Edge-Coloring Technique to Analyze Elementary Trapping Sets of Spatially-Coupled LDPC Convolutional Codes

Mohammad-Reza Sadeghi and Farzane Amirzade

***Abstract*—In this letter, for the first time, an *edge-coloring technique* is proposed to characterize a certain elementary trapping set (ETS) and to obtain sufficient conditions to avoid small ETSs from occurrence in the Tanner graph of SC-LDPC convolutional codes. This technique is applicable to all protograph-based LDPC codes with different girths whose protographs are single-edge, that is, the entries of their base matrices are 0, 1. To further demonstrate the effectiveness of our proposed technique, we apply it to Time-Invariant SC-LDPC-CCs with girths 6 and 8 and column weights up to 5.**

***Index Terms*—SC-LDPC convolutional codes, girth, tanner graph, elementary trapping set, edge coloring.**

I. Introduction

Among different features influencing the performance of an LDPC code we can point out small cycles, the girth, g, and graphical structures like trapping sets (TS) in the Tanner graph (TG) of the code. The TG of spatially-coupled LDPC codes or LDPC convolutional codes, which are constructed by coupling together a series of uncoupled TGs into a single coupled chain, do not contain a number of such graphical features which are broken in the coupling process.

Although the coupling process causes the removal of some TSs, it cannot guarantee the elimination of all TSs with small sizes. A technique to avoid harmful TSs with small sizes is increasing the girth. Recently, many researchers have focused on constructing spatially-coupled LDPC convolutional codes (SC-LDPC-CCs), in two categories single-edge and multiple-edge, with the lowest constraint length and free of short cycles and there are useful results regarding the lower bounds on the *syndrome former memory order* of SC-LDPC-CCs with girth up to 12, [1]–[4].

Since codewords of weight a are just $(a, 0)$ TSs, removing subgraphs of an $(a, 0)$ TS up to a certain size results in a code free of low-weight codewords and consequently with a good minimum distance. Khatami et al. proposed an algorithm to find all low-weight codewords of a quasi-cyclic LDPC code with a certain column weight and girth [5]. More recently, a new approach to design SC-LDPC-CCs free of low-weight codewords was provided by Battaglioni *et al.* [6]. In that approach, column-weight-2 submatrices of the parity-check matrix are associated to cycles of the TG and avoiding some of those cycles results in removing low-weight codewords. Nguyen *et al.* [7] proposed the PEG algorithm to construct (m, n)-regular LDPC codes free of some small TSs and Diouf *et al.* [8] presented an improved PEG algorithm which results in a $(3, n)$-regular LDPC code with girth 8 whose TG is free of $(5, 3)$ TSs and contains a minimum number of $(6, 4)$ TSs.

All algorithms mentioned above are search-based and are implemented on the parity-check matrix of a code. In this letter, by taking a graph theoretical approach, we propose a method named as *edge-coloring technique* (ECT) to characterize and avoid *elementary* TSs (ETSs) in the TG [9]. Obtaining a necessary and sufficient condition to remove ETSs from the TG is difficult specially when the ETS has a complex structure. The ECT contributes to explicitly determine sufficient conditions for an exponent matrix to avoid any ETS from occurrence in the TG. We show that this technique is applicable to all protograph-based LDPC codes whose protographs are single-edge. We apply our technique to Time-Invariant (TI)-SC-LDPC-CCs with girths 6, 8 and column weights 3 to 5.

In the sequel, Section II presents some basic definitions. In Section III, we show the TGs of single-edge fully-connected TI-SC-LDPC-CCs are free of some specific ETSs. Section IV presents the ECT to characterize ETSs in the TG. In Section V, we consider sufficient conditions to have (m, n)-regular TI-SC-LDPC-CCs with girths 6 and 8 whose TGs are free of some small ETSs. In Section VI, we summarize our results.

