \in [t_0, t_f] \end{cases}$$

Based on the aforementioned, a theorem corresponding to the optimal control of fuzzy dynamical linear systems can be derived as follows.

Theorem 4. Consider the fuzzy dynamical linear system (2) and suppose it is stabilizable. The fuzzy optimal control functions vector which satisfies dynamical system (2) and minimizes performance measure (3) is as $\widetilde{U}^*(t) = -\widetilde{K}(t)\widetilde{X}^*(t)$ where the optimal fuzzy feedback gain $\widetilde{K}(t)$ is obtained as

$$\begin{cases} \widetilde{K}(t) = R^{-1}\widetilde{B}^T\widetilde{S}(t) \\ -\dot{\widetilde{S}}(t) = \widetilde{A}^T\widetilde{S}(t) + \widetilde{S}(t)\widetilde{A} - \widetilde{S}(t)\widetilde{B}R^{-1}\widetilde{B}^T\widetilde{S}(t) + Q \\ \widetilde{S}(t_f) = P \\ t \in [t_0, t_f] \end{cases} \tag{21}$$

The optimal fuzzy feedback gain $\widetilde{K}(t) = [\widetilde{k}_{ri}(t)]$ for each $t \in [t_0, t_f]$ is a fuzzy matrix. Then, for applying it to the dynamical system we need to defuzzify $\widetilde{K}(t)$. The defuzzified feedback gain is denoted by $K_c(t) = [k_{ri}(t)]$. It is noteworthy to pinpoint that, defuzzifying the optimal fuzzy feedback gain results in the crisp sub-optimal feedback gain $K_c(t) = [k_{ri}(t)]$ which turns out the sub-optimal control law $\widetilde{U}(t) = -K_c(t)\widetilde{X}(t)$. As a result, according to Note 4, the sub-optimal controlled fuzzy dynamical system with the control law $\widetilde{U}(t) = -K_c(t)\widetilde{X}(t)$ can be considered as a closed loop system and expressed as $\dot{\widetilde{X}}(t) = \left(\widetilde{A} - \widetilde{B}K_c(t)\right)\widetilde{X}(t)$. Fig. 2 shows the block diagram of the sub-optimal controlled fuzzy closed loop dynamical system.

Proposition 4. Applying the optimal control law $\widetilde{U}^*(t) = -\widetilde{K}(t)\widetilde{X}^*(t)$ for controlling the fuzzy dynamical system (2) results in the optimal cost $\widetilde{J}(\widetilde{U}^*) = \frac{1}{2}\widetilde{X}_0^T\widetilde{S}(t_0)\widetilde{X}_0$. □

Proof. The proof is straightforward and hence omitted.

3.1. Restrictions associated to SGH and gH differentiability

Investigating fuzzy optimal control problem under SGH and gH differentiability imposes some restrictions which are expressed in this section.

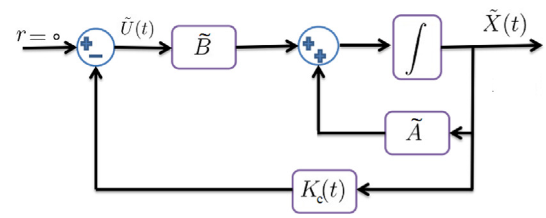

Fig. 2. The block diagram of sub-optimal control of fuzzy dynamical system (2).

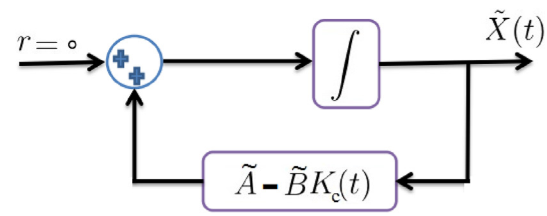

Fig. 3. The closed loop form of the sub-optimal controlled fuzzy dynamical system.

3.1.1. Multiplicity of the solutions for the fuzzy optimal control problem

According to the UBM phenomenon [15], as a whole, the solution of the following fuzzy dynamical systems

$$\begin{cases} \dot{\widetilde{X}}(t) = \widetilde{A}\widetilde{X}(t) + \widetilde{B}\widetilde{U}(t) \\ \widetilde{X}(t_0) = \widetilde{X}_0 \end{cases} \quad \begin{cases} \dot{\widetilde{X}}(t) - \widetilde{A}\widetilde{X}(t) = \widetilde{B}\widetilde{U}(t) \\ \widetilde{X}(t_0) = \widetilde{X}_0 \end{cases} \tag{22}$$

$$\begin{cases} \dot{\widetilde{X}}(t) - \widetilde{B}\widetilde{U}(t) = \widetilde{A}\widetilde{X}(t) \\ \widetilde{X}(t_0) = \widetilde{X}_0 \end{cases} \quad \begin{cases} \dot{\widetilde{X}}(t) - \widetilde{A}\widetilde{X}(t) - \widetilde{B}\widetilde{U}(t) = 0 \\ \widetilde{X}(t_0) = \widetilde{X}_0 \end{cases} \tag{23}$$

are not the same, based on each of the concepts of SGH differentiability and gH differentiability. Therefore, as a whole, there is no same optimal control for fuzzy dynamical systems (22) and (23). In other words, there are multiple optimal controls for a single uncertain dynamical phenomenon. Nevertheless, based on the gr-derivative and RDM fuzzy interval arithmetic approach, the fuzzy dynamical systems (22) and (23) have the same optimal control which can be characterized according to Theorem 4.

3.1.2. Disability in the use of fuzzy lagrange multipliers

According to Note 1, since, as a whole, $\widetilde{A}\widetilde{X}(t) + \widetilde{B}\widetilde{U}(t) - \dot{\widetilde{X}}(t) \neq 0$, then by applying each of the concepts of SGH differentiability and gH differentiability, one cannot use the advantage of the fuzzy Lagrange multipliers as it was used in relation (10).

3.1.3. Incompatibility with the closed loop form of the controlled system

Based on Note 1, the sub-optimal controlled fuzzy dynamical system with the control law $\widetilde{U}(t) = -K_c(t)\widetilde{X}(t)$, as a whole, cannot be represented in the closed loop form as $\dot{\widetilde{X}}(t) = \left(\widetilde{A} - \widetilde{B}K_c(t)\right)\widetilde{X}(t)$,

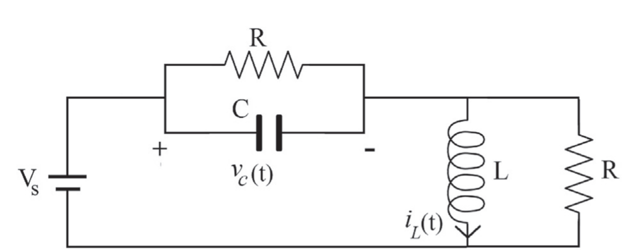

Fig. 4. The RLC circuit considered in Example 1.

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where the fuzzy derivative is considered as SGH or gH derivative. In other words, since $(\widetilde{u}+\widetilde{v})\widetilde{w} \neq \widetilde{u}\widetilde{w}+\widetilde{v}\widetilde{w}$, Fig. 2 is not equivalent with Fig. 3.

3.1.4. Technical difficulties in the process of solving fuzzy optimal control problem

Another more restriction associated SGH or gH differentiability concepts correspond to determine the μ-level sets of fuzzy dynamical systems (22) and (23). As a matter of fact, based on the mentioned concepts, in order to obtain the solution of a fuzzy dynamical system or analyzing its behavior, determining the left and right end-points of the μ-level sets of the fuzzy dynamical system is necessary. However, in the cases that fuzzy dynamical system includes terms in which the multiplication of two fuzzy numbers occurs, specifying the μ-level sets is not always an easy task and may be restricted to some particular cases, see e.g. Ref. [34]. For more illustration, consider the following fuzzy dynamical system

$$\begin{bmatrix}\dot{\widetilde{x}}_1(t)\\ \dot{\widetilde{x}}_2(t)\end{bmatrix} = \begin{bmatrix}\widetilde{a}_{11} & \widetilde{a}_{12}\\ \widetilde{a}_{21} & \widetilde{a}_{22}\end{bmatrix}\begin{bmatrix}\widetilde{x}_1(t)\\ \widetilde{x}_2(t)\end{bmatrix} + \begin{bmatrix}\widetilde{b}_{11}\\ \widetilde{b}_{21}\end{bmatrix}\widetilde{u}(t) \tag{24}$$

