Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2024) 118:167
https://doi.org/10.1007/s13398-024-01665-1

ORIGINAL PAPER

Metric dimension of the complement of the zero-divisor graph

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Received: 20 April 2024 / Accepted: 16 September 2024

Abstract
Let S be the smallest subset of vertices in a graph G such that every vertex outside of S has a unique distance vector with respect to S. Then $|S|$ is defined as the metric dimension of G and it is denoted by $dim_M(G)$. In this paper, the metric dimension of the complement of the zero-divisor graph associated with a commutative ring is discussed. Several formulae for different classes of rings are given.

Keywords Metric dimension · Zero-divisor · Commutative ring

Mathematics Subject Classification 13A99 · 05C78 · 05C12

1 Introduction

Metric dimension of a graph which is an NP-hard problem with many usages in chemistry, combinatorial optimization, robotics, and so on, originates from trilateration in the two dimensional real plane. Some applications of metric dimension in graph theory may be found in [1–3]. Computing the metric dimension in different classes of graphs is interesting not only for graph theorists but also for algebraic graph theorists, see for instance [4–10]. In particular, metric and strong metric dimension of zero-divisor graphs have been studied in [11–13]. In this paper, metric dimension in complement of zero-divisor graphs is investigated.

In this paper, all rings R are assumed to be commutative, non-integral domains with identity and all graphs $G = (V, E)$ are simple. We recall that nodes of a zero-divisor graph associated with a ring R are zero-divisors except 0_R and two different nodes are joined if their product is zero (see [14], for more details). The symbol $\overline{\Gamma}(R)$ stands to denote the complement of a zero-divisor graph associated with R. Moreover, if S is the smallest subset of vertices in a graph G such that all vertices outside of S have different distance vectors with respect to S, then $|S|$ is defined as the metric dimension of G and it is denoted by $dim_M(G)$. The definitions of standard graph and ring theoretical notions are omitted so that there is no

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Published online: 30 September 2024

similarity between this paper and earlier published papers and textbooks. These can be found in [12, 15–17].

2 $diam(\overline{\Gamma}(R))$

First, we need to find the diameter of $\overline{\Gamma}(R)$, as components of distance vectors do not exceed from $diam(\overline{\Gamma}(R))$.

Lemma 2.1 *Let R be a ring and $\overline{\Gamma}(R)$ be a connected graph. Then $diam(\overline{\Gamma}(R)) \leq 4$.*

Proof Let $x, y \in V(\overline{\Gamma}(R))$ and $d(x, y) \neq 1$. If $ann(x) \cup ann(y) \neq Z(R)$, then $\overline{\Gamma}(R)$ contains the path $x - z - y$ from x to y, for every $z \in Z(R) \setminus ann(x) \cup ann(y)$. Now, let $ann(x) \cup ann(y) = Z(R)$. If the equality $ann(t) \cup ann(s) = Z(R)$ holds, for every $t \in N(x)$ and $s \in N(y)$, then $\overline{\Gamma}(R)$ is not connected, a contradiction. So there exist $t \in N(x)$ and $s \in N(y)$ such that $ann(t) \cup ann(s) \neq Z(R)$. Let $k \in Z(R) \setminus ann(t) \cup ann(s)$. Then $x - t - k - s - y$ is a path of length 4 from x to y. □

Using Lemma 2.1 and a similar argument to that of [18, Proposition 2.1], one may prove the following result. Hence, we omit its proof

Proposition 2.1 *If R is a ring, then $dim_M(\overline{\Gamma}(R))$ is finite if and only if R is finite.*

Lemma 2.2 *Suppose that $R \cong R_1 \times \cdots \times R_n$, where R_i is a finite local ring for every $1 \leq i \leq n$.*

1. *If $n = 1$ and $|V(\overline{\Gamma}(R))| \geq 2$ or $n = 2$ and R is reduced, then $\overline{\Gamma}(R)$ is not connected.*
2. *If $n = 2$ and R is non-reduced, then $\overline{\Gamma}(R)$ is connected and $diam(\overline{\Gamma}(R)) = 3$.*
3. *If $n \geq 3$, then $\overline{\Gamma}(R)$ is connected and $diam(\overline{\Gamma}(R)) = 2$.*

Proof (1) If $n = 1$, then $(R, \mathfrak{m})$ is a local ring and since R is finite, $ann(\mathfrak{m}) \neq 0$. Now, it is clear that for every $a \in ann(\mathfrak{m})$, a is not adjacent to any other vertex. Hence, $\overline{\Gamma}(R)$ is not connected. Also, if $n = 2$ and R is reduced, then $R \cong R_1 \times R_2$, where R_i is a field for $1 \leq i \leq 2$. In this case $\overline{\Gamma}(R)$ is not connected, as $\overline{\Gamma}(R) = K_{|R_1|-1} + K_{|R_2|-1}$.