II. Preliminaries

A TI-SC-LDPC-CC with an $m \times n$ binary base matrix W and a syndrome former memory order m_h can be associated to an $m \times n$ exponent matrix $B = [b_{ij}]$, where $b_{ij} \in \{0, 1, \ldots, m_h\}$ or $b_{ij} = (\infty)$. For each integer $t \in \{0, 1, \ldots, m_h\}$ we construct an $m \times n$ binary matrix H_t in which ${H_t}^{(i,j)} = 1$ if $b_{ij} = t$ otherwise ${H_t}^{(i,j)} = 0$. Assuming the matrices H_is are *syndrome former matrices* we have the binary matrix H as the parity-check matrix of a single-edge TI-SC-LDPC-CC. A $2k$-cycle in the TG of a TI-SC-LDPC-CC is associated to the following sum,

$$\sum_{i=0}^{k-1} \pm \left| b_{m_i n_i} - b_{m_i n_{i+1}} \right| = 0, \tag{1}$$

where $n_k = n_0$, $m_i \neq m_{i+1}$, $n_i \neq n_{i+1}$ and $b_{m_i n_i}$ is the (m_i, n_i)-th entry of B [10]. Assuming $\left| b_{m_i n_j} - b_{m_i n_{j'}} \right| = \delta_{i,jj'}$, in Equation (1) $+\delta_{i,jj'} (-\delta_{i,jj'})$ is assigned to a transition from a variable-node to another one along the same check-node to the right, in which $n_j < n_{j'}$ (to the left, in which $n_j > n_{j'}$). Similar equation to consider

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$2k$-cycles is given in [2], where pluses (minuses) are allocated to a transition from a check-node to another one along the same variable-node with a downward (upward) direction.

$$H = \begin{bmatrix} H_0 & 0 & 0 & \ddots \\ H_1 & H_0 & 0 & \ddots \\ H_2 & H_1 & H_0 & \ddots \\ \vdots & H_2 & H_1 & \ddots \\ H_{m_h} & \vdots & H_2 & \ddots \\ 0 & H_{m_h} & \vdots & \ddots \\ 0 & 0 & H_{m_h} & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Definition 1: An (a, b) *trapping set* (TS) is a set of a variable-nodes in the TG which induce a subgraph of the TG with exactly b check-nodes of odd degrees and an arbitrary number of even degree check-nodes. An (a, b) TS is *elementary* (ETS) if all check-nodes are of degree 1 or 2.

Definition 2: Two TSs T_1 and T_2 are isomorphic if there is a bijection f between the nodes of T_1 and nodes of T_2 such that any two nodes v, c of T_1 are adjacent if and only if $f(v), f(c)$ are adjacent in T_2.

Definition 3: For a graph G corresponding to an ETS, a *variable node* (VN) *graph* is constructed by removing all degree-1 check-nodes, defining variable-nodes of G as its vertices and degree-2 check-nodes connecting the variable-nodes in G as its edges.

III. Relation Between the Existence of ETSs and Edge Coloring of VN Graphs

In this section, benefiting from an important topic in graph theory, the edge coloring and its well-known results such as Vizing's theorem [11], we characterize an ETS in the TG of a TI-SC-LDPC-CC whose exponent matrix is B.

Definition 4: An edge coloring of a graph G is an assignment of colors (labels) to the edges of the graph so that no two adjacent edges have the same color (label). The minimum required number of colors for the edges of a given graph is *chromatic index* of the graph which is denoted by $\mathcal{X}'(G)$, or simply $\mathcal{X}'$, which indicates the graph has a $\mathcal{X}'$-edge-coloring.

Theorem 1: [11] If $\Delta(G)$ is the maximum degree of a graph G, then $\Delta(G) \leq \mathcal{X}' \leq \Delta(G) + 1$.

Proposition 1: Given a TI-SC-LDPC-CC with an $m \times n$ exponent matrix B. The necessary condition for the TG to contain an ETS is that its corresponding VN graph has an m-edge-coloring.

Proof: Suppose the TG of a TI-SC-LDPC-CC with an $m \times n$ exponent matrix B contains the ETS. The maximum degree of its VN graph is at most m. Without loss of generality, suppose the maximum degree of the VN graph is equal to m. According to Vizing's Theorem, the chromatic index of the VN graph is m or $m + 1$. Assume (by contradiction) that the VN graph has no m-edge-coloring. Therefore, according to Definition 4, coloring the edges of the VN graph with m labels results in at least two adjacent edges with the same color.

