# Probabilistic Scheduling PERT

#### **Program Evaluation and Review Technique**

Farnaz Sadeghpour

Associate Professor University of Calgary Department of Civil Engineering Project Management Center of Excellence

## PERT

**Program Evaluation and Review Tec mique** 

#### Activity / Project Duration Estimation Deterministic: Probabilistic:

Assumes project conditions remain unc hanged Not always realistic

- PERT: Program Evaluation and Review Technique
- Developed by US Navy 1957
- Polaris nuclear submarine project
- Enables incorporating uncertainty into schedule not knowing precise details and durations

#### PERT

Answers questions like:

#### What is:

- The probability that the project will finish in 61 days?
- The probability that the project will finish in 58 days?
- The probability that the project will take longer than 63 days?
- The probability that the project will finish at least 4 days earlier than expected?
- The completion date to finish with a 95% confidence level?

#### **PERT - Introduction**

- Incorporates uncertainty
- Assuming variability in activity durations
- Three estimates of duration for each activity:
  - 1. Most optimistic duration: t<sub>a</sub>
  - 2. Most pessimistic duration: t<sub>b</sub>
  - 3. Most likely duration:

**Expected duration** 

$$t_{e} = (t_{a} + 4t_{m} + t_{b}) / 6$$

Why? Loosely based on statistical calculations
 → Normal Distribution

#### **Normal Distribution**

Normal Distribution: also "Gaussian distribution"

A family of Continuous Probability Distribution

 $X \sim N ~(\mu, ~\sigma^2)$ 

real-valued random variable X is normally distributed with mean  $\mu$  and variance  $\sigma^2$ 

**Probability Density Function** for a continuous variable

**Density: relative frequency of variables** 

Represented by a continuous curve a

Also called **bell curve**  $\rightarrow$ 

shape of graph of PDF



## **Reminders!**



#### Characterization





Properties of PDF:

- >symmetry about its mean  $\mu$
- $\succ$ mode & median = mean  $\mu$
- >inflection points  $\rightarrow$  One SD from  $\mu$



Larger  $\sigma \rightarrow$  more dispersed Smaller  $\sigma \rightarrow$  less dispersed

#### **Probability Calculation**

Area under curve = sum of expected frequencies



**Z** Table: area under the standard normal probability density curve to the left of z

z	•00	·01	·02	•03	•04	•05	·06
ָיָה אָ אָ	·5000	·5040	•5080	·5120	·5160	·5199	·5239
	·5398	·5438	•5478	·5517	·5557	·5596	·5636
	·5793	·5832	•5871	·5910	·5948	·5987	·6026
	·6179	·6217	•6255	·6293	·6331	·6368	·6406
	·6554	·6591	•6628	·6664	·6700	·6736	·6772
5.6 7.8 9	·6915 ·7257 ·7580 ·7881 ·8159	-6950 -7291 -7611 -7910 -8186	-6985 -7324 -7642 -7939 -8212	•7019 •7357 •7673 •7967 •8238	·7054 ·7389 ·7703 ·7995 ·8264	·7088 ·7422 ·7734 ·8023 ·8289	·7123 ·7454 ·7764 ·8051 ·8315
1.0	·8413	-8438	•8461	·8485	-8508	-8531	·8554
1.1	·8643	-8665	•8686	·8708	-8729	-8749	·8770
1.2	·8849	-8869	•8888	·8907	-8925	-8944	·8962
1.3	·90320	-90490	•90658	·90824	-90988	-91149	·91309
1.4	·91924	-92073	•92220	·92364	-92507	-92647	·92785
1.5	·93319	•93448	·93574	•93699	·93822	•93943	·94062
1.6	·94520	•94630	·94738	•94845	·94950	•95053	·95154
1.7	·95543	•95637	·95728	•95818	·95907	•95994	·96080
1.8	·96407	•96485	·96562	•96638	·96712	•96784	·96856
1.9	·97128	•97193	·97257	•97320	·97381	•97441	·97500

$$\varphi_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

#### Unit Normal Distribution: N (0, 1) mean $\mu = 0$ SD $\sigma = 1$ $\rightarrow$ Z tables



**Probability Density Function** 

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**Z** Table: area under the standard normal probability density curve to the left of z

Z	•00	·01	·02	•03	•04	•05	∙об	
•0 •1 •2 •3 •4	·5000 ·5398 ·5793 ·6179 ·6554	·5040 ·5438 ·5832 ·6217 ·6591	•5080 •5478 •5871 •6255 •6628	•5120 •5517 •5910 •6293 •6664	·5160 ·5557 ·5948 ·6331 ·6700	·5199 ·5596 ·5987 ·6368 ·6736	·5239 ·5636 ·6026 ·6406 ·6772	$\varphi_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
·5 ·6 ·7 ·8 ·9	-6915 -7257 -7580 -7881 -8159	-6950 -7291 -7611 -7910 -8186	-6985 -7324 -7642 -7939 -8212	•7019 •7357 •7673 •7967 •8238	·7054 ·7389 ·7703 ·7995 ·8264	·7088 ·7422 ·7734 ·8023 ·8289	·7123 ·7454 ·7764 ·8051 ·8315	
1.0 1.1 1.2 1.3 1.4	·8413 ·8643 ·8849 ·90320 ·91924	-8438 -8665 -8869 -90490	-8461 -8686 -8888 -90658	-8485 -8708 -8007 -90824 -02364	-8508 -8729 -8925 -90988	-8531 -8749 -8044 -91149 -02647	·8554 ·8770 ·8062 ·91309	
1.5 1.6 1.7 1.8 1.9	·93319 ·94520 ·95543 ·96407 ·97128	•93448 •94630 •95637 •96485 •97193	·93574 ·94738 ·95728 ·96562 ·97257	•93699 •94845 •95818 •96638 •97320	•93822 •94950 •95907 •96712 •97381	·93943 ·95053 ·95994 ·96784 ·97441	·94062 ·95154 ·96080 ·96856 ·97500	-3 -2 -1 <b>1.36</b> 2 3 P(Z ≤z)

