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# A Numerical Study on Plane Contact Problem Involving Inclusions

D D Xie<sup>1</sup>, K Zhu<sup>1\*</sup>, Y H An<sup>2</sup>, P Li<sup>2</sup> and X Q Jin<sup>1,2</sup>

<sup>1</sup> College of Aerospace Engineering, Chongqing University, Chongqing, 400044, China

<sup>2</sup> State Key Laboratory of Mechanical Transmissions, Chongqing University, Chongqing, 400044, China  
E-mail: kaychu0523@foxmail.com

**Abstract.** Hertz's contact theory is widely used in elastic solid contact problems. In this paper, the applicability of Hertz's contact theory in conformal contact problems was examined. Taking deep groove ball bearing and the outer raceway model as an example, both the FEM analyses and the theoretical studies were conducted. It is found that when the radius coefficient of groove curvature is greater than 0.55, Hertz's theory can be practically applied for conformal contact. Contact problems of rolling bearings are generally solved by Hertz's contact theory without considering inclusions and inhomogeneities in the material. In this paper, inclusions are presented in the contact analyses, and the corresponding elastic field distribution of the contact-inclusion model is investigated by using the finite element method.

## 1. Introduction

In recent years, the machinery industry is facing a new round of reshuffle, entering the transition period of the new industrial era, showing the characteristics of intellectualization, informatization and integration. With the rapid development of rail transit, new energy vehicles, aerospace, nuclear power energy, intelligent robots, and other engineering science and technology fields, the requirements for fatigue life and service conditions of mechanical parts are becoming more and more stringent. Therefore, important parts of intelligent manufacturing and advanced equipment manufacturing require engineering materials to have better comprehensive performance. Contact problem is a key research object in the field of machinery and mechanics. As an essential part of machinery manufacturing, the working performance of rolling bearings can directly or indirectly affect the performance of the whole mechanical system. The theoretical solution to this kind of problem usually resorts to Hertz's classical contact theory.

Since Hertz's theory was proposed, it has been widely applied to analyze the contact of elastic solids. As described in Johnson's classical textbook of contact mechanics [1], Hertz's theory is most suitable for counter-conformal contact. For two-dimensional contact problems, the classical Hertz's contact theory applies to the case of counter-conformal contact featuring a small contact area, hence the local area can be regarded as a half-plane, and the profile of the indenter can be approximated by the parabolic equation. Persson [2] derived the integral equation for plane contact problem based on the Airy stress function and presented closed-form formula for the pressure distribution and the relation between the load and the size of the contact area. Hou and Hills [3], and Ciavarella and Decuzzi [4] respectively solved Persson's analytical solution when two contacts are of the same material. Ciavarella and Decuzzi [5] further proposed an approximate solution where two contacts are



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of different materials. Based on the Michell-type stress function [6], Liu [7] used a numerical model to study the frictionless contact between two conformal cylinders. With assistance of Green's function, the analytic function was converted into the influence coefficient to calculate the displacement of the pin and plate, and the general solution of the Fourier series was obtained. The numerical results are consistent with Persson's theory, and the maximum contact pressure calculated by Hertz's theory is small, while the semi-contact angle is large. After that, Liu [8] extended the model to study the influence of geometric irregularities on the contact pressure distribution.

Ball-raceway contact for accurate theoretical analysis usually involves both counter-conformal and conformal contacts. For the counter-conformal contact, Hertz contact theory is usually used, while the plane conformal contact problem generally adopts Persson's method [2]. The present work examines the applicability of the Hertz theory to conformal contact, and the scope of application is determined using the finite element method. Also note that in the process of material processing, the existence of inhomogeneities tends to cause the plastic deformation of the workpiece, therefore may severely affect the elastic field distribution inside the material, and consequently the mechanical performance of the material. A contact model considering the existence of material inhomogeneities is established in this work, where the coupling effects between the inclusion and contact are expected to be of significance in studying the actual bearing contact.