(2) If $n = 2$ and R is non-reduced, then $R \cong R_1 \times R_2$, where $(R_i, \mathfrak{m}_i)$ is a local ring, for $1 \leq i \leq 2$. With no loss of generality, suppose that $\mathfrak{m}_1 \neq 0$. Let $x = (a, 1)$ and $y = (1, 0)$, where $a \in ann(\mathfrak{m}_1)$. Then it is clear that any other vertex is adjacent either to x or y. On the other hand, x is adjacent to y. This implies that $diam(\overline{\Gamma}(R)) \leq 3$. Let $t = (a, 0)$ and $s = (0, 1)$, if $\mathfrak{m}_2 = 0$ and $s = (0, b)$ if $\mathfrak{m}_2 \neq 0$ with $b \in ann(\mathfrak{m}_2)$. Then it is easy to check that $d(t, s) = 3$. Therefore, $diam(\overline{\Gamma}(R)) = 3$.

(3) Since $Z(R) = \mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_n$ and $n \geq 3$, for every $x, y \in Z(R)^*$, $ann(x) \cup ann(y) \neq Z(R)$ and hence by the proof of Lemma 2.1, we have $diam(\overline{\Gamma}(R)) = 2$. □

3 $dim_M(\overline{\Gamma}(R))$; Reduced rings

In this section , we establish some formulas for $dim_M(\overline{\Gamma}(R))$, when R is reduced.

Theorem 3.1 *Let $n \geq 3$ be a positive integer and $R = \prod_{i=1}^{n} \mathbb{Z}_2$. Then the following statements hold.*

1. *$dim_M(\overline{\Gamma}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 2$.*
2. *$dim_M(\overline{\Gamma}(\prod_{i=1}^{n} \mathbb{Z}_2)) = n$, for $n \geq 4$.*

Proof (1) If $n = 3$, then we put $W = \{(1, 0, 0), (0, 1, 0)\}$. Now, we have $D((0, 0, 1)|W) = (2, 2)$, $D((0, 1, 1)|W) = (2, 1)$, $D((1, 0, 1)|W) = (1, 2)$ and $D((1, 1, 0)|W) = (1, 1)$. This implies that $dim_M(\overline{\Gamma}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 2$.

(2) Assume that $n \geq 4$. We show that $dim_M(\overline{\Gamma}(\prod_{i=1}^{n} \mathbb{Z}_2)) = n$. Indeed, we have the following claims:

Claim 1. $dim_M(\overline{\Gamma}(\prod_{i=1}^{n} \mathbb{Z}_2)) \geq n$.

Let $W = \{x_1, x_2, \ldots, x_k\}$ be a metric basis for $\overline{\Gamma}(\prod_{i=1}^{n} \mathbb{Z}_2)$, where k is a non negative integer. Since $n \geq 3$, by Lemma 2.2, $diam(\overline{\Gamma}) \leq 2$ and hence there are exactly 2^k choices for $D(x|W)$, for every $x \in Z(R)^* \setminus W$. Thus $|Z(R)^*| - k \leq 2^k$. Since $|Z(R)^*| = 2^n - 2$, $2^n - 2 - k \leq 2^k$ and so $2^n \leq 2^k + 2 + k$. Since $n \geq 4$, we conclude that $k \geq n$. Therefore $dim_M(\overline{\Gamma}(\prod_{i=1}^{n} \mathbb{Z}_2)) \geq n$.

Claim 2. $dim_M(\overline{\Gamma}(\prod_{i=1}^{n} \mathbb{Z}_2)) \leq n$.

Let $w_i = (0, \ldots, 0, 1, 0 \ldots, 0) \in Z(R)^*$, with 1 in i-th component and put

$$W = \{w_1, w_2, \ldots, w_n\}.$$

We show that W is a resolving set for $\overline{\Gamma}(R)$. Let $u, v \in V(\overline{\Gamma}(R)) \setminus W$ and $u \neq v$. Since $diam(\overline{\Gamma}(R)) \leq 2$, it is not hard to see that each component of $D(u|W)$ is 1 if and only if this component in u is 1 and each component of $D(u|W)$ is 2 if and only if this component in u is 0 (We note that every component of u is 0 or 1). In other words, $D(u|W)$ is obtained by replacing zero components of u by 2 and nonzero components by 1. Since $u \neq v$, we conclude that $D(u|W) \neq D(v|W)$ and so W is the resolving set for $\overline{\Gamma}(R)$. Therefore $dim_M(\overline{\Gamma}(\prod_{i=1}^{n} \mathbb{Z}_2)) \leq n$.