and suppose $\widetilde{x}_1$, $\widetilde{x}_2$ are SGH differentiable functions in the second form. Then, the left and right end-points of the μ-level sets of fuzzy dynamical system (24) are characterized as:

$$\begin{bmatrix}\dot{\underline{x}}_1^\mu(t)\\ \dot{\overline{x}}_1^\mu(t)\\ \dot{\underline{x}}_2^\mu(t)\\ \dot{\overline{x}}_2^\mu(t)\end{bmatrix} = \begin{bmatrix}\max\{\underline{a}_{11}^\mu\underline{x}_1^\mu(t), \underline{a}_{11}^\mu\overline{x}_1^\mu(t), \overline{a}_{11}^\mu\underline{x}_1^\mu(t), \overline{a}_{11}^\mu\overline{x}_1^\mu(t)\} + \max\{\underline{a}_{12}^\mu\underline{x}_2^\mu(t), \underline{a}_{12}^\mu\overline{x}_2^\mu(t), \overline{a}_{12}^\mu\underline{x}_2^\mu(t), \overline{a}_{12}^\mu\overline{x}_2^\mu(t)\}\\ \min\{\underline{a}_{11}^\mu\underline{x}_1^\mu(t), \underline{a}_{11}^\mu\overline{x}_1^\mu(t), \overline{a}_{11}^\mu\underline{x}_1^\mu(t), \overline{a}_{11}^\mu\overline{x}_1^\mu(t)\} + \min\{\underline{a}_{12}^\mu\underline{x}_2^\mu(t), \underline{a}_{12}^\mu\overline{x}_2^\mu(t), \overline{a}_{12}^\mu\underline{x}_2^\mu(t), \overline{a}_{12}^\mu\overline{x}_2^\mu(t)\}\\ \max\{\underline{a}_{21}^\mu\underline{x}_1^\mu(t), \underline{a}_{21}^\mu\overline{x}_1^\mu(t), \overline{a}_{21}^\mu\underline{x}_1^\mu(t), \overline{a}_{21}^\mu\overline{x}_1^\mu(t)\} + \max\{\underline{a}_{22}^\mu\underline{x}_2^\mu(t), \underline{a}_{22}^\mu\overline{x}_2^\mu(t), \overline{a}_{22}^\mu\underline{x}_2^\mu(t), \overline{a}_{22}^\mu\overline{x}_2^\mu(t)\}\\ \min\{\underline{a}_{21}^\mu\underline{x}_1^\mu(t), \underline{a}_{21}^\mu\overline{x}_1^\mu(t), \overline{a}_{21}^\mu\underline{x}_1^\mu(t), \overline{a}_{21}^\mu\overline{x}_1^\mu(t)\} + \min\{\underline{a}_{22}^\mu\underline{x}_2^\mu(t), \underline{a}_{22}^\mu\overline{x}_2^\mu(t), \overline{a}_{22}^\mu\underline{x}_2^\mu(t), \overline{a}_{22}^\mu\overline{x}_2^\mu(t)\}\end{bmatrix}$$
$$+\begin{bmatrix}\max\{\underline{b}_{11}^\mu\underline{u}^\mu(t), \underline{b}_{11}^\mu\overline{u}^\mu(t), \overline{b}_{11}^\mu\underline{u}^\mu(t), \overline{b}_{11}^\mu\overline{u}^\mu(t)\}\\ \min\{\underline{b}_{11}^\mu\underline{u}^\mu(t), \underline{b}_{11}^\mu\overline{u}^\mu(t), \overline{b}_{11}^\mu\underline{u}^\mu(t), \overline{b}_{11}^\mu\overline{u}^\mu(t)\}\\ \max\{\underline{b}_{21}^\mu\underline{u}^\mu(t), \underline{b}_{21}^\mu\overline{u}^\mu(t), \overline{b}_{21}^\mu\underline{u}^\mu(t), \overline{b}_{21}^\mu\overline{u}^\mu(t)\}\\ \min\{\underline{b}_{21}^\mu\underline{u}^\mu(t), \underline{b}_{21}^\mu\overline{u}^\mu(t), \overline{b}_{21}^\mu\underline{u}^\mu(t), \overline{b}_{21}^\mu\overline{u}^\mu(t)\}\end{bmatrix} \tag{25}$$

Due to the fact that, $\widetilde{x}_1(t), \widetilde{x}_2(t)$, and specially $\widetilde{u}(t)$, as a whole, are unknown functions, then the values of the minimum and maximum in the right hand side of relation (25) are not known and cannot be determined in explicit terms, as a whole. Therefore, the process for determining the optimal control has to deal with relation (25) which leads to technical difficulties.

4. Examples

Example 1. Consider the RLC circuit shown in Fig. 4 where $v_c(t), i_L(t)$ and $v_s(t)$ are the voltage of capacitor, the current flowing through the inductor and the source voltage, respectively. The values of resistor, inductor and capacitor are, respectively, R in ohm, L in Henry and C in Farad. The mathematical model of the RLC circuit can be expressed as:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}\frac{-2}{RC} & \frac{1}{C}\\ \frac{-1}{L} & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}\frac{1}{RC}\\ \frac{1}{L}\end{bmatrix}u(t) \tag{26}$$

or equivalently as

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} - \begin{bmatrix}\frac{-2}{RC} & \frac{1}{C}\\ \frac{-1}{L} & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}\frac{1}{RC}\\ \frac{1}{L}\end{bmatrix}u(t) \tag{27}$$

where $[x_1(t) \quad x_2(t)]^T \triangleq [v_c(t) \quad i_L(t)]^T$, $u(t) \triangleq v_s(t)$. With $R = 0.5, L = 1$ and $C = 0.1$ considered, the eigenvalues - i.e. poles - of the system are $\lambda_1 = -39.74$ and $\lambda_2 = -0.25$. Therefore, the system is invariably stable. Now, suppose the initial conditions are fuzzy numbers, i.e. uncertain. With gr-derivative concept and RDM fuzzy interval arithmetic considered, according to Definition 14, the eigenvalues of dynamical systems (26) and (27) with fuzzy initial conditions $\widetilde{x}_1(0), \widetilde{x}_2(0) \in E_1$, are $\lambda_1 = -39.74$ and $\lambda_2 = -0.25$. Thus, based on the proposed approach, the fuzzy dynamical systems (26) and (27) are stable. However, using the notion of SGH or gH differentiability in the first form and fuzzy standard interval arithmetic, dynamical systems (26) and (27) with fuzzy initial conditions are equivalent to the following dynamical systems, respectively:

$$\begin{bmatrix}\dot{\underline{x}}_1^\mu(t)\\ \dot{\overline{x}}_1^\mu(t)\\ \dot{\underline{x}}_2^\mu(t)\\ \dot{\overline{x}}_2^\mu(t)\end{bmatrix} = \begin{bmatrix}0 & \frac{-2}{RC} & \frac{1}{C} & 0\\ \frac{-2}{RC} & 0 & 0 & \frac{1}{C}\\ 0 & \frac{-1}{L} & 0 & 0\\ \frac{-1}{L} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\underline{x}_1^\mu(t)\\ \overline{x}_1^\mu(t)\\ \underline{x}_2^\mu(t)\\ \overline{x}_2^\mu(t)\end{bmatrix} + \begin{bmatrix}\frac{1}{RC}\\ \frac{1}{RC}\\ \frac{1}{L}\\ \frac{1}{L}\end{bmatrix}u(t) \tag{28}$$

and

$$\begin{bmatrix}\dot{\underline{x}}_1^\mu(t)\\ \dot{\overline{x}}_1^\mu(t)\\ \dot{\underline{x}}_2^\mu(t)\\ \dot{\overline{x}}_2^\mu(t)\end{bmatrix} - \begin{bmatrix}\frac{-2}{RC} & 0 & 0 & \frac{1}{C}\\ 0 & \frac{-2}{RC} & \frac{1}{C} & 0\\ \frac{-1}{L} & 0 & 0 & 0\\ 0 & \frac{-1}{L} & 0 & 0\end{bmatrix}\begin{bmatrix}\underline{x}_1^\mu(t)\\ \overline{x}_1^\mu(t)\\ \underline{x}_2^\mu(t)\\ \overline{x}_2^\mu(t)\end{bmatrix} = \begin{bmatrix}\frac{1}{RC}\\ \frac{1}{RC}\\ \frac{1}{L}\\ \frac{1}{L}\end{bmatrix}u(t) \tag{29}$$

The eigenvalues of dynamical system (28) are $\lambda_1 = -39.74, \lambda_2 = -0.25, \lambda_3 = -0.24, \lambda_4 = 40.24$; and those of system (29) are $\lambda_1 = -39.74, \lambda_2 = -0.25, \lambda_3 = -40.24$ and $\lambda_4 = 0.24$. As is seen, dynamical systems (28) and (29) not only are unstable, but also their eigenvalues sets are different from the eigenvalues set of RLC circuit.