Every vertex (edge) of a VN graph corresponds to a column (row) of B. The degree of each vertex in a VN graph determines the number of rows of B which are involved in an ETS. Each edge between two variable-nodes is colored by that row index of B corresponding to the check-node connecting those variable-nodes. To clarify this type of edge-coloring, we replace each 1-component in H_i with row indices of B, that is, for each H_i we replace the 1s of the j-th row by j, for $j = 1, 2, \ldots, m$. We denote the obtained matrices by H'_i. If the VN graph has no m-edge-coloring, then there is a variable-node which is connected to two edges with the same color t. According to the above replacements in all H_is, those two adjacent edges with the same color show the existence of two H'_{i_1} and H'_{i_2} both contain t in their t-th row. So, the t-th row of B contains two values i_1 and i_2 occurring in the same column. Hence, there is an entry like $[i_1, i_2]$ in the t-th row of B which corresponds to a multiple-edge protograph. This contradicts the fact that B is associated to single-edge protographs. ■

We can use a similar approach of Proposition 1 for any single-edge protograph-based LDPC code such as QC-LDPC codes with lifting degree N. Thus, using a generalization of Proposition 1, Corollaries 1, 2, 3 can be applied to any protograph-based LDPC code with an all-one base matrix. In order to prove them we also need some definitions and well-known results in graph theory as well as some results about ETSs in [12].

Definition 5: A complete graph is a graph in which every pair of distinct vertices is connected by a unique edge. A complete graph on n vertices is denoted by K_n.

Corollary 1: A fully-connected $(2\ell, n)$-regular TI-SC-LDPC-CC with girth 6 contains no $(2\ell + 1, 0)$ ETS.

Proof: A complete graph on $2\ell + 1$ vertices has $\mathcal{X}' = 2\ell + 1$ [11]. A $K_{2\ell+1}$ is equivalent to the VN graph of a $(2\ell + 1, 0)$ ETS with girth 6. Since it has no 2ℓ-edge-coloring, Proposition 1 proves the TG of a fully-connected TI-SC-LDPC-CC with column weight 2ℓ and girth 6 contains no $(2\ell + 1, 0)$ ETS. ■

Lemma 1: [12] An (a, b) ETS and its VN graph in an LDPC code with column weight m satisfy the followings. (i) If $\frac{b}{a} < 1$, then the VN graph has at least one vertex of degree m. (ii) If $\frac{b}{a} < 1$, then 4-cycle free TGs contain no (a, b) ETSs with $a \leq m$. (iii) If the VN graph contains $|E|$ edges, then $b = am - 2|E|$. (iv) If a is even, then b is also an even number. (v) If a is odd, then parameters m and b both are even or odd.

Definition 6: An *independent edge set* of a graph G is a subset of the edges such that no two edges in the subset share a vertex of G. An independent edge set with the maximum cardinality is a *maximum independent edge set* whose cardinality is denoted by $\alpha'(G)$.

Theorem 2: [13] Suppose $E(G)$ is the set of edges in a graph G. If $|E(G)| > \alpha'(G)\Delta(G)$, then $\mathcal{X}' = \Delta(G) + 1$.

Corollary 2: A girth-6 fully-connected TI-SC-LDPC-CC contains no $(a, 1)$ ETSs.

Proof: We prove this corollary in two steps. First, we suppose $m = 2\ell$. Lemma 1 part (iii) concludes that if the TG contains an $(a, 1)$ ETS, then the number of edges of its VN graph satisfies in $1 = a(2\ell) - 2|E|$ which is impossible.

Second, suppose the column weight is $m = 2\ell + 1$. If $a = 2k$, then according to Lemma 1 part (iv), b cannot be an odd number. Hence, a $(2\ell + 1, n)$-regular TI-SC-LDPC-CC with girth 6 contains no $(2k, 1)$ ETSs. Now, let $a = 2k + 1$. Lemma 1 part (iii) implies that the number of edges of the VN graph of an $(a, 1)$ ETS in a $(2\ell + 1, n)$-regular TI-SC-LDPC-CC satisfies in $1 = a(2\ell+1) - 2|E|$ which gives $|E| = \frac{(2\ell+1)a-1}{2}$. We denote this VN graph by G. Since $a = 2k+1$ is an odd number, the maximum number of independent edges of the VN graph is $\alpha'(G) \leq \frac{a-1}{2}$. Moreover, according to Lemma 1 part (i), the VN graph of the $(a, 1)$ ETS has the maximum degree $\Delta(G) = 2\ell + 1$. So, we have: $|E| = \frac{(2\ell+1)a-1}{2} > (2\ell + 1)\frac{a-1}{2} \geq (2\ell + 1)\alpha'(G) = \alpha'(G)\Delta(G)$.