By Claims 1, 2, $dim_M(\overline{\Gamma}(\prod_{i=1}^{n} \mathbb{Z}_2)) = n$, for $n \geq 4$. □

To prove Theorem 3.2, the following results are needed.

Remark 3.1 Let G be a connected graph and $V_1, V_2, \ldots, V_k$ be a partition of $V(G)$ such that for every $1 \leq i \leq k$, $x, y \in V_i$, if and only if $d(x, a) = d(y, a)$ for all $a \in V(G) \setminus \{x, y\}$. Then $dim_M(G) \geq |V(G)| - k$.

Proof [10, Theorem 2.1]. □

Lemma 3.1 *Suppose that $R \cong R_1 \times \cdots \times R_n$, where R_i is a finite local ring for every $1 \leq i \leq n$ and $x, y \in V(\overline{\Gamma}(R))$.*

(1) *If $Rx = Ry$, then $N(x) = N(y)$.*
(2) *$N(x) = N(y)$ if and only if $d(x, a) = d(y, a)$ for all $a \in V(G) \setminus \{x, y\}$.*

Proof (1) Suppose that $x - a$ is an edge of $\overline{\Gamma}(R)$. Hence $ax \neq 0$. Since $Rx = Ry$, we have $ann_R(x) = ann_R(y)$ and so we deduce that $ay \neq 0$. This means that $y - a$ is an edge of $\overline{\Gamma}(R)$ and thus $N(x) \subseteq N(y)$. Similarly, $N(y) \subseteq N(x)$, as desired.

(2) ($\Longrightarrow$) Assume that $d(x, a) = 1$ for some $a \in V(G) \setminus \{x, y\}$. Since $N(x) = N(y)$, $d(y, a) = 1$. Now, assume that $d(x, a) = 2$ for some $a \in V(G) \setminus \{x, y\}$. Then there exists $k \in V(G)$ such that $x - k - a$ is a path from x to a. Hence $k \in N(x) = N(y)$. Therefore, $y - k - a$ is a path from y to a. This means that $d(y, a) \leq 2$. Since $d(x, a) = 2$ and $N(x) = N(y)$, we have $d(y, a) = 2$. Finally, if $d(x, a) = 3$ for some $a \in V(G) \setminus \{x, y\}$, then since $diam(\overline{\Gamma}(R)) \leq 3$, we have $d(x, a) = 3$.

($\Longleftarrow$) It is obvious. □

Theorem 3.2 *Suppose that $R \cong F_1 \times \cdots \times F_t \times F'_1 \times \cdots \times F'_s$, where every $F_i \neq \mathbb{Z}_2$ is a finite field, for $1 \leq i \leq t$ and $F'_i \cong \mathbb{Z}_2$, for every $1 \leq i \leq s$. Then the following statements hold.*

(1) *If* $s \geq 2$, *then* $dim_M(\overline{\Gamma}(R)) = |Z(R)^*| - 2^{t+s} + s + 2$.
(2) *If* $s = 1$, *then* $dim_M(\overline{\Gamma}(R)) = |Z(R)^*| - 2^{t+1} + 2$.
(3) *If* $s = 0$, *then* $dim_M(\overline{\Gamma}(R)) = |Z(R)^*| - 2^{t} + 2$.

Proof (1) Let
$A = \{(x_1, \ldots, x_{t+s}) \in V(\overline{\Gamma}(R)) \mid x_i \in \{0, 1\}$ for every $1 \leq i \leq t+s\}$ and $B = \{(\underbrace{1, \ldots, 1}_{t}, x_1, \ldots, x_s) \in A\}$.

We complete the proof in two steps.

Step 1. We show that $dim_M(\overline{\Gamma}(R)) \geq |Z(R)^*| - 2^{t+s} + s + 2$.