Example 2. Consider the RLC circuit shown in Fig. 4. Suppose, the values of the inductor and capacitor are uncertain. Then, we have:

$$\begin{bmatrix}\dot{\widetilde{x}}_1(t)\\ \dot{\widetilde{x}}_2(t)\end{bmatrix} = \begin{bmatrix}-\frac{2}{R\widetilde{C}} & \frac{1}{\widetilde{C}}\\ -\frac{1}{\widetilde{L}} & 0\end{bmatrix}\begin{bmatrix}\widetilde{x}_1(t)\\ \widetilde{x}_2(t)\end{bmatrix} + \begin{bmatrix}\frac{1}{R\widetilde{C}}\\ \frac{1}{\widetilde{L}}\end{bmatrix}\widetilde{u}(t) \tag{30}$$

Let $R = 1$ and the uncertain values of the inductor and capacitor be respectively as $\widetilde{L} = \widetilde{C} = \widetilde{2}$, where $\widetilde{2} = (1, 2, 3)$ is the triangular fuzzy number. Moreover, the initial conditions may be uncertain.

Based on Proposition 3, the fuzzy dynamical system (30) is not controllable. The granular dynamical system corresponding to (30) is as

$$\begin{bmatrix} \dfrac{\partial x_1^{gr}(t,\mu,\alpha_{x_1})}{\partial t} \\ \dfrac{\partial x_2^{gr}(t,\mu,\alpha_{x_2})}{\partial t} \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{2^{gr}(\mu,\alpha_2)} & \dfrac{1}{2^{gr}(\mu,\alpha_2)} \\ -\dfrac{1}{2^{gr}(\mu,\alpha_2)} & 0 \end{bmatrix} \begin{bmatrix} x_1^{gr}(t,\mu,\alpha_{x_1}) \\ x_2^{gr}(t,\mu,\alpha_{x_2}) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{2^{gr}(\mu,\alpha_2)} \\ \dfrac{1}{2^{gr}(\mu,\alpha_2)} \end{bmatrix} u^{gr}(t,\mu,\alpha_u) \tag{31}$$

where $2^{gr}(\mu,\alpha_2) \triangleq \mathcal{H}(\widetilde{2})$. Using the state-coordinate change (i.e. similarity transformation) $Z^{gr} = TX^{gr}$ where

$$T = \begin{bmatrix} -\dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \\ \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{bmatrix}$$

the following granular dynamical system is obtained which is in fact controllability normal form.

$$\begin{bmatrix} \dfrac{\partial z_1^{gr}(t,\mu,\alpha_{z_1})}{\partial t} \\ \dfrac{\partial z_2^{gr}(t,\mu,\alpha_{z_2})}{\partial t} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{2^{gr}(\mu,\alpha_2)} & 0 \\ \dfrac{2}{2^{gr}(\mu,\alpha_2)} & -\dfrac{1}{2^{gr}(\mu,\alpha_2)} \end{bmatrix} \begin{bmatrix} z_1^{gr}(t,\mu,\alpha_{z_1}) \\ z_2^{gr}(t,\mu,\alpha_{z_2}) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{\sqrt{2}}{2^{gr}(\mu,\alpha_2)} \end{bmatrix} u^{gr}(t,\mu,\alpha_u) \tag{32}$$

As is seen, the eigenvalue of the uncontrollable part is $\frac{-1}{2^{gr}(\mu,\alpha_2)}$ which means the uncontrollable mode is stable. Then, the granular dynamical system (32) or equivalently (31) is stabilizable. Thus, based on Definition 16, the fuzzy dynamical system (30) - where $R = 1, \widetilde{L} = \widetilde{C} = \widetilde{2}$ - are also stabilizable.

Example 3. Consider the following fuzzy dynamical system of motion for the Boeing 747 in longitudinal direction adopted from Refs. [15,35].

$$\begin{bmatrix} \dot{\widetilde{u}}(t) \\ \dot{\widetilde{\alpha}}(t) \\ \dot{\widetilde{q}}(t) \\ \dot{\widetilde{\theta}}(t) \end{bmatrix} = \begin{bmatrix} \widetilde{X}_u & \widetilde{X}_w & 0 & \dfrac{-g\cos(\theta_0)}{\widetilde{u}_0} \\ \dfrac{\widetilde{Z}_u}{1-\widetilde{Z}_{\dot{w}}} & \dfrac{\widetilde{Z}_w}{1-\widetilde{Z}_{\dot{w}}} & \dfrac{\widetilde{u}_0+\widetilde{Z}_q}{\widetilde{u}_0(1-\widetilde{Z}_{\dot{w}})} & \dfrac{-g\sin(\theta_0)}{\widetilde{u}_0(1-\widetilde{Z}_{\dot{w}})} \\ \widetilde{u}_0\left(\widetilde{M}_u+\dfrac{\widetilde{M}_{\dot{w}}\widetilde{Z}_u}{1-\widetilde{Z}_{\dot{w}}}\right) & \widetilde{u}_0\left(\widetilde{M}_w+\dfrac{\widetilde{M}_{\dot{w}}\widetilde{Z}_w}{1-\widetilde{Z}_{\dot{w}}}\right) & \widetilde{M}_q+\dfrac{(\widetilde{u}_0+\widetilde{Z}_q)\widetilde{M}_{\dot{w}}}{1-\widetilde{Z}_{\dot{w}}} & \dfrac{-\widetilde{M}_{\dot{w}}g\sin(\theta_0)}{1-\widetilde{Z}_{\dot{w}}} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{u}(t) \\ \widetilde{\alpha}(t) \\ \widetilde{q}(t) \\ \widetilde{\theta}(t) \end{bmatrix} + \begin{bmatrix} \dfrac{X_{\delta_e}}{\widetilde{u}_0} & \dfrac{X_{\delta_T}}{\widetilde{u}_0} \\ \dfrac{Z_{\delta_e}}{\widetilde{u}_0(1-\widetilde{Z}_{\dot{w}})} & \dfrac{Z_{\delta_T}}{\widetilde{u}_0(1-\widetilde{Z}_{\dot{w}})} \\ M_{\delta_e}+\dfrac{\widetilde{M}_{\dot{w}}Z_{\delta_e}}{1-\widetilde{Z}_{\dot{w}}} & M_{\delta_T}+\dfrac{\widetilde{M}_{\dot{w}}Z_{\delta_T}}{1-\widetilde{Z}_{\dot{w}}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{\delta}_e(t) \\ \widetilde{\delta}_T(t) \end{bmatrix} \tag{33}$$

Table 1
Variables and parameters of the uncertain linear model of the Boeing 747 in longitudinal direction.

$g = 32.174$	$\widetilde{X}_u = \frac{-5.936}{\widetilde{u}_0}$	$X_{\delta_T} = 0$
$\widetilde{X}_w = \frac{13.048}{\widetilde{u}_0}$	$\widetilde{Z}_u = \frac{-64.568}{\widetilde{u}_0}$	$\theta_0 = 0$
$\widetilde{Z}_w = \frac{-169.064}{\widetilde{u}_0}$	$\widetilde{Z}_{\dot{w}} = \frac{-2673.4}{\widetilde{u}_0^2}$	$M_{\delta_e} = -0.5769$
$\widetilde{Z}_q = \frac{-2148.7}{\widetilde{u}_0}$	$\widetilde{M}_w = \frac{-0.532}{\widetilde{u}_0}$	$Z_{\delta_T} = -7.854$
$\widetilde{M}_u = 0$	$\widetilde{M}_{\dot{w}} = \frac{-15.68}{\widetilde{u}_0^2}$	$X_{\delta_e} = 0$
$\widetilde{M}_q = \frac{-122.668}{\widetilde{u}_0}$	$Z_{\delta_e} = -9.817$	$M_{\delta_T} = -0.634$