#### Pay attention to the legend

for (0 <u><</u> z < ∞)

i	
Φ(z)=   φ(t) dt _	

Z	.00	-01	-02	·03	•04	·05	·06	·07	·08	·09					-	0 ž									
0	·5000	·4960	·4920	·4880	·4840	·4801	·4761	•4721	·4681	·4641	z	•00	·OI	.02	·03	•04	·05	·06	·07	·08	•09				
- •r	·4602	·4562	.4522	·4483	·4443	.4404	·4364	·4325	4286	.4247	.0	.5000	.5040	.5080	.5120			.5220	.5270	.5210	.5250				
2	·4207	·4168	·4129	·4090	.4052	.4013	·3974	·3936	·3897	·3859	·I	.5308	.5438	.5478	-5517	-5557	-5506	-5439	-5675	-5714	5753				
3	.3821	·3783	·3745	.3707	·3669	.3632	·3594	·3557	.3520	·3483	•2	.5793	.5832	.5871	.5010	.5048	.5087	.6026	.6064	6103	·6141				
4	·3446	·3409	·3372	.3330	.3300	·3264	.3228	.3192	-3156	-3121	•3	.6179	.6217	.6255	.6203	.6331	.6368	.6406	.6443	-6480	6517				
5	·3085	·3050	.3015	·2981	·2946	.2012	·2877	·2843	·2810	·2776	•4	.6554	·6591	.6628	·6664	.6700	.6736	.6772	-6808	-6844	-6879				
6	·2743	·2709	·2676	·2643	2611	·2578	·2546	·2514	·2483	.2451	.5	-6075	-6050	-6085	-7010										
- 7	·2420	·2389	·2358	·2327	·2297	·2266	·2236	·2206	.2177	·2148	•6	-7257	·720I	.7324	.7357	7054	-7422	7454	-7486	-7517	-7540				
8	·2119	·2090	·2061	·2033	·2005	·1977	·1949	·1922	·1894	·1867	.7	.7580	.7611	-7642	.7673	.7703	.7734	.7764	-7704	-7823	7852				
9	·1841	·1814	-1788	·1762	.1736	·1711	·1685	·1660	·1635	·1611	·8	·7881	.7910	.7939	.7967	7995	-8023	.8051	·8078	.8106	·8133				
-1.0	·1587	·1562	·1539	-1515	·1492	·1469	·1446	·1423	·1401	·1379	•9	·8159	·8186	.8212	·8238	·8264	-8289	.8315	.8340	·8365	·8389				
-1.1	·1357	·1335	·1314	·1292	·1271	.1251	.1230	·1210	·1190	.1170	1.0	.8413	.8438	·8461	-8485	-8508	-8531	-8554	-8577	-8500	-8621				
-1.5	-1151	·1131	·1112	·1093	·1075	·1056	·1038	·1020	.1003	·09853	I·I	·8643	.8665	-8686	·8708	.8720	-8740	.8770	·8700	-8810	-8830				
-1.3	.09680	.09510	·09342	·09170	.09012	.08851	·08691	-08534	.08379	-08226	1.2	·8849	·8869	-8888	.8907	-8025	-8044	-8062	·8980	-8997	·90147				
-1.4	08070	·07927	.07780	.07030	·07493	·07353	·07215	·07078	·06944	-06811	1.3	·90320	·90490	·90658	.90824	-90988	.01140	.01300	·91466	.91621	91774				
-1.2	·06681	·06552	·06426	·06301	·06178	·06057	·05938	·05821	·05705	.05592	1.4	·91924	·92073	.92220	·92364	.92507	·92647	92785	.92922	·93056	·93189				
-1.6	·05480	·05370	·05262	·05155	·05050	·04947	·04846	·04746	·04648	.04551	1.5	.03310	.03448	03574	.03600	.03822	.02043	.04062	.04170	.04205	.04408				
-1·7	·04457	·04363	·04272	·04182	·04093	·04006	·03920	·03836	·03754	·03673	1.6	935-9	.04630	.04738	·04845	.04050	·05053	05154	·05254	05352	05440				
-1.8	·03593	.03515	·03438	.03362	03288	·03216	·03144	·03074	·03005	·02938	1.7	95543	95637	95728	-95818	.05007	·95994	•96080	-96164	·96246	.96327				
-1.0	.02872	·02807	·02743	02080	.02019	·02559	·02500	·02442	02385	·02330	1.8	·96407	·96485	.96562	·96638	.96712	·96784	·96856	.96926	.96995	.97062				
-2.0	·02275	·02222	·02169	·02118	·02068	·02018	·01970	·01923	·01876	·01831	1.9	.97128	.97193	·97257	.97320	·97381	·9744I	.97500	.97558	97615	·97670				
-2·I	·01786	·01743	·01700	·01659	-01618	·01578	.01539	01500	01463	·01426	2.0	.07725	.07778	.07827	07882	107022	107082	108020	08077	·08124	08760				
-2.2	·01390	01355	·01321	·01287	.01255	.01222	.01191	01160	·01130	.01101	2.1	-97/25	08257	-08200	·0824T	·9/932 ·08282	08422	08461	08500	08537	08574				
-2.3	.01072	.01044	.01017	-02 9903	·02 9642	·0² 9387	·029137	·02 8894	·0² 8656	·02 8424	2.2	08610	08645	08670	08713	08745	08778	08800	08840	·08870	·08800				
-2.4	·028198	·0 <sup>2</sup> 7976	·0² 7760	·0 <sup>2</sup> 7549	·0² 7344	·0² 7143	·0² 6947	·0² 6756	•0² 6569	·0²6387	2.3	.98928	.98956	.98983	·9 <sup>2</sup> 0097	·9² 0358	·9² 0613	·9² 0863	·9 <sup>2</sup> 1106	·9 <sup>2</sup> 1344	·92 1576				
-2.5	·026210	·02 6037	·0 <sup>2</sup> 5868	·0² 5703	·0 <sup>2</sup> 5543	·0² 5386	·02 5234	·02 5085	·0 <sup>2</sup> 4940	·0 <sup>2</sup> 4799	2.4	·92 1802	·92 2024	·92 2240	·92 2451	·9² 2656	·92 2857	·92 3053	·9² 3244	·9 <sup>2</sup> 3431	·9² 3613				
-2.6	·0² 4661	·02 4527	·0² 4396	·02 4269	·0 <sup>2</sup> 4145	·0² 4025	·02 3907	·02 3793	·02 3681	·0 <sup>2</sup> 3573	2.5	.02 2700	102 2062	02 4722	-02 4207	-02 4457	102 46T 4	·02 4766	·02 4015	·02 5060	·02 520T				
-2.7	•0² 3467	·0² 3364	∙o² 3264	·0² 3167	·02 3072	·0² 2980	·02 2890	·0² 2803	·0² 2718	·0² 2635	2.6	·0 <sup>2</sup> 5330	9 3903 02 5473	·02 5604	9 4~9/ ·0 <sup>2</sup> 573T	9 4457 0° 5855	·02 5075	·0 <sup>2</sup> 6003	·0 <sup>2</sup> 6207	·0 <sup>2</sup> 6310	·0 <sup>2</sup> 6427				
-2.8	·0 <sup>2</sup> 2555	·02 2477	·0² 2401	·02 2327	·02 2256	·0 <sup>2</sup> 2186	·0 <sup>2</sup> 2118	·02 2052	·02 1988	·02 1926	2.7	·0 <sup>2</sup> 6533	·02 6636	·0 <sup>2</sup> 6736	·0 <sup>2</sup> 6833	·0 <sup>2</sup> 6028	·0 <sup>2</sup> 7020	·Q <sup>2</sup> 7IIO	·Q2 7107	·92 7282	·92 7365				