Assume that $x = (x_1, \ldots, x_{t+s})$ and $y = (y_1, \ldots, y_{t+s})$ are vertices of $\overline{\Gamma}(R)$. Define the relation $\sim$ on $V(\overline{\Gamma}(R))$ as follows: $x \sim y$, whenever "$Rx = Ry$". It is easily seen that $\sim$ is an equivalence relation on $V(\overline{\Gamma}(R))$. By $[x]$, we denote the equivalence class of x. Let $a = (a_1, \ldots, a_{t+s})$ and $b = (b_1, \ldots, b_{t+s})$ be two elements of $[x]$. Since $a \sim b$, $Ra = Rb$, and so by Lemma 3.1, $N(a) = N(b)$. This, together with the fact that the number of equivalence classes is $2^{t+s} - 2$ and Remark 3.1, imply that
$(Z(R)^* \setminus A) \subseteq W'$, where W' is a metric basis for $\overline{\Gamma}(R)$, because $|[x] \cap A| = 1$.

Let $a \in B$. Then $d(a, b) = 1$ for every $b \in Z(R)^* \setminus A$. This means that to resolve the elements of B the set W' must have some more elements of A. Since $|B| = 2^s - 1$, W' has at least s more elements. Thus $dim_M(\overline{\Gamma}(R)) \geq |Z(R)^*| - 2^{t+s} + s + 2$.

Step 2. We show that $dim_M(\overline{\Gamma}(R)) \leq |Z(R)^*| - 2^{t+s} + s + 2$. Let
$C = \{(\underbrace{0, \ldots, 0}_{t}, 1, 0, \ldots, 0), \ldots, (\underbrace{0, \ldots, 0}_{t}, 0, \ldots, 0, 1)\}$ and $W = (Z(R)^* \setminus A) \cup C$.

We show that W is a resolving set for the graph $\overline{\Gamma}(R)$. Indeed, we prove that $D(x|W) \neq D(y|W)$, for all $x, y \in Z(R)^* \setminus W$ with $x \neq y$. For this purpose, we divide the set of vertices of $Z(R)^* \setminus W$ into following subsets.
$V_1 = \{(\underbrace{x_1, \ldots, x_t}_{t}, \underbrace{y_1, \ldots, y_s}) \mid x_i = 0$ for some $1 \leq i \leq s\}$ and
$V_2 = \{(\underbrace{1, \ldots, 1}_{t}, \underbrace{y_1, \ldots, y_s})\}$.

We continue the proof in the following cases.

Case 1. Let $x = (\underbrace{x_1, \ldots, x_t}_{t}, \underbrace{y_1, \ldots, y_s}) \in V_1$ and $u_i \in U(F_i)$ with $u_i \neq 1$ for every $1 \leq i \leq t$. Put $z = (\underbrace{a_1, \ldots, a_t}_{t}, \underbrace{b_1, \ldots, b_s})$, where $x_i = 0$ and $y_j = 0$ if and only if $a_i = u_i$ and $b_j = 1$, respectively and $x_i = 1$ and $y_j = 1$ if and only if $a_i = 0$ and $b_j = 0$, respectively. Then we can easily get $z \in W$ and $d(x, z) \neq d(y, z)$ for every $y \in V_1 \cup V_2$. This implies that $D(x|W) \neq D(y|W)$.

Case 2. If $x, y \in V_2$, then by the proof of Theorem 3.1, for some $z \in C$, $d(x, z) \neq d(y, z)$ where $C = \{(\underbrace{0, \ldots, 0}_{t}, 1, 0, \ldots, 0), \ldots, (\underbrace{0, \ldots, 0}_{t}, 0, \ldots, 0, 1)\}$. Hence $D(x|W) \neq D(y|W)$. Therefore, W is a resolving set.

By Steps 1,2, $dim_M(\overline{\Gamma}(R)) = |Z(R)^*| - 2^{t+s} + s + 2$.

(2, 3) If $s \leq 1$, then in the proof of part 1, we have $|B| = 1$ or $B = \varnothing$. So it is easy to see that (2) and (3) hold. □

Example 3.1 (1) Let $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $(2, 0, 1) \in [(1, 0, 1)]$, $(2, 1, 0) \in [(1, 1, 0)]$ and $(2, 0, 0) \in [(1, 0, 0)]$, $\{(2, 0, 1), (2, 1, 0), (2, 0, 0)\} \subseteq W$, where W is a resolving set for the graph $\overline{\Gamma}(R)$. On the other hand, if we put $B = \{(1, 0, 1), (1, 1, 0), (1, 0, 0)\}$, then $d(a, b) = 1$ for every $a \in B$ and $b \in \{(2, 0, 1), (2, 1, 0), (2, 0, 0)\}$. So we need to add at least 2 elements in $\{(2,0,1),(2,1,0),(2,0,0)\}$ to resolve the elements of B. Now, assume that $W=\{(2,0,1),(2,1,0),(2,0,0),(0,1,0),(0,0,1)\}$. Then we have $D((1,0,1)|W)=(1,1,1,2,1)$, $D((1,1,0)|W)=(1,1,1,1,2)$, $D((1,0,0)|W)=(1,1,1,2,2)$ and $D((0,0,1)|W)=(1,2,2,2,1)$. Therefore, $dim_M(\overline{\Gamma}(R))=5$.