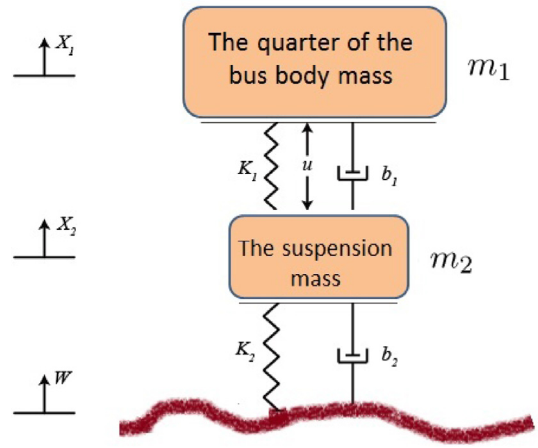

Fig. 5. The model of a quarter of a bus suspension system.

where $\widetilde{u}(t), \widetilde{\alpha}(t), \widetilde{q}(t)$, and $\widetilde{\theta}(t)$ are the linear velocity in the direction of X-axis in $(\frac{ft}{s})$, angle of attack in (rad), pitch rate in $(\frac{rad}{s})$, and pitch angle in (rad), respectively. The aircraft is in powered approach at Mach number *about* $M = 0.25$ and standard see level conditions. Expressing the Mach number as *about* $M = 0.25$ means that it is uncertain and is not precisely known. It is well known that the speed of sound depends on temperature and it increases as the ambient temperature increases. Then, the actual speed of the aircraft traveling at Mach number $M = 0.25$ will depend on the temperature of the fluid through which it is passing. Therefore, considering such an uncertainty is reasonable. In this paper, the Mach number is considered as $\widetilde{M} = (0.23, 0.25, 0.29)$ and the initial conditions

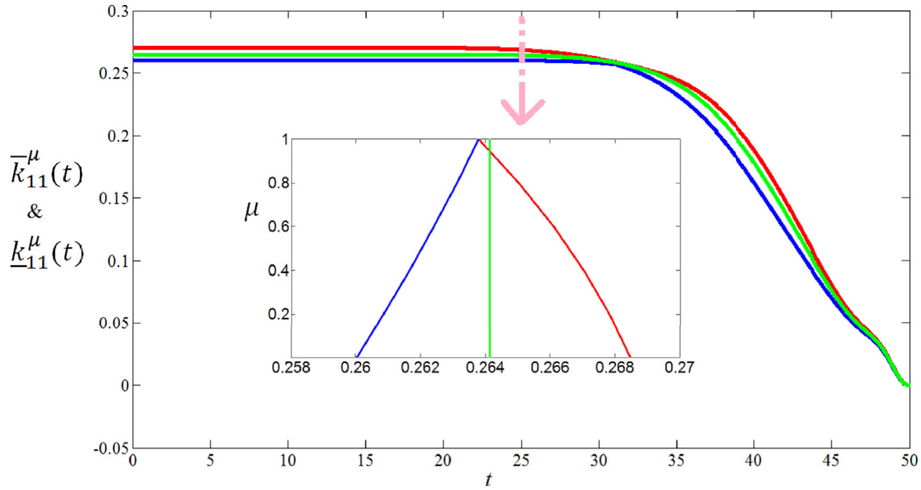

Fig. 6. The sub-optimal feedback gain $k_{c11}(t)$ (green curve), and fuzzy optimal feedback gain $\widetilde{k}_{11}(t)$ corresponding to Example 2. The blue and red curves show the left and right end-points of the support of $\widetilde{k}_{11}(t)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

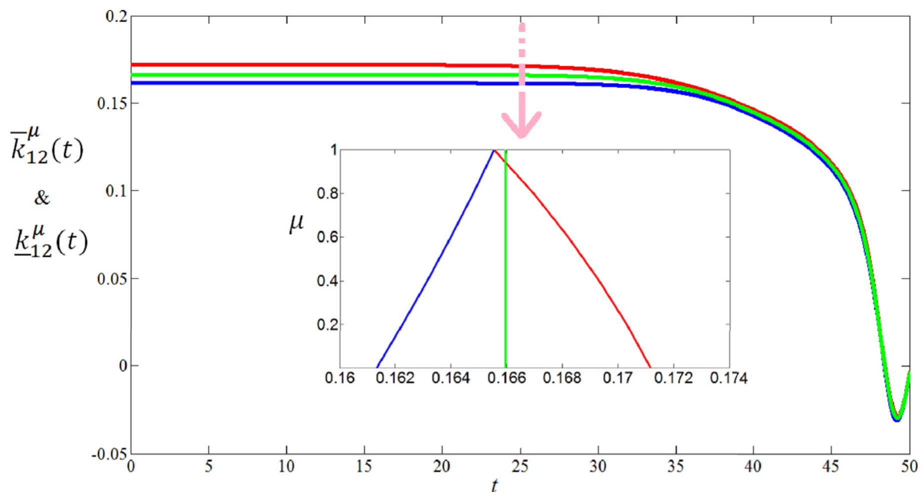

Fig. 7. The sub-optimal feedback gain $k_{c12}(t)$ (green curve), and fuzzy optimal feedback gain $\widetilde{k}_{12}(t)$ corresponding to Example 2. The blue and red curves show the left and right end-points of the support of $\widetilde{k}_{12}(t)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

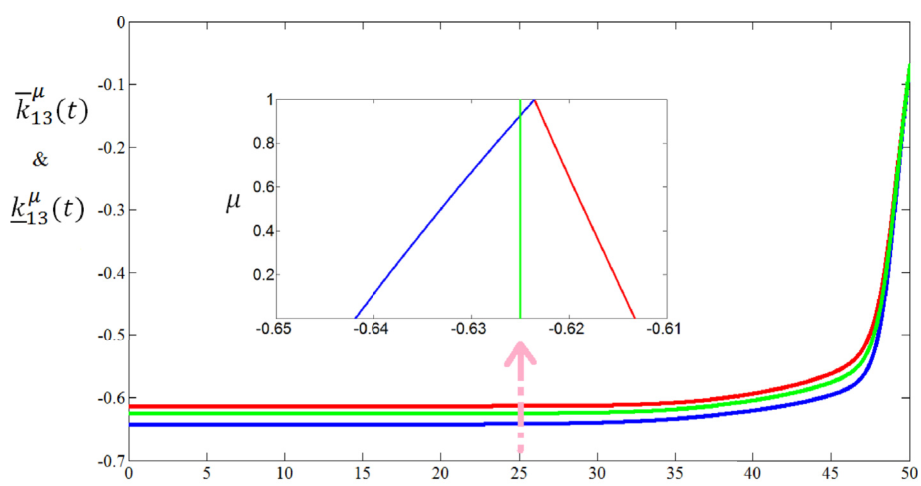

Fig. 8. The sub-optimal feedback gain $k_{c13}(t)$ (green curve), and fuzzy optimal feedback gain $\widetilde{k}_{13}(t)$ corresponding to Example 2. The blue and red curves show the left and right end-points of the support of $\widetilde{k}_{13}(t)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

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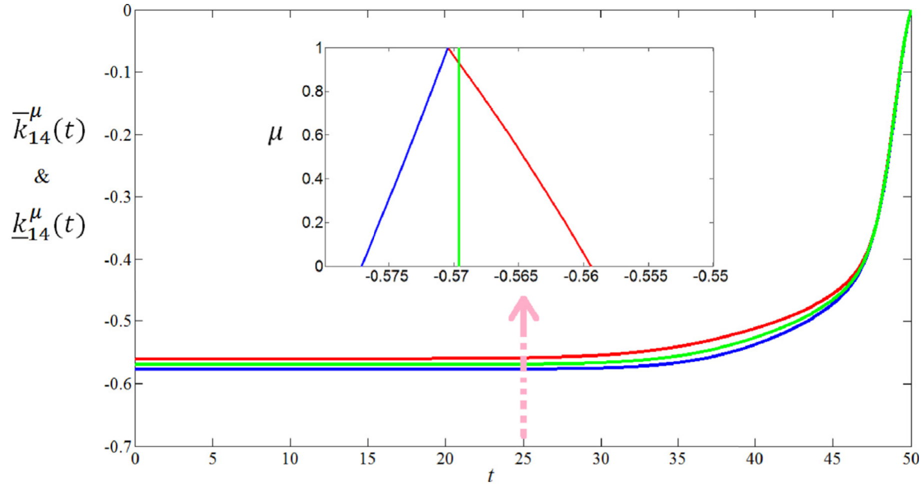

Fig. 9. The sub-optimal feedback gain $k_{c14}(t)$ (green curve), and fuzzy optimal feedback gain $\widetilde{k}_{14}(t)$ corresponding to Example 2. The blue and red curves show the left and right end-points of the support of $\widetilde{k}_{14}(t)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