-2.9	·0² 1866	·0² 1807	·02 1750	·02 1695	·0² 1641	·0² 1589	02 1538	·02 1489	·0² 1441	·0² 1395	2.8	·9 <sup>2</sup> 7445	·9² 7523	·9 <sup>2</sup> 7599	·92 7673	·9 <sup>2</sup> 7744	·92.7814	·9 <sup>2</sup> 7882	·9² 7948	·92 8012	·92 8074				
-3·0	·02 1350	·02 1306	·0² 1264	·0² 1223	·02 1183	·02 1144	·0² 1107	·02 1070	·02 1035	·0 <sup>2</sup> 1001	2.9	·928134	·92 8193	·928250	·928305	·9 <sup>2</sup> 8359	·928411	·9² 8462	·9² 8511	·9 <sup>2</sup> 8559	·9² 8605				
-3·1	·03 9676	·03 9354	·03 9043	·03 8740	·03 8447	·03 8164	·03 7888	·03 7622	·03 7364	·037114	3.0	·028650	·02 860 4	02 8726	02 8777	·028817	·028856	·02 8803	·02 8030	·02 8065	·02 8000				
-3.2	·036871	·03 6637	·03 6410	·03 6190	·03 5976	·03 5770	·03 5571	·03 5377	·03 5190	·03 5009	3.1	-03 0324	03 0646	030057	·03 T260	·03 1553	·03 1836	·0 <sup>3</sup> 2112	·0 <sup>3</sup> 2378	·03 2636	·0 <sup>3</sup> 2886				
-3.3	·03 4834	·03 4665	·03 4501	·03 4342	·03 4189	·03 404 I	·0³ 3897	·03 3758	·03 3624	·0³ 3495	3.2	·033120	·03 3363	·03 3500	·03 3810	·0 <sup>3</sup> 4024	·0 <sup>3</sup> 4230	·0 <sup>3</sup> 4420	·03 4623	·Q3 4810	·03 4001				
3.4	·03 3369	·0 <sup>3</sup> 3248	-03 3131	-03 3018	·03 2909	·03 2803	·03 2701	·03 2002	·03 2507	03 2415	3.3	·93 5166	·Q <sup>3</sup> 5335	·93 5499	·93 5658	·93 5811	·93 5959	·936103	93 6242	·936376	·93 6505				
3.5	·03 2326	·0 <sup>3</sup> 2241	·03 2158	·03 2078	·0 <sup>3</sup> 2001	·03 1926	·03 1854	·03 1785	·03 1718	·03 1653	3.4	·936631	·93 6752	·93 6869	·936982	·93 7091	·937197	·93 7299	·93 7398	·937493	·93 7585				
-3.6	·03 1591	·03 1531	·03 1473	·03 1417	·03 1363	·03 1311	03 1261	·03 1213	·03 1166	·03 1121	2.5		010000	-01-78 40	-01 -000	-01 7000	018074	103 87 46	103 8275	.03 8282	.03 82 47				
-3.7.	·03 1078	·03 1036	·04 9961	·0 <sup>4</sup> 9574	049201	·048842	·o+8496	·048162	·047841	·0 <sup>4</sup> 7532	3.2	-937074	·9·7759	03 8527	038682	·9°7999	-03 8680	03 8720	03 8787	03 8834	03 8870				
-3.8	·0+7235	·o• 6948	·0+6673	·0 <b>*</b> 6407	·0 <sup>4</sup> 6152	-04 5906	·o <b>•</b> 5669	·0 <sup>4</sup> 5442	·0 <sup>4</sup> 5223	·04 5012	3.7	-03 8022	·03 8064	-0+0030	-0+0426	·0*0700	·04 II58	·0 <sup>4</sup> I 504	·04 1838	·0 <sup>4</sup> 2150	·0 <sup>4</sup> 2468				
-3.9	·044810	·0 <b>*</b> 4615	·0 <b>*</b> 4427	·04 4247	·0 <sup>4</sup> 4074	·o• 3908	·0 <sup>4</sup> 3747	·o• 3594	·o• 3446	•04 3304	3.8	-0+ 2765	·04 3052	·0 <sup>4</sup> 3327	·04 3503	·0 <sup>4</sup> 3848	·0 <sup>4</sup> 4004	·Q <sup>4</sup> 433I	·9+ 4558	·9 <sup>4</sup> 4777	·9 <sup>4</sup> 4988				
-4.0	.0+3167	·04 3036	·04 2010	·0+2780	·04 2673	·0+2561	·0 <sup>4</sup> 2454	·04 2351	·0 <sup>4</sup> 2252	·0 <sup>4</sup> 2157	3.9	-9+5190	·9 <sup>+</sup> 5385	·9 <sup>+</sup> 5573	·9 <sup>4</sup> 5753	·9 <sup>4</sup> 5926	·9+6092	·9+6253	·9 <sup>4</sup> 6406	·9+6554	·94 6696				
-4.1	·0+2066	·04 1978	·04 1894	·04 1814	·04 1737	·04 1662	041591	·04 1523	·04 1458	·04 I395										04 77 48	.04 78 42				
-4.2	·04 1335	·04 1277	·04 1222	·04 1168	·04 III8	·04 1069	-04 1022	·05 9774	·05 9345	·05 8934	4.0	-9+0833	·9* 0904	·9* 7090	·9*7211	·9•7327	·9•7439	·9*7540	04 8477	-9+7/40	-04 8605				
-4.3	·05 8540	05 8163	·05 7801	·05 7455	·05 7124	·056807	·o\$ 6503	·05 6212	·05 5934	·05 5668	4.1	·9*7934	·9* 8022	-04 8778	·04 8822	-04 8882	·0+8031	-04 8078	05 0226	050655	·05 1066				
-4.4	·05 5413	05 5169	<sup>•0⁵</sup> 4935	·05 4712	·os 4498	·05 4294	•0 <b>s</b> 4098	·05 3911	·05 3732	·05 3561	4.3	·05 1460	·05 1837	05 2100	05 2545	·05 2876	-05 3103	·Q <sup>5</sup> 3407	·95 3788	·9 <sup>s</sup> 4066	·95 4332				
-4.5	·05 3308	·05 3241	·05 3002	·0 <sup>5</sup> 2040	·05 2813	·05 2682	·05 2558	·05 2430	·052325	·05 2216	4.4	·9 <sup>5</sup> 4587	·9 <sup>5</sup> 4831	·95 5065	·9 <sup>5</sup> 5288	·95502	·95 5706	·95 5902	·95 6089	·95 6268	·95 6439				
-4.6	·05 2112	05 2013	-05 1010	05 1828	·05 1742	·05 1660	-05 1581	-05 1506	·05 1434	·05 1366							05 00 79	-05 7 4 4 7	-05 7567	105 7675	105 778 4				
-4.7	·05 1301	·05 1239	·05 1179	·05 1123	05 1069	·05 1017	·06 9680	·06 9211	·06 8765	·06 8339	4.5	·9·0002	·9·0759	-9-0908	-9-7051	-9-7107	-9-7310	9 7442	-05 8404	·05 8566	05 8624				
-4.8	·0 <sup>6</sup> 7933	·06 7547	·0 <sup>6</sup> 7178	·066827	066492	06173	·06 5869	·06 5580	·0 <sup>6</sup> 5304	·0 <sup>6</sup> 5042	4.0	-9-7000	19-7907	-05 8821	-05 8877	-05 802T	05 8082	000320	·0 <sup>6</sup> 0780	·06 1235	06 1661				
-4.9	·06 4792	-0 <sup>6</sup> 4554	·06 4327	·06 4111	·0 <sup>6</sup> 3906	·06 3711	·0 <sup>6</sup> 3525	•06 3348	·06 3179	·06 30 19	4.8	·06 2067	-06 2453	06 2822	·06 3173	06 3508	·Q6 3827	·0 <sup>6</sup> 4131	·0 <sup>6</sup> 4420	·9 <sup>6</sup> 4696	·9 <sup>6</sup> 4958				
			Example	e: Ø(_3	$(57) = (0^3)$	1785 = 0.0	0001785				4.9	·9 <sup>6</sup> 5208	·9 <sup>6</sup> 5446	·9 <sup>6</sup> 5673	·9 <sup>6</sup> 5889	·9 <sup>6</sup> 6094	·96289	·9º 6475	96652	·966821	·96 6981				
			r	- ( •													-								