(2) Let $R\cong\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_2$. Then with a same method as in (1), $\{(2,2,0),(2,0,0),(2,0,1),(2,1,0),(1,2,0),(0,2,1),(0,2,0)\}\subseteq W$. Since $|B|=1$, we put $\{(2,2,0),(2,0,0),(2,0,1),(2,1,0),(1,2,0),(0,2,1),(0,2,0)\}=W$. Then
$D((1,1,0)|W)=(1,1,1,1,1,1,1)$,
$D((1,0,1)|W)=(1,1,1,1,1,1,2)$,
$D((0,1,1)|W)=(1,2,1,1,1,1,1)$,
$D((1,0,0)|W)=(1,1,1,1,1,2,2)$,
$D((0,1,0)|W)=(1,2,2,1,1,1,1)$,
$D((0,0,1)|W)=(2,2,1,2,2,1,2)$.
Therefore, $dim_M(\overline{\Gamma}(R))=7$.

4 $dim_M(\overline{\Gamma}(R))$; Non-reduced rings

In this section, we compute $dim_M(\overline{\Gamma}(R))$ in case R is a non-reduced ring. We start with the following result.

Theorem 4.1 *Suppose that $R\cong R_1\times\cdots\times R_n$, where $(R_i,\mathfrak{m}_i)$ is a finite local ring such that $\mathfrak{m}_i^2=0$ and $|\mathfrak{m}_i|=2$ for every $1\le i\le n$ and $n\ge 2$. Then $dim_M(\overline{\Gamma}(R))=4^n-3^n-2^n+n+1$.*

Proof For every $1\le i\le n$, suppose that $\mathfrak{m}_i=\{0,a_i\}$, where a_i is a non-zero zero divisor of R_i. Hence $R_i=\{0,a_i,1,1+a_i\}$, where $U(R_i)=\{1,1+a_i\}$ (note that $|Z(R)|=2$ if and only if $R\cong\mathbb{Z}_4$ or $R\cong\mathbb{Z}_2[X]/(X^2)$). Assume that $1+a_i=b_i$ and consider the following sets.

$A=\{(x_1,\ldots,x_n)\in V(\overline{\Gamma}(R))\mid x_i=b_i$ for some $1\le i\le n\}$ and
$B=\{(x_1,\ldots,x_n)\in V(\overline{\Gamma}(R))\setminus A\mid x_i\ne 0$ for every $1\le i\le n\}$.

We complete the proof in two steps.

Step 1. We show that $dim_M(\overline{\Gamma}(R))\ge 4^n-3^n-2^n+n+1$.

Let $x=(x_1,\ldots,x_n)\in A$ and $y=(y_1,\ldots,y_n)\in V(\overline{\Gamma}(R))\setminus A$ such that $y_i=1$ if and only if $x_i=b_i$ for every $1\le i\le n$ and other components of y are the same as the components of x. In this case it is clear that $N[x]=N[y]$, and hence by Remark 3.1, $x\in W'$ or $y\in W'$, for every metric basis W'. Let W'' be a metric basis such that $A\subseteq W''$. Now, let $a\in B$. Then $d(a,b)=1$ for every $b\in A\cup B$. This means that to resolve the elements of B the set W'' must have some more elements of $V(\overline{\Gamma}(R))\setminus A$. Since $|B|=2^n-1$, W'' has at least n more elements. Thus $dim_M(\overline{\Gamma}(R))\ge 4^n-3^n-2^n+n+1$.

Step 2. We show that $dim_M(\overline{\Gamma}(R))\le 4^n-3^n-2^n+n+1$. Let

$C=\{(a_1,0,\ldots,0),(0,a_2,0,\ldots,0),\ldots,(0,\ldots,0,a_n)\}$ and $W=A\cup C$. We show that W is a resolving set for the graph $\overline{\Gamma}(R)$. We divide the set of vertices of $V(\overline{\Gamma}(R))\setminus A\cup C$ as below.