## 2. Two-dimensional Hertz Contact Theory

A plane strain contact model of two cylinders under normal force  $P$  is considered in the present work [1]. The contact area is denoted by contact semi-width  $a$ . According to Hertz's contact theory, the initial gap  $h$  between the corresponding points on the mating surfaces is

$$h = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) x^2 = \frac{1}{2R} x^2 \quad (1)$$

where  $R_1$  and  $R_2$  are the radii of cylinders in contact, and  $R$  is the equivalent radius. In the contact zone, the following geometric condition is enforced,

$$\bar{u}_{z1} + \bar{u}_{z2} = \delta - h \quad (2)$$

where  $\delta$  is the penetration depth at  $x=0$ . Displacements  $\bar{u}_{z1}$  and  $\bar{u}_{z2}$  of the two points could be determined by Hertz's theory. The contact pressure is then obtained as follows

$$p(x) = \frac{2P}{\pi a^2} \sqrt{a^2 - x^2} \quad (3)$$

The maximum contact pressure  $p_0$  could be written as

$$p_0 = \frac{2P}{\pi a} = \sqrt{\frac{PE^*}{\pi R}} \quad (4)$$

where  $E^*$  is the equivalent modulus, and satisfies the following relationship

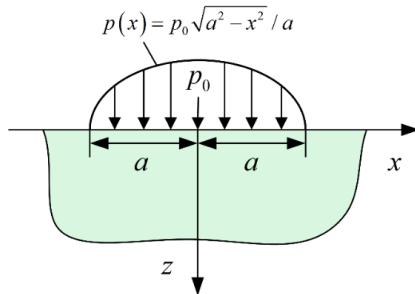
$$\frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \quad (5)$$

where  $E_1, E_2$  are Young's moduli, and  $\nu_1, \nu_2$  are Poisson's ratios of the two cylinders respectively.

Under the contact load shown in figure 1, the stress components in the elastic half-plane could be written in closed-form

$$\begin{cases} \sigma_x = -\frac{zp_0}{\pi} \left( \frac{a^2 + 2x^2 + 2z^2}{a} \phi - \frac{2\pi}{a} - 3x\psi \right) \\ \sigma_z = -\frac{zp_0}{\pi} (a\phi - x\psi) \\ \sigma_{xz} = -\frac{z^2 p_0}{\pi} \psi \end{cases} \quad (6)$$

where  $\phi = \pi(1+k_3)/k_1 k_4$ ,  $\psi = \pi(1-k_3)/k_1 k_4$ , and  $k_1 = (x+a)^2 + z^2$ ,  $k_2 = (x-a)^2 + z^2$ ,  $k_3 = \sqrt{k_2/k_1}$ ,  $k_4 = \sqrt{2k_3 + (k_1 + k_2 - 4a^2)/k_1}$ .



**Figure 1.** Pressure distribution in the contact area

The above discussion aims at cases of two-dimensional counter-conformal contact. However, for two-dimensional conformal contact, Hertz's contact theory generally is not guaranteed due to the large contact zone. To examine the applicability of Hertz theory for conformal contact, it is noted that the radius of the concave contact surface is considered to be negative, and accordingly the equivalent radius is determined by  $1/R = 1/R_1 - 1/R_2$ . Details of the verification is presented in the next section.