Φ(z)=∫φ(t)dt \_\_\_\_\_

(–∞ (z <u>{</u>0)

for



### **Standardization (Coding)**

What about normal distributions that are not Standard (z)?



Any  $X \sim N(\mu, \sigma^2) \rightarrow \text{transformed to } Z$ 

### **Standardization (Coding)**

The duration of a project has a normal distribution with **mean of 65 days** and **SD of 5 days**.

What is the probability that the project will finish **between 55 & 72.5 days**?

$$Z = \frac{X_i - \mu}{\sigma}$$

P (
$$55 < \mathbf{X} < 72.5$$
) =  
P ( $\underline{55 - 65}_{5} < \mathbf{Z} < \underline{72.5 - 65}_{5}$ ) =  
 $5$   
P ( $-2 < \mathbf{Z} < 1.5$ ) =  
0.93319 - 0.02275 = 0.91044

#### PERT

Three estimates of duration for each activity:

- 1. Most optimistic duration: t<sub>a</sub>
- 2. Most pessimistic duration: t<sub>b</sub>
- 3. Most likely duration: t<sub>m</sub>

Expected duration  
= Mean (µ) 
$$t_e = (t_a + 4t_m + t_b) / 6$$
  
Variance  $\sigma^2 = [(t_b - t_a) / 6]^2$ 

P	FRT																
		2	3			$\rightarrow / \rightarrow 8 \rightarrow 9$											
					+ - (+	<b>⊥</b> /+ ⊥ + ) / 6											
	Μ	ost F	Proba	ble	ι <sub>e</sub> – (ι <sub>a</sub>	$+4l_m + l_b/70$											
	Optimistic	2		Pessimistic $\sigma^2 = [(t_b - t_a) / 6]^2$													
		7	•		<b>E</b>												
	Activity	t <sub>a</sub>	t <sub>m</sub>	t <sub>b</sub>	Mean $\mu$ (t <sub>e</sub> )	Variance $\sigma^2$											
	1	1	3	5													
	1		5	5													
	2	3	6	9													
	3	10	13	19													

2	3	6	9	
3	10	13	19	
4	3	9	12	
5	1	3	8	
6	8	9	16	
7	4	7	13	
8	3	6	9	
9	1	3	8	
Sum				

## **Central Limit Theorem**

 $\underline{x}_1$  to  $\underline{x}_n$  :random variables with normal distributions  $t_1$  to  $t_n$  : mean of durations  $v_1$  to  $v_n$  :variances