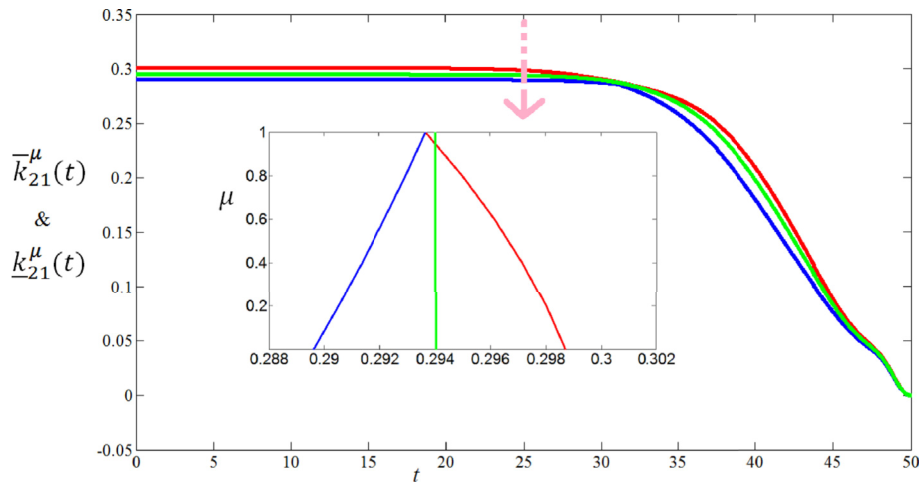

Fig. 10. The sub-optimal feedback gain $k_{c21}(t)$ (green curve), and fuzzy optimal feedback gain $\widetilde{k}_{21}(t)$ corresponding to Example 2. The blue and red curves show the left and right end-points of the support of $\widetilde{k}_{21}(t)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

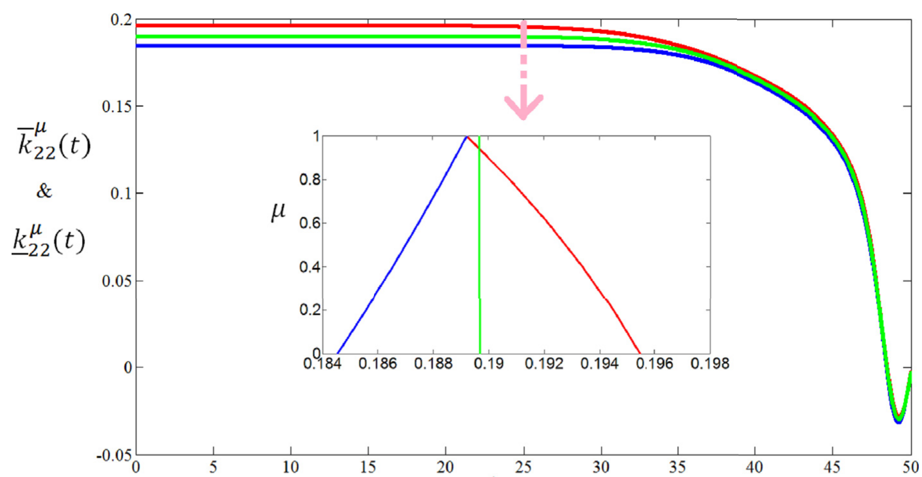

Fig. 11. The sub-optimal feedback gain $k_{c22}(t)$ (green curve), and fuzzy optimal feedback gain $\widetilde{k}_{22}(t)$ corresponding to Example 2. The blue and red curves show the left and right end-points of the support of $\widetilde{k}_{22}(t)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

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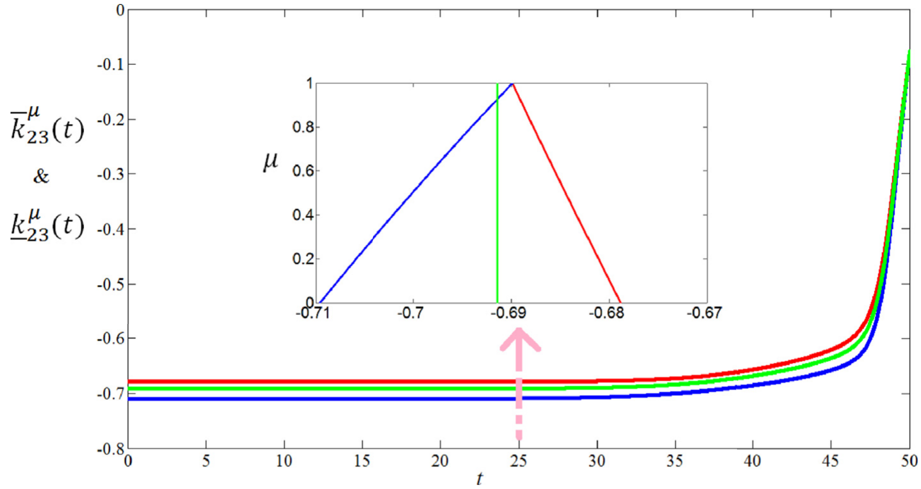

Fig. 12. The sub-optimal feedback gain $k_{c23}(t)$ (green curve), and fuzzy optimal feedback gain $\widetilde{k}_{23}(t)$ corresponding to Example 2. The blue and red curves show the left and right end-points of the support of $\widetilde{k}_{23}(t)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

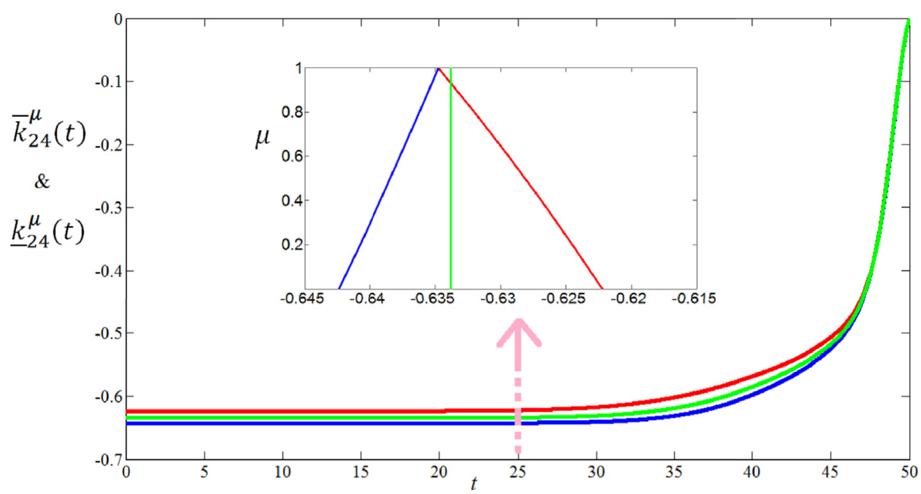

Fig. 13. The sub-optimal feedback gain $k_{c24}(t)$ (green curve), and fuzzy optimal feedback gain $\widetilde{k}_{24}(t)$ corresponding to Example 2. The blue and red curves show the left and right end-points of the support of $\widetilde{k}_{24}(t)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

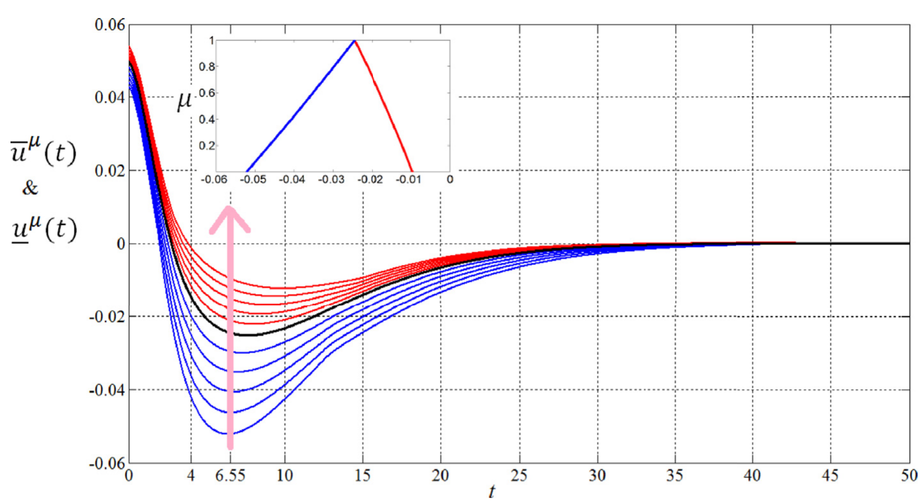

Fig. 14. The μ-level sets of the linear velocity, $\widetilde{u}(t)$. The blue and red curves show the left and right end-points of the μ-level sets. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