From $|E| > \alpha'(G)\Delta(G)$ and Theorem 2 we conclude that the VN graph of the $(a, 1)$ ETS has $\mathcal{X}' = \Delta(G) + 1$. Hence, the VN graph of a $(2k + 1, 1)$ ETS has no $(2\ell + 1)$-edge-coloring and Proposition 1 implies that the TG of a TI-SC-LDPC-CC with $g = 6$ and $m = 2\ell + 1$ contains no $(2k + 1, 1)$ ETSs. ∎

Corollary 3: A fully-connected TI-SC-LDPC-CC of column weight 2ℓ, $l > 1$, contains no $(2\ell + 1, 2)$ ETSs.

Proof: We prove this corollary in two steps. First, suppose the girth of the TG is 6. The VN graph of a $(2\ell+1, 2)$ ETS in LDPC codes with $m = 2\ell$ and $g = 6$ is obtained by removing an edge from $K_{2\ell+1}$. We denote this VN graph by G. Since $g = 6$ and the VN graph has triangle, $|E(G)| = \ell(2\ell+1) - 1$, $\Delta(G) = 2\ell$ and $\alpha'(G) = 2$. Therefore, for each $\ell \geq 2$ we have $|E(G)| > \alpha'(G)\Delta(G)$ and Theorem 2 shows that $\mathcal{X}' = 2\ell + 1$. Thus, from Proposition 1 we conclude that a $(2\ell, n)$-regular TI-SC-LDPC-CC with girth 6 contains no $(2\ell+1, 2)$ ETS. Second, suppose the girth is 8. In LDPC codes with $m = 2\ell$ and $g = 8$ the parameters of (a, b) ETSs with $\frac{b}{a} < 1$ fulfill the inequalities $a \geq 4\ell - 1$ and $b \geq 2\ell^2 - 1$ [12]. Thus, a $(2\ell, n)$-regular TI-SC-LDPC-CC with $g = 8$ contains no $(2\ell + 1, 2)$ ETSs. Similarly, it is proved for codes with $g \geq 10$ [12]. ∎

IV. Edge-Coloring Technique to Remove ETSs

Suppose a single-edge protograph-based LDPC code with an $m \times n$ exponent matrix B contains an (a, b) ETS. So, its VN graph has an m-edge-coloring with colors $1, \ldots, m$, the row indices of B. In this section, in order to avoid the occurrence of such ETS in the TG we propose an *edge-coloring technique* (ECT) by proceeding the following steps.

Step 1: We obtain all non-isomorphic ETSs and their corresponding VN graphs.

Step 2: We try on different ways to color the edges of each VN graph.

Step 3: We look for k-cycles with the same edge coloring in all VN graphs colored in step 2.

If the colors of the edges of a k-cycle in the VN graph are $i_1, \ldots, i_k \in \{1, \ldots, m\}$, then there is a $2k$-cycle obtained from the rows $i_1, \ldots, i_k$ of B. If by applying all coloring methods to all VN graphs, in the second step, we end up with k-cycles with edge colors $i_1, \ldots, i_k$ in colored VN graphs, in the third step, then avoiding $2k$-cycles obtained from the rows $i_1, \ldots, i_k$ of B results in an $(m + 1)$-edge-coloring for each VN graph. Thus, Proposition 1 and its generalized form give a protograph-based LDPC code free of the desired (a, b) ETS.

Hereafter, a $2k$-cycle obtained from k rows $i_1, \ldots, i_k$ of an exponent matrix B is denoted by $2k$-cycle$_{\{i_1,\ldots,i_k\}}$. For example, a 6-cycle obtained from three rows i, j, k of B is denoted by 6-cycle$_{\{i,j,k\}}$. In addition, each 4-cycle in a VN graph corresponds to an 8-cycle in the TG. Hence, 8-cycles obtained from two rows i, j and 8-cycles obtained from three rows i, j, k of B, where i is used twice in the 8-cycle, are denoted by 8-cycle$_{\{i,j\}}$ and 8-cycle$_{\{i,j,i,k\}}$, respectively.