If 
$$X = x_1 + x_2 + ... + x_n$$
  
Then  $T_x = t_1 + t_2 + ... + t_n$   
 $V_x = v_1 + v_2 + ... + v_n$ 

#### In other words:

The mean of the sum is the sum of means

The variance of the sum is the sum of the variances

#### <u>Also:</u>

The distribution of the sum of distributions will be normal; regardless of the shape of each distribution

### **Central Limit Theorem**



## Changes in s<sup>2</sup> & SD with increasing N





Activity	t <sub>a</sub>	t <sub>m</sub>	t <sub>b</sub>	Mean $\mu$ (t <sub>e</sub> )	Variance $\sigma^2$
1	1	3	5	3	0.44
2	3	6	9	6	1.00
3	10	13	19	13.5	2.25
4	3	9	12	8.5	2.25
5	1	3	8	3.5	1.36
6	8	9	16	10	1.77
7	4	7	13	7.5	2.25
8	3	6	9	6	1.00
9	1	3	8	3.5	1.36
Sum				61.5	13.69



Activity	t <sub>a</sub>	t <sub>m</sub>	t <sub>b</sub>	Mean $\mu$ (t <sub>e</sub> )	Variance $\sigma^2$
1	1	3	5	3	0.44
2	3	6	9	6	1.00
3	10	13	19	13.5	2.25
4	3	9	12	8.5	2.25
5	1	3	8	3.5	1.36
6	8	9	16	10	1.23
7	4	7	13	7.5	2.25
8	3	6	9	6	1.00
9	1	3	8	3.5	1.36
Sum				61.5	13.14

?

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**PERT - Example**  

$$t_e = 61.5 \text{ days}$$
;  $\sigma^2 = 13.14 \text{ days}$ ;  $\sigma = 3.5 \text{ days}$   
 $Z = (X - \text{Mean})/\sigma \rightarrow Z$ -Table to determine *P*

a) The probability that the project will finish before 61.5 days.



b) The probability that the project will finish before end of day 58.





e) Completion date to finish with at least a 95% confidence level.

#### **Approximate Probabilities**

#### How many SD from Mean?

P (
$$\mu$$
 - 1 $\sigma$  < X <  $\mu$  + 1 $\sigma$ ) ~ 0.68  
P ( $\mu$  - 2 $\sigma$  < X <  $\mu$  + 2 $\sigma$ ) ~ 0.95  $\longrightarrow$  ~ 95% of cases lie w/in 2 SD of  $\mu$   
P ( $\mu$  - 3 $\sigma$  < X <  $\mu$  + 3 $\sigma$ ) ~ 0.99 ~ ALL cases lie w/in 3 SD of  $\mu$ 



#### **Example:**

P 
$$(\mu - 1\sigma < X < \mu + 1\sigma) \sim 0.68$$
~ 2/3 of cases lie w/in1 SD of  $\mu$ P  $(\mu - 2\sigma < X < \mu + 2\sigma) \sim 0.95$  $\sim 95\%$  of cases lie w/in2 SD of  $\mu$ P  $(\mu - 3\sigma < X < \mu + 3\sigma) \sim 0.99$ ~ ALL cases lie w/in3 SD of  $\mu$ 

The duration of a certain project has a mean of 100 and SD of 15.



#### **PERT** $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9$

$$t_{e} = 61.5 \text{ days}$$
;  $\sigma^{2} = 13 \text{ days}$ ;  $\sigma = 3.5 \text{ days}$ 

 $Z = (X - Mean) / \sigma \rightarrow Z$ -Table to determine %

What is:

- a) The probability that the project will finish within 61.5 days.
- b) The probability that the project will finish before day 58.
- c) The probability that the project will finish more than **3.5 days late**.
- d) The probability that the project will finish at least 7 days early.
- e) The completion date with 95% confidence level.



#### PERT

#### **Program Evaluation & Review Technique**

- Expected duration  $t_e = (t_a + 4t_m + t_b) / 6$
- Variance  $\sigma^2 = [(t_b t_a) / 6]^2$
- Standard deviation σ
- Probability of completion with certain days  $Z = (X - Mean) / \sigma \rightarrow Z$ -Table to determine P

## **PERT - Notation**

ADM: Activity Diagram Method commonly used for PERT



What is the probability of finishing this project within 25 days?

## **Floats in PERT**

In PERT floats are commonly referred to as "Slack"



 $\rightarrow$  Z-Table to determine P

## Float (Slack) in PERT

Example: What is the probability of "C complete Start D" slack > 0?



## **Closing Notes:**

Shortcoming of PERT:

Does not consider all scenarios Focus on CP e.g. subcritical paths

#### **Monte Carlo Simulation**

- A class of computational algorithms
- Rely on repeated random sampling
- Projects associated with high degree of uncertainty

 $\rightarrow$ unpredictable nature of events

• Application example: Probabilistic Scheduling

History: Invented in1940's - nuclear weapon projects Manhattan Project: US, UK, Canada 1<sup>st</sup> atomic bomb World War II Named after gambling uncle!

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#### PERT

#### **Program Evaluation & Review Technique**

- Expected duration  $t_e = (t_a + 4t_m + t_b) / 6$
- Variance  $\sigma^2 = [(t_b t_a) / 6]^2$
- Standard deviation σ
- Probability of completion with certain days Z = (X – Mean) /  $\sigma \rightarrow$  Z-Table to determine %