$V_1=\{(x_1,\ldots,x_n)\in V(\overline{\Gamma}(R))\setminus A\cup C\mid x_i\ne 0$ for every $1\le i\le n\}$,
$V_2=\{(x_1,\ldots,x_n)\in V(\overline{\Gamma}(R))\setminus A\cup C\mid x_i\in\{0,1\}$ for every $1\le i\le n\}$,
$V_3=\{(x_1,\ldots,x_n)\in V(\overline{\Gamma}(R))\setminus A\cup C\mid x_i\in\{0,a_i\}$ for every $1\le i\le n\}$ and
$V_4=\{(x_1,\ldots,x_n)\in V(\overline{\Gamma}(R))\setminus A\cup C\mid x_i=0, x_j=1, x_k=a_k$ for some $1\le i,j,k\le n\}$.

Also, we consider the following partition for W:
$E=\{(b_1,0,\dots,0),(0,b_2,0,\dots,0),\dots,(0,\dots,0,b_n)\}$,
$C=\{(a_1,0,\dots,0),(0,a_2,0,\dots,0),\dots,(0,\dots,0,a_n)\}$ and
$D=W\setminus E\cup C$.
In fact, we arrange the elements of W as follows.

$$W=\{(\underbrace{(b_1,0,\dots,0),\dots,(0,\dots,0,b_n)}_{\in E}),(\underbrace{(a_1,0,\dots,0),\dots,(0,\dots,0,a_n)}_{\in C}),\underbrace{d_1,\dots,d_{|D|}}_{\in D})\}.$$

Now we check the distance vectors of the elements of $V_1,\dots,V_4$ with respect to W.
Let $x=(x_1,\dots,x_n)\in V_1$. Then since $x_i\neq 0$, we can easily get

$$D(x|W)=(\underbrace{1,\dots,1}_{n\,components},\underbrace{s_1,\dots,s_n}_{n\,components},\underbrace{d_1,\dots,d_{|D|}}_{|D|\,components})$$

such that $s_i=1$ if and only if $x_i=1$ and $s_j=2$ if and only if $x_j=a_j$ for every $1\le i,j\le n$.
Next, let $x=(x_1,\dots,x_n)\in V_2$. Then

$$D(x|W)=(\underbrace{t_1,\dots,t_n}_{n\,components},\underbrace{s_1,\dots,s_n}_{n\,components},\underbrace{d_1,\dots,d_{|D|}}_{|D|\,components})$$

such that $t_i=s_i=1$ if and only if $x_i=1$ and $t_j=s_j=2$ if and only if $x_j=a_j$ for every $1\le i,j\le n$.
Let $x=(x_1,\dots,x_n)\in V_3$. Then

$$D(x|W)=(\underbrace{t_1,\dots,t_n}_{n\,components},\underbrace{2,,2,\dots,2}_{n\,components},\underbrace{d_1,\dots,d_{|D|}}_{|D|\,components})$$

such that $t_i=1$ if and only if $x_i=a_i$ and $t_j=2$ if and only if $x_j=0$ for every $1\le i,j\le n$.
Finally, let $x=(x_1,\dots,x_n)\in V_4$. Then

$$D(x|W)=(\underbrace{t_1,\dots,t_n}_{n\,components},\underbrace{s_1,\dots,s_n}_{n\,components},\underbrace{d_1,\dots,d_{|D|}}_{|D|\,components})$$

such that $t_i=1$ if and only if $x_i=1$ or $x_i=a_i$ and $s_j=2$ if and only if $x_j=a_j$ or $x_j=0$ for every $1\le i,j\le n$.
Now, it is easy to see that for all $x,y\in V=\cup V_i$, with $x\neq y$, we have $D(x|W)\neq D(y|W)$. Hence W is a resolving set for the graph $\overline{\Gamma}(R)$.
By Steps 1,2, $dim_M(\overline{\Gamma}(R))=4^n-3^n-2^n+n+1$. □

Theorem 4.2 *Suppose that $R\cong R_1\times\cdots\times R_n$, where $(R_i,\mathfrak{m}_i)$ is a finite local ring such that $\mathfrak{m}_i^2=0$ and $|\mathfrak{m}_i|\ge 3$ for every $1\le i\le n$. Then $dim_M(\overline{\Gamma}(R))=|Z(R)^*|-3^n+2$.*