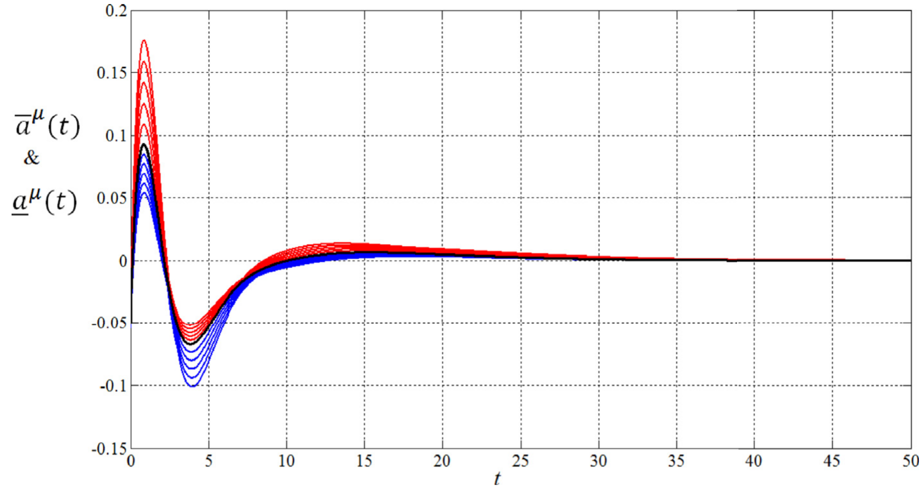

Fig. 15. The μ-level sets of the angle of attack, $\widetilde{a}(t)$. The blue and red curves show the left and right end-points of the μ-level sets. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

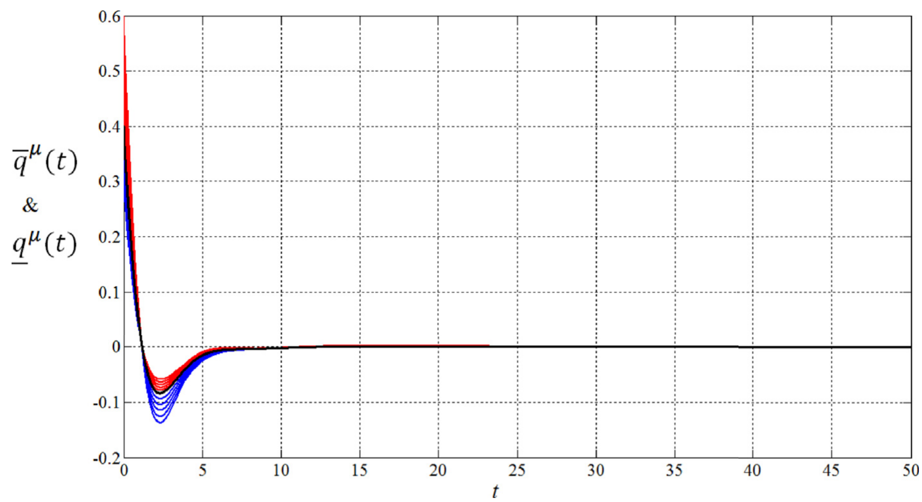

Fig. 16. The μ-level sets of the pitch rate, $\widetilde{q}(t)$. The blue and red curves show the left and right end-points of the μ-level sets. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

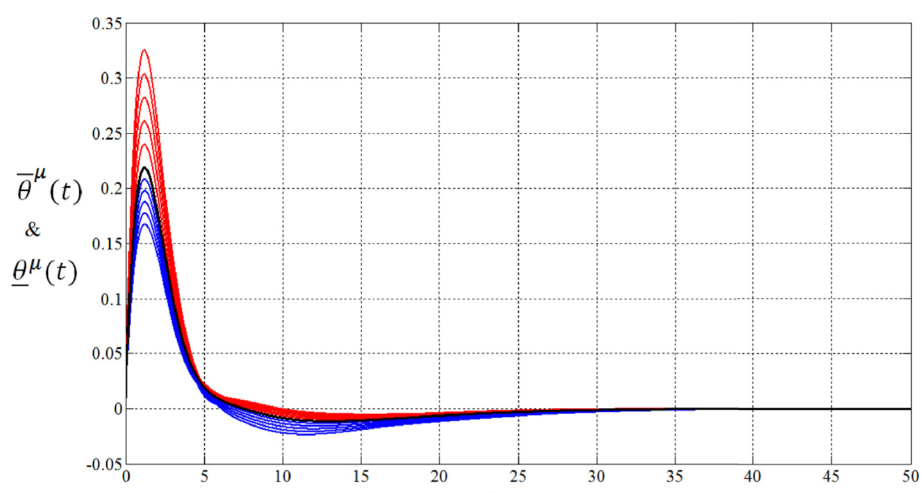

Fig. 17. The μ-level sets of the pitch angle, $\widetilde{\theta}(t)$. The blue and red curves show the left and right end-points of the μ-level sets. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

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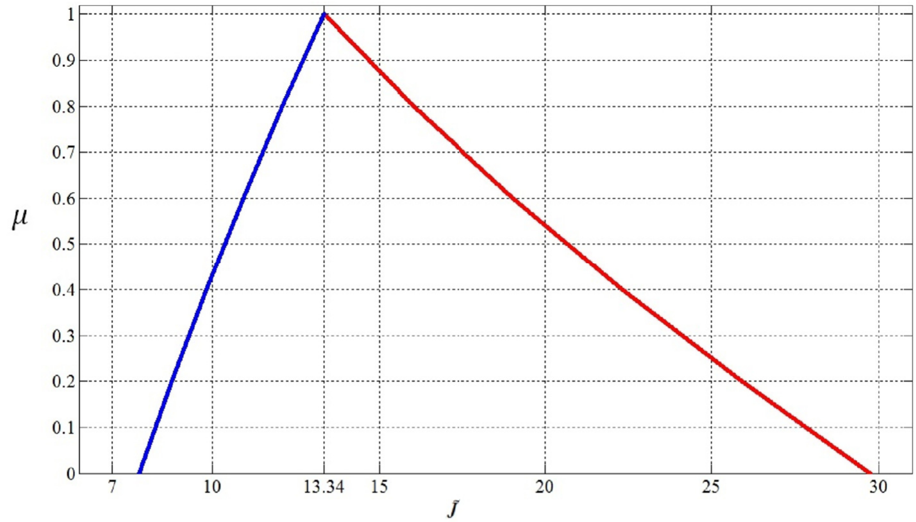

Fig. 18. The membership functions of the sub-optimal performance index corresponding to Example 2.

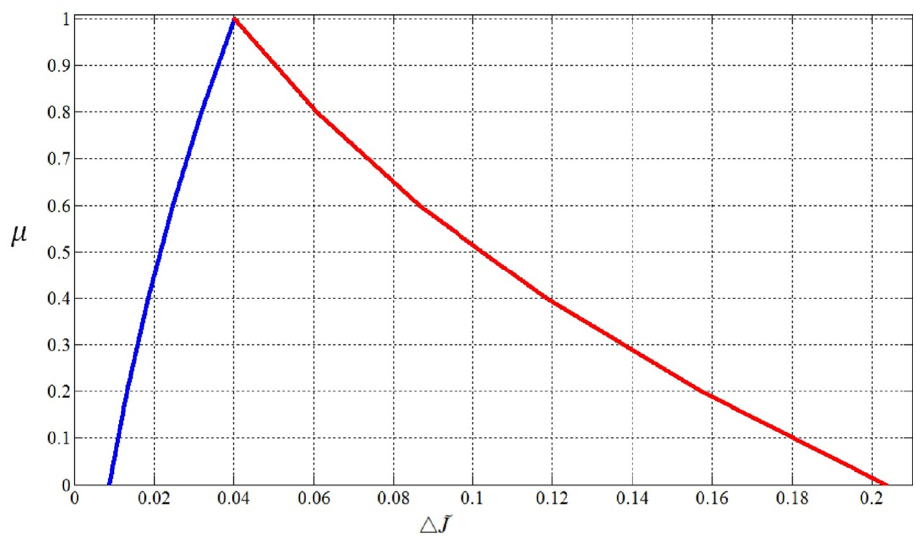

Fig. 19. The membership function of the difference between the optimal and sub-optimal performance indices corresponding to Example 2.