	60-	·5359 ·5753 ·6141 ·6517 ·6879	-7224 -7549 -7852 -8133 -8389	-8621 -8830 -90147 -91774 -93189	-94408 -95449 -96327 -97062 -97670	-98169 -98574 -98899 -9 <sup>2</sup> 1576 -9 <sup>2</sup> 3613	9 <sup>2</sup> 5201 9 <sup>2</sup> 6427 9 <sup>2</sup> 7365 9 <sup>2</sup> 8074 9 <sup>2</sup> 8605	.9 <sup>2</sup> 8999 .9 <sup>3</sup> 2886 .9 <sup>3</sup> 4991 .9 <sup>3</sup> 6505 .9 <sup>3</sup> 7585	.9 <sup>3</sup> 8347 .9 <sup>3</sup> 8879 .9 <sup>4</sup> 2468 .9 <sup>4</sup> 4988 .9 <sup>4</sup> 6696	9 <sup>4</sup> 7843 9 <sup>4</sup> 8605 9 <sup>5</sup> 1066 9 <sup>5</sup> 4332 9 <sup>5</sup> 6439	95 7784 95 8634 96 1661 96 4958 96 6981
	.08	-5319 -5714 -6103 -6480 -6844	-7190 -7517 -7823 -8106 -8365	-8599 -8810 -8997 -91621 -93056	-94295 -95352 -96246 -96995 -97615	-98124 -98537 -98870 -9 <sup>2</sup> 1344 -9 <sup>2</sup> 13431	-9 <sup>2</sup> 5060 -9 <sup>2</sup> 6319 -9 <sup>2</sup> 7282 -9 <sup>2</sup> 8012	-9 <sup>2</sup> 8965 -9 <sup>3</sup> 2636 -9 <sup>3</sup> 4810 -9 <sup>3</sup> 6376 -9 <sup>3</sup> 7493	.9 <sup>3</sup> 8282 .9 <sup>3</sup> 8834 .9 <sup>4</sup> 2159 .9 <sup>4</sup> 4777 .9 <sup>4</sup> 6554	-9 <sup>4</sup> 7748 -9 <sup>4</sup> 8542 -9 <sup>5</sup> 0655 -9 <sup>5</sup> 4066 -9 <sup>5</sup> 6268	.9 <sup>5</sup> 7675 .9 <sup>5</sup> 8566 .9 <sup>6</sup> 1235 .9 <sup>6</sup> 4696 .9 <sup>6</sup> 6821
8	£0:	5279 5675 6644 6808	.7157 -7486 -7794 -8078 -8340	-8577 -8790 -8980 -91466 -92922	-94179 -95254 -96164 -96926 -97558	-98077 -98500 -98840 -9 <sup>2</sup> 1106 -9 <sup>2</sup> 3244	-9 <sup>2</sup> 4915 -9 <sup>2</sup> 6207 -9 <sup>2</sup> 7197 -9 <sup>2</sup> 7948 -9 <sup>2</sup> 8511	-9 <sup>2</sup> 8930 -9 <sup>3</sup> 2378 -9 <sup>3</sup> 4623 -9 <sup>3</sup> 6242 -9 <sup>3</sup> 7398	938215 9387 941838 944558 94558 94558	.94 7649 .94 8477 .95 0226 .95 3788 .95 6089	.9 <sup>5</sup> 7561 .9 <sup>5</sup> 8494 .9 <sup>6</sup> 0789 .9 <sup>6</sup> 4420 .9 <sup>6</sup> 6652
hor (0 <u>⊊</u> z (	90:	-5239 -5636 -6026 -6406 -6772	-7123 -7454 -7764 -8051 -8315	.8554 .8770 .8962 .91309 .92785	-94062 -95154 -96080 -96856 -97500	-98030 -98461 -98809 -9 <sup>2</sup> 0863 -9 <sup>2</sup> 3053	-9 <sup>2</sup> 4766 -9 <sup>2</sup> 6093 -9 <sup>2</sup> 7110 -9 <sup>2</sup> 7882 -9 <sup>2</sup> 8462	9 <sup>2</sup> 8893 9 <sup>3</sup> 2112 9 <sup>3</sup> 4429 9 <sup>3</sup> 6103 9 <sup>3</sup> 7299	-9 <sup>3</sup> 8146 -9 <sup>3</sup> 8739 -9 <sup>4</sup> 1504 -9 <sup>4</sup> 4331 -9 <sup>4</sup> 6253	-9* 7546 -9* 8409 -9* 8978 -9 <sup>\$</sup> 3497 -9 <sup>\$</sup> 5902	.9 <sup>5</sup> 7442 .9 <sup>5</sup> 8419 .9 <sup>6</sup> 0320 .9 <sup>6</sup> 4131 .9 <sup>6</sup> 475
/	50.	-5199 -5596 -5387 -6368 -6736	-7088 -7422 -7734 -8023 -823	-8531 -8749 -8944 -91149 -92647	-93943 -95053 -95994 -96784 -97441	-97982 -98422 -98778 -920613 -92257	-9 <sup>2</sup> 4614 -9 <sup>2</sup> 5975 -9 <sup>2</sup> 7020 -9 <sup>2</sup> 7814 -9 <sup>2</sup> 8411	-9 <sup>2</sup> 8856 -9 <sup>3</sup> 1836 -9 <sup>3</sup> 4230 -9 <sup>3</sup> 5959	.9 <sup>3</sup> 8074 .9 <sup>3</sup> 8689 .9 <sup>4</sup> 1158 .9 <sup>4</sup> 4094 .9 <sup>4</sup> 6092	.9 <sup>4</sup> 7439 .9 <sup>4</sup> 8338 .9 <sup>4</sup> 8931 .9 <sup>5</sup> 3193	.95 7318 .95 8340 .95 8983 .96 3827 .96 6289