### 3. Study on Contact-inclusion Coupling Model

#### 3.1. The Applicability of Conformal Contact

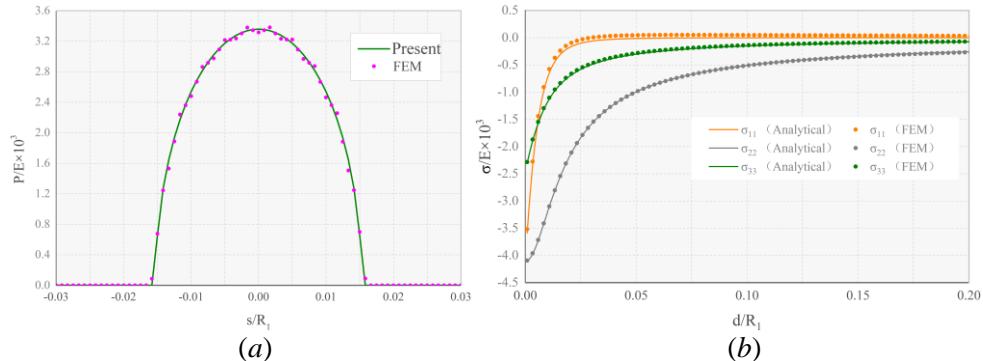
As a typical case of conformal contact, the model of a bearing ball under and the outer raceway is considered. The radius of the ball and the outer raceway is 6mm and 30mm respectively, both elastic moduli are 207GPa, and Poisson's ratios are 0.3. A concentrated force of 100N is applied on the ball. The values of the maximum contact pressure  $P_0$  and contact semi-width  $a$  obtained by FEM simulation are presented in table 1. Due to the negligible errors, it could conclude that the Hertz theory is applicable for conformal contact. The contact pressure is shown in figure 2(a), and the variation of stresses along the z-axis is presented in figure 2(b). By comparison, it is found that the values of FEM are in good agreement with that obtained by the analytical method, which indicates the effectiveness of the Hertz contact theory in the analysis of conformal contact. Since the contact semi-width of conformal contact is larger than that of counter-conformal contact, it is necessary to further explore the valid range of Hertz's theory.

The application of Hertz's theory is limited by the radius coefficient of groove curvature, which is the ratio of raceway radius to ball diameter as

$$\phi = \frac{R_2}{2R_1} \quad (7)$$

**Table 1.** Results and error of conforming contact

| Parameter                                 | Theoretical Value | FEM     | Error  |
|---|-------------------|---------|--------|
| Contact semi-width $a/\text{mm}$          | 0.0916            | 0.0939  | 2.511% |
| Maximum contact pressure $P_0/\text{MPa}$ | 694.951           | 699.914 | 0.714% |

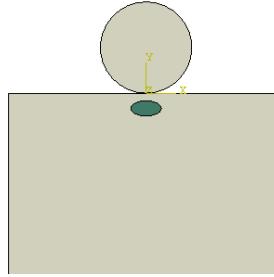
**Figure 2.** (a) Contact pressure distribution; (b) Stress components along the z-axis.**Table 2.** The variation of contact semi-width and maximum pressure with radius coefficient of groove curvature.

| Ball radius $R_1/\text{mm}$ | Raceway radius $R_2/\text{mm}$ | Radius coefficient of groove curvature | Analytical contact semi-width $a/\text{mm}$ | Contact semi-width (FEM) $a/\text{mm}$ | Contact semi-width error | Analytical maximum pressure $P_0/\text{GPa}$ | Maximum pressure (FEM) $P_0/\text{GPa}$ | $P_0$ error |
|-----------------------------|--------------------------------|--|---|--|--------------------------|--|---|-------------|
| 6                           | 30                             | 2.50                                   | 0.0916                                      | 0.0939                                 | 2.511%                   | 694.951                                      | 699.914                                 | 0.714%      |
| 6                           | 18                             | 1.50                                   | 0.1004                                      | 0.1036                                 | 3.287%                   | 634.400                                      | 640.160                                 | 0.907%      |
| 6                           | 12                             | 1.00                                   | 0.1159                                      | 0.1199                                 | 3.538%                   | 549.406                                      | 556.645                                 | 1.317%      |
| 6                           | 9                              | 0.75                                   | 0.1420                                      | 0.1472                                 | 3.662%                   | 448.589                                      | 456.956                                 | 1.865%      |
| 6                           | 7.2                            | 0.60                                   | 0.2008                                      | 0.1919                                 | 4.383%                   | 317.200                                      | 325.672                                 | 2.671%      |
| 6                           | 6.6                            | 0.55                                   | 0.2719                                      | 0.2500                                 | 8.054%                   | 234.268                                      | 246.659                                 | 5.289%      |

To obtain the valid range of Hertz's contact theory, the radius of the ball is proposed to be constant. The radius coefficient of groove curvature varies with the radius of the outer raceway. As shown in table 2, the errors of contact radius and the maximum contact pressure are less than 5% when the radius coefficient of groove curvature is greater than 0.55. The present study shows that prediction by the Hertz theory is better suited for larger radius coefficient of groove curvature, and once its value drops below 0.55, the errors may increase significantly, implying that under such circumstance the Hertz solution tends to be not reliable anymore.