Proof Assume that $a_i\in\mathfrak{m}_i$, for some non-zero zero-divisor element a_i of R_i, and
$A=\{(x_1,\dots,x_n)\in V(\overline{\Gamma}(R))\,|\,x_i\in\{0,1,a_i\}$ for every $1\le i\le n\}$.
We complete the proof in two steps.
Step 1. We show that $dim_M(\overline{\Gamma}(R))\ge|Z(R)^*|-3^n+2$. By a similar proof to that of Theorem 4.1, there exists $x\in A$ such that $N[x]=N[y]$, for every $y\in Z(R)^*\setminus A$. Hence by Remark 3.1, $Z(R)^*\setminus A\subseteq W'$, where W' is a metric basis for $\overline{\Gamma}(R)$. Thus $dim_M(\overline{\Gamma}(R))\ge |Z(R)^*|-3^n+2$.
Step 2. We show that $dim_M(\overline{\Gamma}(R))\le|Z(R)^*|-3^n+2$.

Let $W = Z(R)^* \setminus A$. We show that W is a resolving set for the graph $\overline{\Gamma}(R)$. We divide the set of vertices of A as below.

$V_1 = \{(x_1, \ldots, x_n) \in A \mid x_i \neq 0 \text{ for every } 1 \leq i \leq n\}$,

$V_2 = \{(x_1, \ldots, x_n) \in A \mid x_i \in \{0, 1\} \text{ for every } 1 \leq i \leq n\}$,

$V_3 = \{(x_1, \ldots, x_n) \in A \mid x_i \in \{0, a_i\} \text{ for every } 1 \leq i \leq n\}$ and

$V_4 = \{(x_1, \ldots, x_n) \in A \mid x_i = 0, x_j = 1, x_k = a_k \text{ for some } 1 \leq i, j, k \leq n\}$.

Also assume that $u_i \in U(R_i)$ and $0 \neq b_i \in \mathfrak{m}_i$ with $u_i \neq 1$ and $b_i \neq a_i$ for every $1 \leq i \leq n$. Then we consider the following partition for W.

$E = \{(u_1, 0, \ldots, 0), (0, u_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, u_n)\}$,

$C = \{(b_1, 0, \ldots, 0), (0, b_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, b_n)\}$ and

$D = W \setminus E \cup C$.

In fact, we arrange the elements of W as follows.

$W = \{(\underbrace{(u_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, u_n)}_{\in E}), (\underbrace{(b_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, b_n)}_{\in C}), \underbrace{d_1, \ldots, d_{|D|}}_{\in D})\}$. Now, by a similar proof to that of Theorem 4.1, $D(x|W) \neq D(y|W)$, for all $x, y \in V = \cup V_i$, with $x \neq y$.

By Steps 1,2, $dim_M(\overline{\Gamma}(R)) = |Z(R)^*| - 3^n + 2$. □

Theorem 4.3 *Suppose that* $R \cong R_1 \times \cdots \times R_n \times R'_1 \times \cdots \times R'_m$, $(R_i, \mathfrak{m}_i)$ *is a finite local ring such that* $\mathfrak{m}_i^2 = 0$ *and* $|\mathfrak{m}_i| \geq 3$ *for every* $1 \leq i \leq n$ *and* $(R'_i, \mathfrak{m}'_i)$ *is a local ring with* $|\mathfrak{m}'_i| = 2$ *for every* $1 \leq i \leq m$ *with* $m \geq 2$. *Then* $dim_M(\overline{\Gamma}(R)) = |Z(R)^*| - 3^{n+m} + m + 2$.

Proof Assume that $0 \neq a_i \in \mathfrak{m}_i$, $0 \neq a'_i \in \mathfrak{m}'_i$, for some elements $a_i \in R_i$, $a'_i \in R'_i$,

$A = \{(x_1, \ldots, x_{n+m}) \in V(\overline{\Gamma}(R)) \mid x_i \in \{0, 1, a_i, a'_j\}\}$,

$B = \{(\underbrace{1, \ldots, 1}_{n}, x_1, \ldots, x_m) \in A \mid x_i \neq 0 \text{ for every } 1 \leq i \leq m\}$ and

$C = \{(\underbrace{0, \ldots, 0}_{n}, a'_1, 0, \ldots, 0), \ldots, (\underbrace{0, \ldots, 0}_{n}, 0, \ldots, 0, a'_m)\}$.

If we let $W = (Z(R)^* \setminus A) \cup C$, then a similar argument to that of the proof of Theorem 4.1 shows that W is a metric basis for the graph $\overline{\Gamma}(R)$ and hence $dim_M(\overline{\Gamma}(R)) = |Z(R)^*| - 3^{n+m} + m + 2$. □

We end this paper with the following result.