The following Lemma is provided to clarify how to use the edge-coloring technique.

Lemma 2: A sufficient condition to remove $(6, 4)$ ETSs from a fully-connected TI-SC-LDPC-CC with $g = 6$ and $m = 5$ is to avoid 6-cycle$_{\{1,2,3\}}$, 6-cycle$_{\{1,2,4\}}$ and 6-cycle$_{\{2,3,4\}}$.

Proof: The VN graph of a $(6, 4)$ ETS contains 6 vertices of degree at least 3, 13 edges and the maximum degree 5. The VN graphs of two non-isomorphic $(6, 4)$ ETSs, shown in Fig. 1 (a), (b), have the degree sequences $d_1 = \{5, 5, 4, 4, 4, 4\}$ and $d_2 = \{5, 5, 5, 4, 4, 3\}$, respectively.

We first investigate the $(6, 4)$ ETS in Fig. 1 (a). Similar scenario holds for the ETS in Fig. 1 (b). We call the VN graph of $(6, 4)$ ETS in Fig. 1 (a) as G_1. Suppose u, v are the nodes of degree 5 and w_1, w_2, w_3, w_4 are the nodes of degree 4. We denote the color of an edge e by $c(e)$. For the edge uv we have $c(uv) \in \{1, 2, 3, 4, 5\}$. We prove that avoiding triangles with colors $(1, 2, 3)$, $(1, 2, 4)$, $(2, 3, 4)$ yields $c(uv) \notin \{1, 2, 3, 4, 5\}$.

In Fig. 2 (a) we illustrate a subgraph of G_1 with an edge coloring such that $c(uv) = 1$. In order to construct G_1, we have to add other edges to Fig. 2 (a). However, if we add the edges w_3w_4 or w_1w_4, then the only color for these edges is 1 by which we have a triple $(1, 2, 3)$ or $(1, 2, 4)$. So, the only vertex which can be connected to w_4 is w_2. As a result, the degree of w_4 will be at most 3. Generally, regardless of the colors we choose for the edges uw_i; $i \in \{1, 2, 3, 4\}$, we conclude that if $c(uv) = 1$, then by avoiding triangles with colors $(1, 2, 3), (1, 2, 4), (2, 3, 4)$ we construct a graph from Fig. 2 (a) such that the degree of each vertex w_1, w_2, w_3, w_4 is at most 3. So, extending Fig. 2 (a) to G_1 is impossible. The same condition happens when we choose $c(uv) = 3, 4$.

If $c(uv) = 2$, Fig. 2 (b), then avoiding triangles with colors $(1, 2, 3)$, $(1, 2, 4)$ and $(2, 3, 4)$ results in a vertex v with three edges with color 5, which is impossible.

Finally, suppose $c(uv) = 5$, Fig. 2 (c). Since the degree of w_2 in G_1 is 4, it is connected to two vertices of w_1, w_3, w_4. However, the mentioned restrictions result in $c(w_1w_2) = c(w_2w_3) = c(w_2w_4) = 5$. So, connecting w_2 to two vertices of w_1, w_3, w_4 causes two adjacent edges with the same color. Generally, if $c(uv) = 5$, then for each $i \in \{1, 2, 3, 4\}$ we

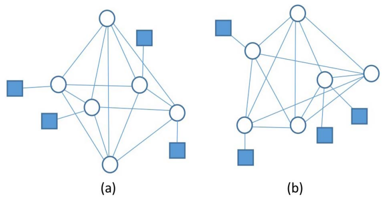

Fig. 1. Two non-isomorphic $(6, 4)$ ETSs in the TG of a girth-6 LDPC code with column weight 5. Each edge corresponds to a degree-2 check node.