### 3.2. Half-plane Contact-inclusion Model

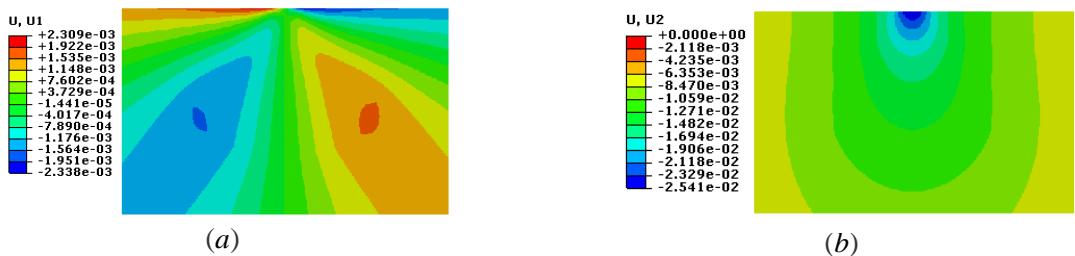
Hertz's contact theory is utilized to solve the contact problem of Rolling bearing usually when the inclusions or inhomogeneities inside the materials are disregarded. However, the influences of inclusions or inhomogeneities on the response fields are usually not negligible. To explore this issue, a coupled contact-inclusion model is proposed in the following, as shown in figure 3. In this work, a semi-infinite plane containing a thermal inclusion is considered, and the resultant field distribution is studied by using the finite element method.



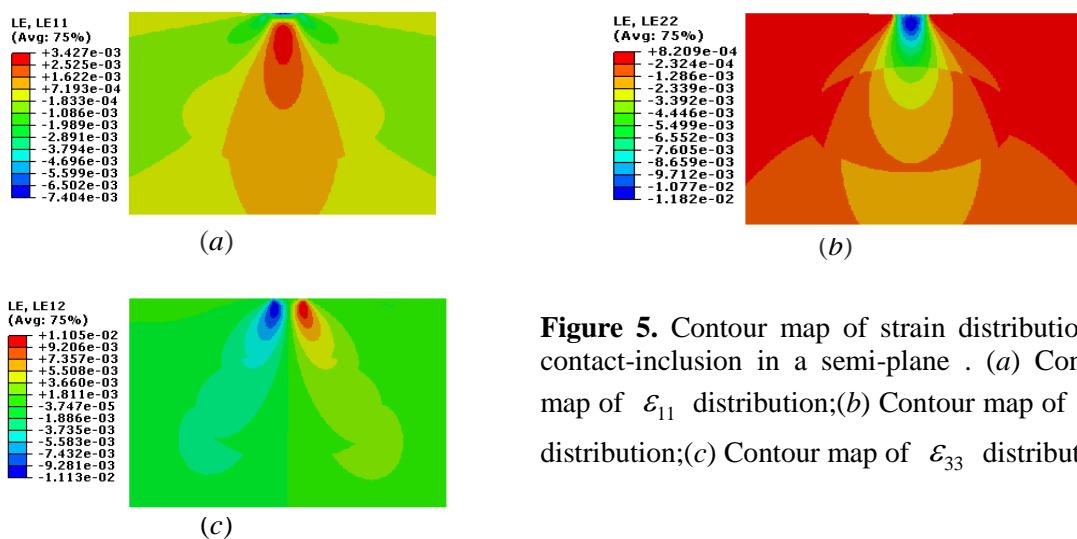
**Figure 3.** Contact - inclusion coupling model.