Corollary 4.1 *Let* $R \cong S_1 \times S_2 \times S_3 \times S_4$, *where* $S_1 \cong R_1 \times \cdots \times R_n$, $S_2 \cong R'_1 \times \cdots \times R'_m$, $S_3 \cong F_1 \times \cdots \times F_t$, $S_4 = F'_1 \times \cdots \times F'_s$ *such that* $(R_i, \mathfrak{m}_i)$ *is a finite local ring with* $\mathfrak{m}_i^2 = 0$, $|\mathfrak{m}_i| \geq 3$ *for every* $1 \leq i \leq n$, $(R'_i, \mathfrak{m}'_i)$ *is a local ring with* $|\mathfrak{m}'_i| = 2$ *for every* $1 \leq i \leq m$, F_i *is a finite field with* $|F_i| > 2$, *for every* $1 \leq i \leq t$ *and* $F'_i \cong \mathbb{Z}_2$, *for every* $1 \leq i \leq s$, $m, s \geq 2$. *Then* $dim_M(\overline{\Gamma}(R)) = |Z(R)^*| - 3^{n+m}2^{t+s} + m + s + 2$.

Proof Assume that $0 \neq a_i \in \mathfrak{m}_i$, $0 \neq a'_i \in \mathfrak{m}'_i$ for some elements $a_i \in R_i$, $a'_i \in R'_i$,

$A = \{(x_1, \ldots, x_{n+m+t+s}) \in V(\overline{\Gamma}(R)) \mid x_i \in \{0, 1, a_i, a'_j\}\}$,

$B = \{(\underbrace{1, \ldots, 1}_{n}, x_1, \ldots, x_m, \underbrace{1, \ldots, 1}_{t+s}) \in A \mid x_i \neq 0 \text{ for every } 1 \leq i \leq m\}$ and

$C = \{(\underbrace{1, \ldots, 1}_{n+m+t}, x_1, \ldots, x_s) \in A \mid x_i = 0, 1\ 1 \leq i \leq s\}$.

We complete the proof in two steps.

Step 1. We show that $dim_M(\overline{\Gamma}(R)) \geq |Z(R)^*| - 3^{n+m}2^{t+s} + m + s + 2$.

A similar proof to that of Theorem 4.1 shows that $Z(R)^* \setminus A \subseteq W'$, where W' is a metric basis for $\overline{\Gamma}(R)$. Again, by a similar argument used in the proof of Theorem 4.1, let $a \in B \cup C$. Then $d(a, b) = 1$ for every $b \in Z(R)^* \setminus A$. This means that to resolve the elements of $B \cup C$ the set W' must have some more elements of $V(\overline{\Gamma}(R)) \setminus A$. Since $|B| = 2^m - 1$ and $|C| = 2^s - 1$, W' has at least $m + s$ more elements. Thus $dim_M(\overline{\Gamma}(R)) \geq |Z(R)^*| - 3^{n+m}2^{t+s} + m + s + 2$.

Step 2. We show that $dim_M(\overline{\Gamma}(R)) \leq |Z(R)^*| - 3^{n+m}2^{t+s} + m + s + 2$.

Let

$B' = \{(\underbrace{0, \ldots, 0}_{n}, \underbrace{a_1', 0, \ldots, 0}_{m}, \underbrace{0, \ldots, 0}_{t+s}), \ldots, (\underbrace{0, \ldots, 0}_{n}, \underbrace{0, \ldots, 0, a_m'}_{m}, \underbrace{0, \ldots, 0}_{t+s})\}$,

$C' = \{(\underbrace{0, \ldots, 0}_{n+m+t}, 1, 0, \ldots, 0), \ldots, (\underbrace{0, \ldots, 0}_{n+m+t}, 0, \ldots, 0, 1)\}$ and $W = \{Z(R)^* \setminus A\} \cup \{B' \cup C'\}$. By similar proofs to those of Theorems 4.3 and 3.2, we get $\{Z(R)^* \setminus A\} \cup \{B' \cup C'\}$ is a resolving set. Thus $dim_M(\overline{\Gamma}(R)) \leq |Z(R)^*| - 3^{n+m}2^{t+s} + m + s + 2$.

By Steps 1,2 $dim_M(\overline{\Gamma}(R)) = |Z(R)^*| - 3^{n+m}2^{t+s} + m + s + 2$. □

Acknowledgements The authors express their deep gratitude to the referee for the careful reading and valuable suggestions which have definitely improved the paper.

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