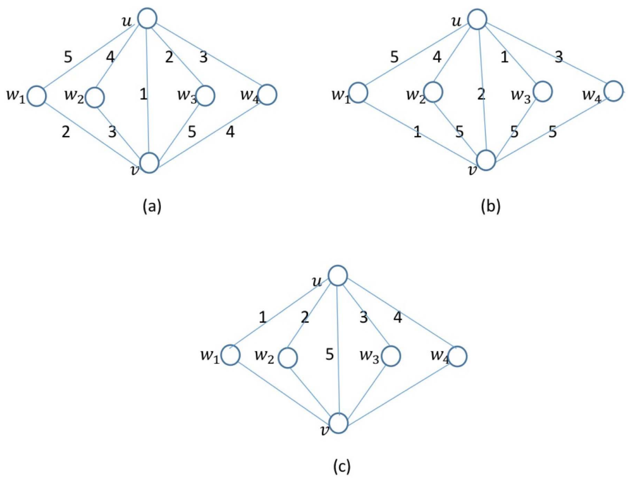

Fig. 2. Figures (a), (b), (c) are subgraphs of the VN graph of the $(6, 4)$ ETS in Fig. 1 (a) with an edge coloring such that $c(uv) = 1, 2, 5$, respectively.

have $c(uw_i) \in \{1, 2, 3, 4\}$ and by taking any coloring method for these edges, we end up with two adjacent edges w_iw_js, $i \neq j \in \{1, 2, 3, 4\}$ with the same color. Consequently, $c(uv) \neq 5$ and extending Fig. 2 (c) to G_1 is impossible. ■

V. Fully-Connected TI-SC-LDPC-CCs Free of Small ETSs

The smallest size of an (a, b) ETS in the TG of a variable-regular LDPC code with column weight m and girths 6 and 8 are $m + 1$ and $2m - 1$, respectively, [12]. In this section, we apply the ECT to girth-6 TI-SC-LDPC-CCs with $m = 3, 4, 5$ to increase the smallest size of an (a, b) ETS, $a < b$, from $m + 1$ to $m + 2$. In fact, instead of applying the PEG algorithm in [7], we use our ECT to provide sufficient conditions to have a TG whose smallest ETSs are $(m + 2, 3)$ ETSs if m is odd and $(m + 2, 4)$ ETSs if m is even. We also apply ECT to girth-8 TI-SC-LDPC-CCs with $m = 3, 4$.

Theorems 3, 6, 7 are based on checking 8-cycles in the exponent matrix. To avoid 8-cycles on two rows of B all $2 \times r$ submatrices of B with $2 \leq r \leq 4$ have to be checked. Equation (1) implies that if the absolute values of the right sides of Equation (1) in two 2×2 submatrices of B are equal, then the TG contains 8-cycles. Suppose a set $A_{(i_1,i_2)}$ consists of absolute values of the right side of Equation (1) when 2×2 submatrices of B with row indices i_1, i_2 are considered. Thus, from Equation (1) we conclude that if $A_{(i_1,i_2)}$ contains repeated elements or $A_{(i_1,i_2)} \cap A_{(i_1,i_3)} \neq \emptyset$, then there exists an 8-cycle$_{\{i_1,i_2\}}$ and 8-cycle$_{\{i_1,i_2,i_1,i_3\}}$, respectively.

Theorem 3: Let B with $m = 3$ be an exponent matrix of a TI-SC-LDPC-CC with $g = 6$ and column weight 3. Avoiding 8-cycles obtained from two rows of B yields a TG free of (a, b) ETSs with $a \leq 5$ and $b < 3$ in which $m_h \geq \frac{n(n-1)}{4}$.

Proof: The smallest (a, b) ETSs, $b < a$, in variable-regular LDPC codes with column weight 3 and $g = 6$ are $(4, b)$ ETSs [12]. According to the ECT, a sufficient condition to remove $(4, 0)$ and $(4, 2)$ ETSs is to avoid 8-cycles obtained from each two rows of B. Corollary 2 proves that the TG of TI-SC-LDPC-CCs is free of $(5, 1)$ ETSs. Hence, removing the mentioned 8-cycles results in a TG free of (a, b) ETSs with $a \leq 5$ and $b < 3$. Since to avoid 8-cycles obtained from two rows of B we only check the right side of Equation (1) in all 2×2 submatrices of B and the number of these submatrices in each two rows of B is $\binom{n}{2}$ we have $|A_{(1,2)}| = |A_{(1,3)}| = |A_{(2,3)}| = \binom{n}{2}$. Since they are subsets of $\{1, 2, \ldots, 2m_h\}$, we have $2m_h \geq \binom{n}{2}$. So, $m_h \geq \frac{n(n-1)}{4}$. ■

Theorem 4: Let B with $m = 4$ be an exponent matrix of a girth-6 TI-SC-LDPC-CC. A sufficient condition to have a TG free of $(5, b)$ ETSs with $b < 5$ and $(6, b)$ ETSs with $b \leq 2$ is to avoid 6-cycle$_{\{1,2,3\}}$ and 6-cycle$_{\{1,2,4\}}$.