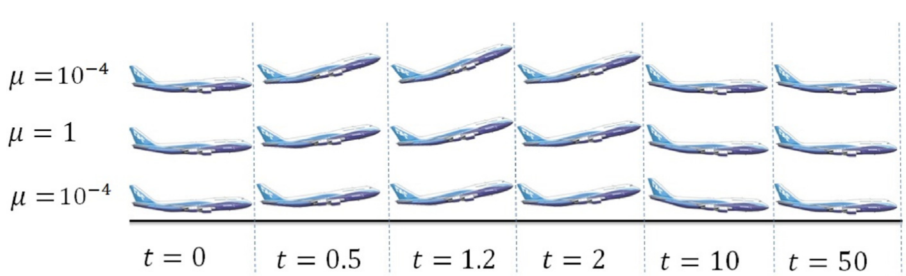

Fig. 20. The possible motions of controlled Boeing 747 in the presence of uncertainty for some degrees of possibility.

Table 2
The parameters of a quarter of the bus suspension system.

$m_1 = 2500(Kg)$
$m_2 = 3200(Kg)$
$\widetilde{k}_1$ *is about* $80000(\frac{N}{m}) = (72000, 80000, 85333)(\frac{N}{m})$
$\widetilde{k}_2$ *is about* $500000(\frac{N}{m}) = (450000, 500000, 533333)(\frac{N}{m})$
$\widetilde{b}_1$ *is about* $350(\frac{Ns}{m}) = (315, 350, 385)(\frac{Ns}{m})$
$\widetilde{b}_2$ *is about* $15020(\frac{Ns}{m}) = (13768, 15020, 16522)(\frac{Ns}{m})$

of the aircraft are as $\widetilde{u}(0) = \frac{14}{\widetilde{u}_0}, \widetilde{\alpha}(0) = \frac{-14}{\widetilde{u}_0}, \widetilde{q}(0) = (0.3, 0.4, 0.6)$, and $\widetilde{\theta}(0) = (0.01, 0.01, 0.01)$ where $\widetilde{u}_0 = 1116\widetilde{M}$. The coefficients have been determined in Table 1. Moreover, the deviation of elevator and thrust, i.e. $\widetilde{\delta}_e(t)$ and $\widetilde{\delta}_T(t)$ are the control inputs.

The aim is to find the optimal control law for the system (33) to minimize the performance measure (3) with the weighting matrices $P = 17I_{4\times4}, Q = 70I_{4\times4}$, and $R = 140I_{2\times2}$. It should be noted that, based on Proposition 3, the granular controllability of fuzzy dynamical system (33) can be verified. Then, according to Theorem 4 for obtaining the fuzzy optimal control law, the fuzzy feedback gains must be determined. The fuzzy feedback gains depend on solving fuzzy matrix differential equation (21) whose the final condition is $\widetilde{S}(t_f) = 17I_{4\times4}$. In order to obtain the solution of fuzzy matrix differential equation (21), the granular matrix differential equation corresponding to (21), i.e.

$$\begin{cases} -\dfrac{\partial S^{gr(t,\mu,\alpha_S)}}{\partial t} = \\ A^{gr^T}(\mu,\alpha_A)S^{gr}(t,\mu,\alpha_S) + S^{gr}(t,\mu,\alpha_S)A^{gr}(\mu,\alpha_A) \\ -S^{gr}(t,\mu,\alpha_S)B^{gr}(\mu,\alpha_B)R^{-1}B^{gr^T}(\mu,\alpha_B)S^{gr}(t,\mu,\alpha_S) \\ +70I_{4\times4} \\ S^{gr}(t_f,\mu,\alpha_S) = 17I_{4\times4} \\ t\in[0,50] \end{cases} \tag{34}$$

has been solved, and $\widetilde{S}(t) = \mathcal{H}^{-1}(S^{gr}(t,\mu,\alpha_S))$ has been obtained. Then, using $\widetilde{K}(t) = R^{-1}B^T\widetilde{S}(t)$, the optimal fuzzy feedback gain $\widetilde{K}(t) = [\widetilde{k}_{ri}(t)]_{m\times n}, r = 1, 2, i = 1, \ldots, 4$ can be determined.

Figs. 6–13 show the left and right end-points of the μ-level sets of the optimal fuzzy feedback gains $\widetilde{k}_{ri}(t)$ for $\mu = 0$. Additionally, by the aid of the center of gravity method, the optimal fuzzy feedback gains have been defuzzified and shown by green curves in the figures. The membership functions of the fuzzy feedback gains at $t = 25$ with their defuzzified values have been also depicted in the center of the figures.

Figs. 14–17 also show the left and right end-points of the μ-level sets of the fuzzy trajectories $\widetilde{u}, \widetilde{\alpha}, \widetilde{q}$, and $\widetilde{\theta}$ which have been obtained by applying the sub-optimal control law $\widetilde{U}(t) = -K_c(t)\widetilde{X}(t)$. It should be noted that, the value of each of the system states in any time is a fuzzy number. Additionally, using these figures one can predict the possible trajectories that the system states during the time pass. Furthermore, the values and features of each of the states in any time can be expressed as a linguistic variable. As an illustration, on the basis of Fig. 14 one can express that the linear velocity has an undershoot at $t = 6.55$ which is *about* "−0.025".

The sub-optimal performance index obtained by applying the sub-optimal control law $\widetilde{U}(t) = -K_c(t)\widetilde{X}(t)$ has been shown in Fig. 18. Based on the figure, it can be interpreted that the performance index is *approximately* "13.34". If the performance is not satisfactory, then the weighting matrices can be changed. Moreover, by determining the optimal performance index using Proposition 4, the difference between the sub-optimal performance index and the optimal performance index, i.e. $\triangle\widetilde{J} = J\left(-K_c(t)\widetilde{X}(t)\right) - \frac{1}{2}\widetilde{X}_0^T\widetilde{S}(t_0)\widetilde{X}_0$ has been obtained and shown in Fig. 19. Simply put, Fig. 19 shows that the sub-optimal performance index is *about* "0.04" far from the optimal value of the performance index. Furthermore, based on the Fig. 17, the possible motions of controlled Boeing 747 in the presence of uncertainty has been illustrated in Fig. 20 for some degrees of possibility.

Example 4. Consider the suspension system of one of the four wheels of a bus shown in Fig. 5. In the suspension system m_1, m_2, K_1, K_2, b_1, and b_2 are a quarter of the bus body mass, the suspension mass, spring stiffness constant of suspension system, spring stiffness constant of wheel and tire, damping constant of suspension system, damping constant of wheel and tire, respectively. The control force and road displacement were denoted by u and w. The quarter of the bus body displacement is determined by x_1, and x_2 shows non-sprung mass displacement. The values of the bus suspension system parameters have been determined in Table 2.

With spring stiffness and damping constants considered as uncertain parameters, the fuzzy state space model of the suspension system is expressed as

$$\begin{bmatrix} \dot{\widetilde{x}}_1(t) \\ \dot{\widetilde{x}}_2(t) \\ \dot{\widetilde{y}}_1(t) \\ \dot{\widetilde{y}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{\widetilde{b}_1\widetilde{b}_2}{m_1m_2} & 0 & \dfrac{\widetilde{b}_1^2m_1 + \widetilde{b}_1^2m_2 + \widetilde{b}_1\widetilde{b}_2m_1 - \widetilde{k}_1m_1m_2}{m_1^2m_2} & -\dfrac{\widetilde{b}_1}{m_1} \\ \dfrac{\widetilde{b}_2}{m_2} & 0 & -\dfrac{\widetilde{b}_1m_2 + (\widetilde{b}_1+\widetilde{b}_2)m_1}{m_1m_2} & 1 \\ \dfrac{\widetilde{k}_2}{m_2} & 0 & -\dfrac{\widetilde{k}_1m_2 + (\widetilde{k}_1+\widetilde{k}_2)m_1}{m_1m_2} & 0 \end{bmatrix} \times \begin{bmatrix} \widetilde{x}_1(t) \\ \widetilde{x}_2(t) \\ \widetilde{y}_1(t) \\ \widetilde{y}_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \dfrac{1}{m_1} & \dfrac{\widetilde{b}_1\widetilde{b}_2}{m_1m_2} \\ 0 & -\dfrac{\widetilde{b}_2}{m_2} \\ \dfrac{1}{m_1} + \dfrac{1}{m_2} & -\dfrac{\widetilde{k}_2}{m_2} \end{bmatrix} \begin{bmatrix} \widetilde{u}(t) \\ \widetilde{w}(t) \end{bmatrix} \tag{35}$$

where $\widetilde{x}_2 = \frac{d\widetilde{x}_1(t)}{dt}, \widetilde{y}_1 = \widetilde{x}_1 - \widetilde{x}_2$ and $\widetilde{y}_2 = \frac{d\widetilde{y}_1(t)}{dt}$. The aim is to find the sub-optimal control law for the system (35) so as to keep the quarter of bus body displacement near zero in the presence of road displacement as an uncontrolled input to the system. The performance measure considered in this case is as:

$$\widetilde{J}(\widetilde{U}) = \frac{1}{2}\left[10000\widetilde{x}_1^2(3) + 5000\widetilde{x}_2^2(3)\right] + \frac{1}{2}\int_0^3 \left(10000\widetilde{x}_1^2(t) + 5000\widetilde{x}_2^2(t) + 0.00002\widetilde{u}^2(t)\right)dt \tag{36}$$