, in the second	·04	-5160 -5557 -5331 -6331 -6700	-7054 -7389 -7703 -8264	-8508 -8729 -8925 -90988 -92507	-93822 -94950 -95907 -96712 -97381	.97932 .98382 .98745 .9 <sup>2</sup> 0358	-9 <sup>2</sup> 4457 -9 <sup>2</sup> 5855 -9 <sup>2</sup> 6928 -9 <sup>2</sup> 7744 -9 <sup>2</sup> 8359	.9 <sup>2</sup> 8817 .9 <sup>3</sup> 1553 .9 <sup>3</sup> 4024 .9 <sup>3</sup> 5811 .9 <sup>3</sup> 7091	.9 <sup>3</sup> 7999 .9 <sup>3</sup> 8637 .9 <sup>4</sup> 0799 .9 <sup>4</sup> 3848	.9 <sup>4</sup> 7327 .9 <sup>4</sup> 8263 .9 <sup>4</sup> 8882 .9 <sup>4</sup> 8882 .9 <sup>5</sup> 5502	.957187 .958258 .958931 .963508 .96094
** ¢(1)dt	£0:	-5120 -5517 -5910 -6293 -6664	-7019 -7357 -7673 -7673 -7967 -8238	-8485 -8708 -8907 -90824 -92364	-93699 -94845 -95818 -96638	-97882 -98341 -98713 -9 <sup>2</sup> 0097 -9 <sup>2</sup> 2451	-9 <sup>2</sup> 4297 -9 <sup>2</sup> 5731 -9 <sup>2</sup> 6833 -9 <sup>2</sup> 7673 -9 <sup>2</sup> 8305	.9 <sup>2</sup> 8777 .9 <sup>3</sup> 1260 .9 <sup>3</sup> 3810 .9 <sup>3</sup> 5658 .9 <sup>3</sup> 5982	-9 <sup>3</sup> 7922 -9 <sup>3</sup> 8583 -9 <sup>4</sup> 0426 -9 <sup>4</sup> 3593 -9 <sup>4</sup> 5753	.9 <sup>4</sup> 7211 .9 <sup>4</sup> 8186 .9 <sup>4</sup> 8832 .9 <sup>5</sup> 2545 .9 <sup>5</sup> 288	95 7051 95 8172 95 8877 95 8877 96 3173 96 5889
•	-02	-5080 -5478 -5871 -6255 -6628	-6985 -7324 -7642 -7939 -8212	-8461 -8686 -8888 -9658 -92220	-93574 -94738 -95728 -96562 -97257	-97831 -98300 -98679 -98983 -9 <sup>2</sup> 2240	-9 <sup>2</sup> 4132 -9 <sup>2</sup> 5604 -9 <sup>2</sup> 6736 -9 <sup>2</sup> 7599 -9 <sup>2</sup> 8250	.9 <sup>2</sup> 8736 .9 <sup>3</sup> 0957 .9 <sup>3</sup> 5590 .9 <sup>3</sup> 5499	-9 <sup>3</sup> 7842 -9 <sup>3</sup> 8527 -9 <sup>4</sup> 0039 -9 <sup>4</sup> 3327 -9 <sup>4</sup> 5573	.9* 7090 .9* 8106 .9* 8778 .9* 2199 .95 5065	.9 <sup>5</sup> 6908 .9 <sup>5</sup> 8081 .9 <sup>5</sup> 8821 .9 <sup>6</sup> 2822
	10.	-5040 -5438 -5832 -6217 -6591	-6950 -7291 -7611 -7910 -8186	-8438 -8665 -8869 -90490 -92073	-93448 -94630 -95637 -96485 -97193	-97778 -98257 -98645 -98956 -9 <sup>2</sup> 2024	-9 <sup>2</sup> 3963 -9 <sup>2</sup> 5473 -9 <sup>2</sup> 6636 -9 <sup>2</sup> 7523 -9 <sup>2</sup> 8193	-9 <sup>2</sup> 8694 -9 <sup>3</sup> 0646 -9 <sup>3</sup> 3363 -9 <sup>3</sup> 5335 -9 <sup>3</sup> 5752	·9 <sup>3</sup> 7759 ·9 <sup>3</sup> 8469 ·9 <sup>3</sup> 8964 ·9 <sup>4</sup> 3052 ·9 <sup>4</sup> 5385	9+6964 9+8022 9+8723 951837 954831	.9 <sup>5</sup> 6759 .9 <sup>5</sup> 7987 .9 <sup>5</sup> 8761 .9 <sup>5</sup> 2453 .9 <sup>6</sup> 2453
	0.	5000 5398 5793 6179 6554	-6915 -7257 -7580 -7881 -7881 -8159	-8413 -8643 -8849 -90320 -91924	-93319 -94520 -95543 -95407 -97128	-97725 -98214 -98610 -98928 -98928	-9 <sup>2</sup> 3790 -9 <sup>2</sup> 5339 -9 <sup>2</sup> 6533 -9 <sup>2</sup> 6533 -9 <sup>2</sup> 8134	-9 <sup>2</sup> 8650 -9 <sup>3</sup> 0324 -9 <sup>3</sup> 3129 -9 <sup>3</sup> 5166 -9 <sup>3</sup> 6631	9 <sup>3</sup> 7674 9 <sup>3</sup> 8409 9 <sup>3</sup> 8922 9 <sup>4</sup> 2765 9 <sup>4</sup> 2765	-9+ 6833 -9+ 7934 -9+ 8665 -9* 1460 -9* 1460	.9 <sup>5</sup> 6602 .9 <sup>5</sup> 7888 .9 <sup>5</sup> 8699 .9 <sup>6</sup> 2067 .9 <sup>6</sup> 5208
	Z	о́ н и́ ю́ 4	νο έν όν	11 12 12 14 14 14 14 14 14 14 14 14 14 14 14 14	1.5 7 1.6 1.6 1.6 1.6 1.6 1.6 1.6 1.6 1.6 1.6	2:2 2:2 2:4 3:4	8 8 7 6 6 9 8 7 8 6	3.3 3.5 3.4 3.5	8.5 2.5 2.5 2.5 2.5 2.5 2.5 2.5 2.5 2.5 2	4.4.4.4 0.1.5.6.4.4	4 4 4 4 4 6 6 6 6