Proof: The smallest (a, b) ETS, $b < a$, which satisfies in Lemma 1 part (iii) is $(5, 0)$ ETSs. Corollaries 1 and 3 imply that the TG of a $(4, n)$-regular TI-SC-LDPC-CC with girth 6 is free of $(5, 0)$ and $(5, 2)$ ETSs. According to the ECT, a sufficient condition to remove $(5, 4)$ and $(6, 2)$ ETSs is to avoid 6-cycle$_{\{1,2,3\}}$ and 6-cycle$_{\{1,2,4\}}$. Since eliminating $(5, 4)$ ETSs causes the removal of $(6, 0)$ ETSs, we conclude that avoiding the mentioned 6-cycles results in a TG free of $(5, b)$ ETSs with $b \leq 4$ and $(6, b)$ ETSs with $b \leq 2$. ■

Theorem 5: The smallest (a, b) ETSs, $b < a$, in a $(5, n)$-regular TI-SC-LDPC-CC with girth 6 and without 6-cycle$_{\{1,2,3\}}$, 6-cycle$_{\{1,2,4\}}$ and 6-cycle$_{\{2,3,4\}}$ are $(7, 3)$ ETSs.

Proof: The TG of a $(5, n)$-regular TI-SC-LDPC-CC with girth 6 contains $(6, b)$ ETSs. If we connect two nonadjacent variable-nodes of degree 4 in the VN graph of a $(6, 4)$ ETS, then the VN graph of a $(6, 2)$ ETS is obtained. Similarly, a $(6, 0)$ ETS can be obtained from a $(6, 2)$ ETS. Hence, the VN graph of the $(6, 4)$ ETS is a subgraph of the VN graphs of $(6, 2)$ and $(6, 0)$ ETSs and eliminating $(6, 4)$ ETSs results in the removal of $(6, 2)$ ETSs and $(6, 0)$ ETSs. As shown in Lemma 2, avoiding 6-cycle$_{\{1,2,3\}}$, 6-cycle$_{\{1,2,4\}}$ and 6-cycle$_{\{2,3,4\}}$ is sufficient to remove all $(6, 4)$ ETSs and consequently to eliminate all $(6, b)$ ETSs with $b < a$. From Corollary 2 we imply that the TG is free of $(7, 1)$ ETSs. Thus, removing the desired 6-cycles gives a TG whose smallest (a, b) ETSs with $b < a$ is $(7, 3)$ ETSs. ■

Theorem 6: Let B with $m = 3$ be an exponent matrix of a girth-8 TI-SC-LDPC-CC with column weight 3. Avoiding 8-cycle$_{\{1,2\}}$, 8-cycle$_{\{1,3\}}$ and 8-cycle$_{\{1,2,1,3\}}$ results in a TG free of (a, b) ETSs, $5 \leq a \leq 8$, $b \leq 3$ in which $m_h \geq \frac{n(n-1)}{2}$.

Proof: According to the ECT, $(5, 3)$ ETSs can be avoided from the TG by removing 8-cycle$_{\{1,2,1,3\}}$. So, $A_{(1,2)} \cap A_{(1,3)} = \emptyset$. The non-existence of $(5, 3)$ and $(7, 3)$ ETSs guarantees the non-existence of (a, b) ETSs, $5 \leq a \leq 8$ and $b \leq 3$, [7]. The roots of $(7, 3)$ ETSs are $(5, 3)$ and $(6, 4)$ ETSs.

TABLE I

AN EXPONENT MATRIX OF A $(3, n)$-REGULAR TI-SC-LDPC-CC (QC-LDPC CODE) WITH GIRTH g BASED ON THEOREMS 3, 6.

n	g	SC(QC)	$m_h(N)$	B with an all-zero first column and first row
6	6	SC	17	1 4 10 12 17 4 16 9 17 6
6	6	QC	31	1 3 8 12 18 3 9 24 5 23
7	6	SC	25	1 4 10 18 23 25 2 24 25 6 14 9
8	6	SC	36	1 3 13 21 27 32 36 2 15 34 35 23 9 5
9	6	SC	56	1 3 14 18 23 30 49 55 2 6 5 56 47 25 18 39
9	6	QC	85	1 3 8 14 29 33 49 76 29 2 62 66 76 22 61 79
5	8	SC	22	1 13 20 22 3 8 14 18
6	8	SC	35	1 12 16 21 34 3 10 27 29 35
7	8	SC	55	1 4 14 26 32 55 2 9 17 36 47 52