### 3.3. Contact-inclusion Response Field Analysis

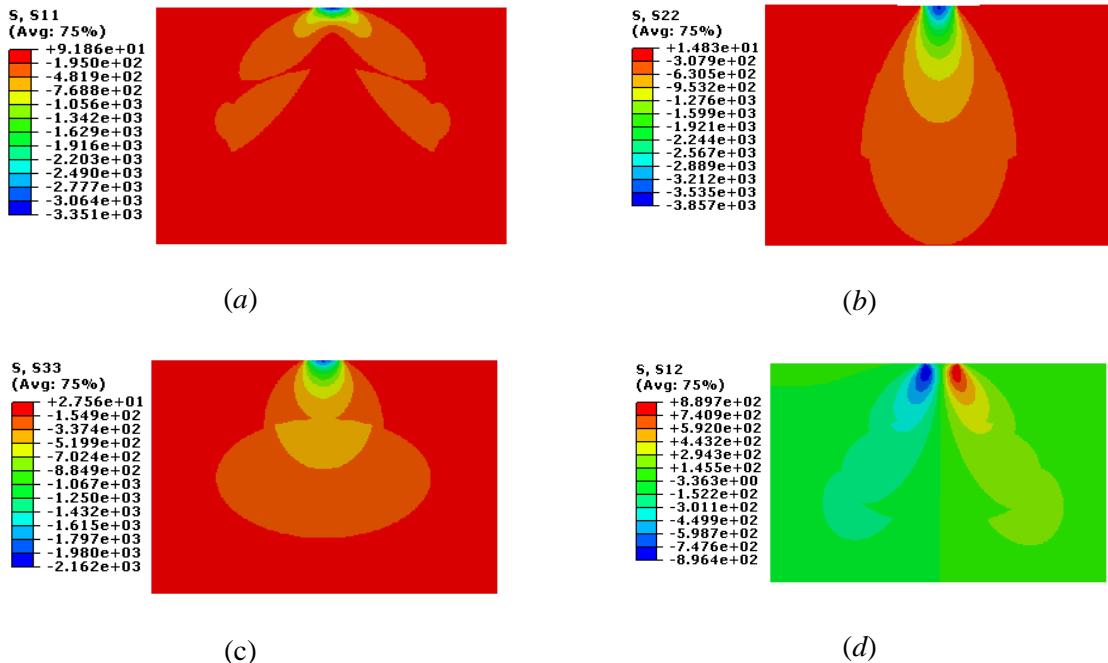
Finite element analyses are conducted for the inclusion-contact model, where the contour plot of the displacement, strain and stress are respectively shown in figures 4-6. The response characteristics, which are induced by the coupling effect of thermal expansion and contacts of the inclusion, are presented in the contour plot of the contact and inclusion area. It is seen that the resulting elastic fields may be significantly influenced in the vicinity of the inclusion. Such influence would diminish when the field location moves away from the contact and inclusion zones, and gradually vanish at infinity. From the figures, it is also obvious to find that the displacement, strain, and stress are either symmetric or antisymmetric, as the indenter is aligned with the inclusion.



**Figure 4.** Contour map of displacement distribution of a contact-inclusion in a semi-plane. (a) Contour map of  $u_1$  distribution; (b) Contour map of  $u_2$  distribution.



**Figure 5.** Contour map of strain distribution of contact-inclusion in a semi-plane . (a) Contour map of  $\epsilon_{11}$  distribution;(b) Contour map of  $\epsilon_{22}$  distribution;(c) Contour map of  $\epsilon_{33}$  distribution.



**Figure 6.** Contour map of stress distribution of contact-inclusion in a semi-plane. (a) Contour map of  $\sigma_{11}$  distribution; (b) Contour map of  $\sigma_{22}$  distribution; (c) Contour map of  $\sigma_{33}$  distribution; (d) Contour map of  $\sigma_{12}$  distribution.

#### 4. Conclusion

Hertz's contact theory is usually used for counter-conformal contact problems. In this paper, the applicability of Hertz's theory to counter-conformal contact problems is firstly verified, and its application range is studied. The results show that the applicability of Hertz's theory is better suited when the radius coefficient of groove curvature is greater than 0.55. In addition, the coupled contact-inclusion model is investigated by the finite element analysis, and the distribution of the elastic fields and the interactions between the contact and inclusion are presented.

#### Acknowledgments

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