Example:  $\Phi(3.57) = \cdot 9^3 8215 = 0.9998215$ .

	60.	.4641 .4247	.3859	.3483 .3121	-2776	-2451	·2148	1191.	0001	6/21.	-09853	-08226		76CCo	·03673	.02938	-02330	.0183I	-01420	-02 8424	-02 6387	·0 <sup>2</sup> 4799	·0 <sup>2</sup> 3573	·0 <sup>2</sup> 2635	-02 1395	1001 20-	·03 7114	-03 2405	·03 2415	-0 <sup>3</sup> 1653	1211 fo.	-04 5012	·0+ 3304	-04 2157	·0+ I395	-05 8934 -05 5668	-05 3561	-05 2216	·0 <sup>5</sup> I366	-06 8339	610£ .0.	
	80.	.4681 .4286	2685.	·3520 ·3156	-2810	-2483	LL12.	·1894	1011	10011.	500I.	-06044	200220	04648	·03754	-03005	-02385	-01876	·014b3	-02 8656	·0 <sup>2</sup> 6569	-02 4940	·02 3681	-02 2718	-02 1441	·02 1035	·017364	0615 °0.	-03 2507	61 II to.	·03 II66	·0* 5223	-0* 3446	•0* 2252	·04 1458	-05 50345	·05 3732	·05 2325	·05 1434	-06 8765	-0° 3179	
	20.	.4721 .4325	9866.	·3557	-2843	-2514	.2206	.1922 1660	C011.	0121.	·1020	-08534 -07078	.06891	.04746	-03836	-03074	-02442	·01923	01500	02 8894	·0 <sup>2</sup> 6756	·0 <sup>2</sup> 5085	·0 <sup>2</sup> 3793	·0 <sup>2</sup> 2803	02 1489	02 1070	-03 7622	0. 5377	2092 60.	-03 I785	·0 <sup>3</sup> 1213	0.5442	-0* 3594	.0+2351	·0+ I523	-05 9774	1168 50.	05 2439	·0° 1506	00 9211	-0° 3348	
(0 <sup>7</sup> z)(	-0Ç	.4761 .4364	.3974	·3594 ·3228	-2877	-2546	-2236	·1949 ·1685	yrrt.	.1230	·1038	-08691	Scool.	-04846	03920	·03144	-02500	02610-	-01539	-02 0137	-02 6947	·0 <sup>2</sup> 5234	-02 3907	02 2890	021538	L011 20.	·0 <sup>3</sup> 7888	1025 50.	10/2 10.	·0 <sup>3</sup> 1854	1921 10.	0450.0	-0+ 3747	•0* 2454	·0+ I 29I	-04 I022	·05 4098	-05 2558	·05 I58I	-06 9680	·0° 3525	001785.
- for (-a	-05	.4801 .4404	.40I3	-3032 -3264	2012.	-2578	-2266	11/11.		1251.	·1056	-08851	-solo.	-04047	-04006	-03216	-02559	-02018	-01578	02 9387	-02 7143	·0 <sup>2</sup> 5386	·0 <sup>2</sup> 4025	-02 2980	·02 1589	·01144	·03 8164	1707 60-	·0 <sup>3</sup> 2803	·01 1926	IISI to.	0, 5006	·0+ 3908	.0+2561	·04 1662	-04 I069	·02 4294	·0 <sup>5</sup> 2682	·03 1660	LI01 50.	11/2 00.	785 = 0.0
	-04	-4840 -4443	-4052	.3300	-2046	-2611	1622.	-1736	CULT.	1/21.	52oi.	·09012		-02020	·04093	·03288	02019	-02068	21010-	-0 <sup>2</sup> q642	·0 <sup>2</sup> 7344	·0 <sup>2</sup> 5543	-02 4145	-02 3072	1491 20.	·02 1183	·0 <sup>3</sup> 8447	0265 -0-	6062 FO.	1002 to.	-01 1363	1026-0-	-0* 4074	.0+ 2673	-0+ 1737	-04 III8	-05 4498	·05 2813	·05 1742	6901 so.	-0, 3906	$57) = \cdot 0^{3}1$
	£0:	-4880 -4483	-4090	-3336 -3336	1802.	-2643	-2327	·1762	.1676	2621.	·1093	-07636	TOEGO.	-05155	-04182	-03362	02020	02118	-9010-	-02 0003	-02 7549	-0 <sup>2</sup> 5703	·0 <sup>2</sup> 4269	-023107	·02 1695	·0 <sup>2</sup> 1223	-03 8740	0610 -0-	01 3018	·0 <sup>3</sup> 2078	7141 to-	4/c6-0-	-0* 4247	.0+2789	•0* I814	-04 II68	-05 4712	-05 2949	·05 1828	-05 1123	-06 4111	: ⊉ (−3·
Φ(z)= -	-02	-4920 -4522	-4129	·3745 ·3372	5102.	-2676	-2358	1002.	.1620	·1314	·1112	-09342 -07780	yeryo.	-05262	.04272	-03438	-02743	-02169	00/10.	L1010-	-03 7760	·o <sup>2</sup> 5868	-02 4396	·02 3264	-02 I750	·02 1264	-03 9043	0140 -0-	1515 to-	-0 <sup>3</sup> 2158	·03 1473	1066.0-	-0+ 4427	0162+0.	·0* 1894	-05 7801	-05 4935	-05 30Q2	6161 so.	6/11 so.	-06 4327	Example
	10-	.4960 .4562	4168	:3703 ·3409	0505.	60/2.	•2389	·2090	. 1660	.1335	1811.	-09510		-02370	-04363	-03515	-02807	-02222	.01743	01044	9262 20-	·02 6037	·0 <sup>2</sup> 4527	-02 3304	-02 1807	·0 <sup>2</sup> 1306	·0 <sup>3</sup> 9354	-03 4665	-03 3248	·0 <sup>3</sup> 2241	·0 <sup>3</sup> I53I	04 6048	·0+4615	•0+3036	·0+ 1978	-04 1277	6915 so.	-05 3241	-05 2013	021230	-06 4554	
	00-	·5000 ·4602	.4207	·3821 ·3446	-3085	-2743	-2420	-2119 -1841	. тевл	1357	·ISII	-09680 08076	-06681	-05480	-04457	-03593	-02872	-02275	09210-	01072	·02 8198	-02 6210	·0 <sup>2</sup> 4661	-023467	0,1866	-07 1350	-0, 9676	1600 -0-	6985 ro-	·03 2326	·0 <sup>3</sup> I59I	-0+ 7235	-0+48I0	L015+0-	-0+2066	-04 1335	·055413	8055 20.	.052112	1021 301	·0° 4792	
	2	9 H 	 4	 မ် 4	۰ ب	ې ۱	1	х <del>с</del>		1.I.I	-1.2	-1:3		9.1	2.1-	-1.8	6.1-	-2.0	-2.1	-2.3	-2.4	-2.5	-2.6	-2.7	-2.0	-3:0	-3.1	13.2	-3.5	-3:5	-3.6	13.7	-3.9	-4.0	-4.1	-4-2	+ + - 4 - 4 -	-4.5	-4.6	-4-7	- 4.9	