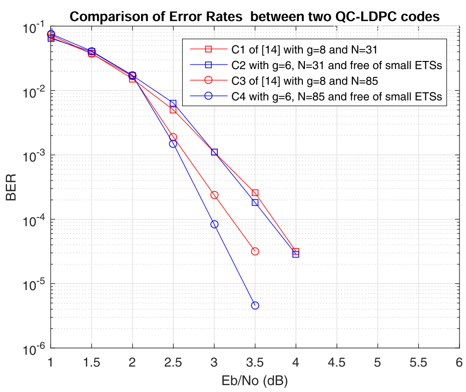

Fig. 3. Comparison of performance curves between QC-LDPC codes, one from [14] and the other constructed based on Theorem 3.

There are two non-isomorphic $(6, 4)$ ETSs. The VN graph of the $(5, 3)$ ETS is a subgraph of the VN graph of one of these two $(6, 4)$ ETSs and eliminating the $(5, 3)$ ETS proves that this type of ETSs is not in the TG. As we concentrate on the edge-coloring of the VN graph of the other $(6, 4)$ ETS, we imply that avoiding 8-cycle$_{\{1,2\}}$, 8-cycle$_{\{1,3\}}$ and 8-cycle$_{\{1,2,1,3\}}$ causes the removal of all $(6, 4)$ ETSs which can be extended to a $(7, 3)$ ETS. Therefore, the sufficient condition to remove $(7, 3)$ ETSs is $|A_{(1,2)}| = |A_{(1,3)}| = \binom{n}{2}$, $A_{(1,2)} \cap A_{(1,3)} = \emptyset$. Since $A_{(1,2)}$ and $A_{(1,3)}$ are subsets of $\{1, 2, \ldots, 2m_h\}$, $2m_h \geq 2\binom{n}{2}$. Thus, $m_h \geq \frac{n(n-1)}{2}$. ■

Theorem 7: Let B with $m = 4$ be an exponent matrix of a TI-SC-LDPC-CC with $g = 8$ and column weight 4. A sufficient condition to remove $(7, 4)$ ETSs is $|A_{(1,2)}| = \binom{n}{2}$, $A_{(1,2)} \cap A_{(1,3)} = \emptyset$ and $A_{(1,2)} \cap A_{(1,4)} = \emptyset$.

Proof: Applying the ECT to $(7, 4)$ ETSs, we conclude that the sufficient condition to remove these ETSs is to avoid 8-cycle$_{\{1,2\}}$, 8-cycle$_{\{1,2,1,3\}}$ and 8-cycle$_{\{1,2,1,4\}}$. ■

In Table I, some exponent matrices of TI-SC-LDPC-CCs and QC-LDPC codes with $m = 3$ and $g = 6, 8$ with minimum m_h, N satisfying Theorems 3 and 6 are provided.

We also present simulation results to show the impact of removing ETSs. We compare the performance curves of two $(3, 6)$-regular QC-LDPC codes C_1, C_2 with $N = 31$ and two $(3, 9)$-regular QC-LDPC codes C_3, C_4 with $N = 85$. The exponent matrices of C_2, C_4 are proposed in Table I and those of C_1, C_3 are provided in [14] as the best known codes with girth 8. Their performances decoded using the sum-product algorithm with 50 iterations are shown in Fig. 3. As can be seen, C_1 and C_2 have almost the same performances, C_2 slightly outperforms C_1. Moreover, C_4 with $g = 6$ and free of small size ETSs significantly outperforms C_3 with $g = 8$.

VI. CONCLUSION

We presented an edge-coloring technique (ECT) to characterize ETSs in the TG of protograph-based LDPC codes whose protographs are single-edge. We applied the ECT to TI-SC-LDPC-CCs in order to provide sufficient conditions for the removal of ETSs with small size from the TG of TI-SC-LDPC-CCs with column weights 3 to 5 and girths 6 and 8.

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