Similar to the previous example, we first obtain the fuzzy optimal control law by solving fuzzy matrix differential equations shown in (21). Then the fuzzy feedback gains are defuzzified and used for control of the system. The sub-optimal control law obtained using the defuzzified feedback gains has been shown in Fig. 21. The possible quarter of bus body displacements and its derivative have been illustrated in Figs. 22 and 23. The initial condition is as $\left(x_1(0), x_2(0), y_1(0), y_2(0)\right) = (0, 0, 0, 0)$. The road displacement has been considered as a pulse whose amplitude is about 0.1 m, i.e. $(0.08, 0.1, 0.12)$, for $0.1 \le t \le 0.2$. It should be noted that for obtaining the optimal control law, the matrix $B \times [1 \quad 0]^T$ is used instead B. The reason comes from this fact that only the control force u can be modified.

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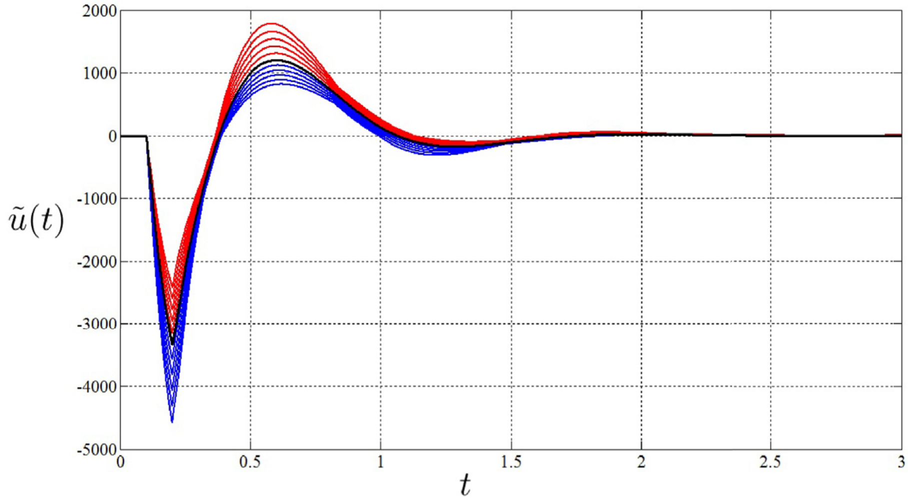

Fig. 21. The μ-level sets of the sub-optimal control, $\widetilde{u}(t)$. The blue and red curves show the left and right end-points of the μ-level sets. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

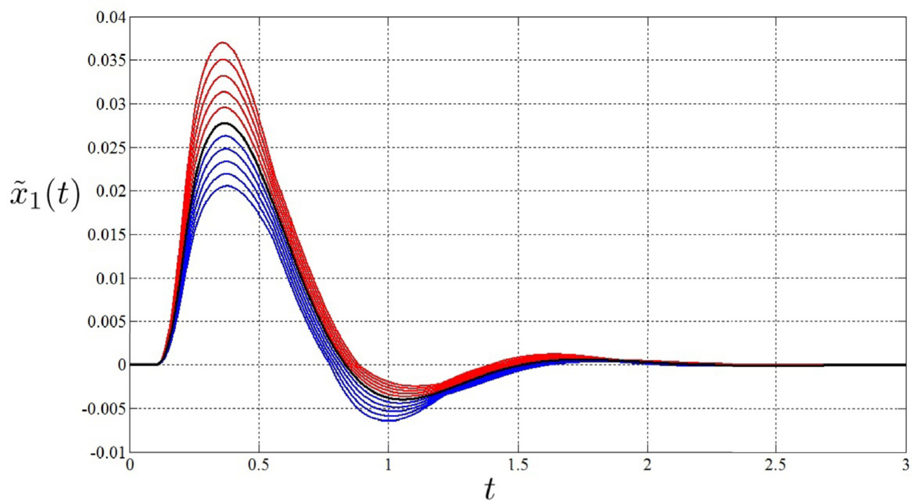

Fig. 22. The μ-level sets of the quarter of bus body displacement, $\widetilde{x}_1(t)$. The blue and red curves show the left and right end-points of the μ-level sets. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

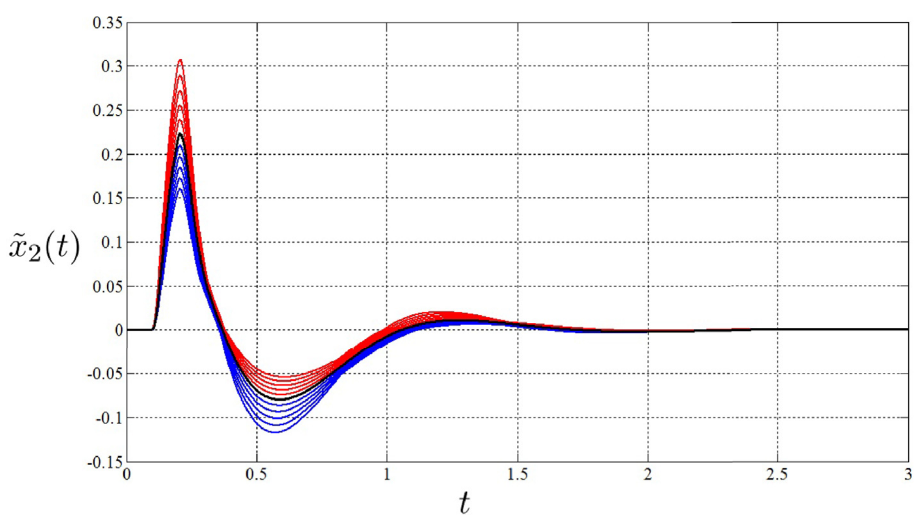

Fig. 23. The μ-level sets of the speed of quarter of bus body displacement, $\widetilde{x}_2(t)$. The blue and red curves show the left and right end-points of the μ-level sets. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

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5. Conclusions

In this paper, the uncertain linear dynamical system was considered to be regulated in an optimal manner. The uncertainties were expressed as the fuzzy numbers, and the fuzzy derivative was regarded as gr-derivative. Thanks to gr-derivative and RDM fuzzy interval arithmetic the fuzzy optimal control problem was converted to an equivalent optimal control problem dealing with multivariable functions. The optimal control law was obtained in the form of state variables fuzzy feedback. Although the optimal control with fuzzy feedback gains satisfied the conditions and minimized the criteria, the fuzzy feedback gains needed to be defuzzified in practice. As a result, the sub-optimal control law was obtained. The concepts of granular controllability and granular stabilizability of the uncertain dynamical system were also given. Using Example 1 we showed that the advantage of the concept of granular eigenvalues defined based on RDM fuzzy interval arithmetic in comparison with other approaches which are on the basis of FSIA. It was demonstrated that the approaches - SGH differentiability and gH differentiability - based on FSIA in dealing with the fuzzy optimal control problem suffer from some drawbacks outlined below:

1. Disability in the use of fuzzy Lagrange multipliers.
2. Multiplicity of the solutions for the fuzzy optimal control problem.
3. Incompatibility with the closed loop form of the controlled system.
4. Technical difficulties in the process of solving fuzzy optimal control problem.

The superiorities of the proposed approach arise from this key point that the proposed approach does not have the mentioned shortcomings. It is noteworthy to pinpoint that, robust control methods seek to bound the uncertainty rather than express it in the form of a distribution. Simply put, given a bound on the uncertainty, the control can deliver results that meet the crisp control system requirements in all cases. Then, a combination of the obtained results in this paper and robust control methods would be considered as a future work.

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