Probabilistic Scheduling

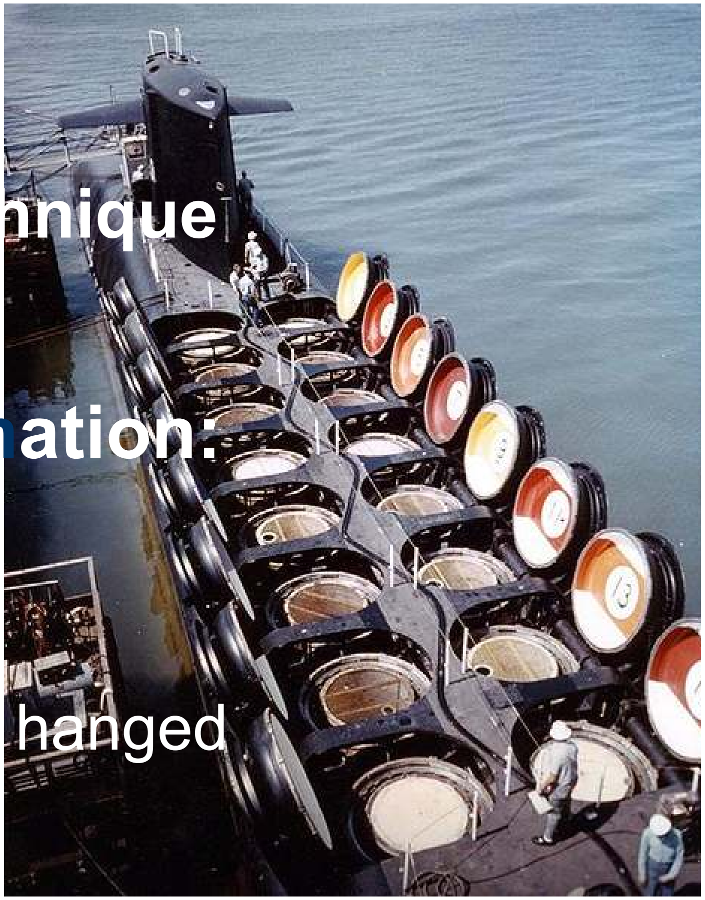
PERT

Program Evaluation and Review Technique

Farnaz Sadeghpour
Associate Professor
University of Calgary
Department of Civil Engineering
Project Management Center of Excellence

PERT

Program Evaluation and Review Technique



Activity / Project Duration Estimation:

Deterministic:

Probabilistic:

Assumes project conditions remain unchanged

Not always realistic

- PERT: Program Evaluation and Review Technique
- Developed by US Navy 1957
- Polaris nuclear submarine project
- Enables incorporating uncertainty into schedule
  not knowing precise details and durations

PERT

Answers questions like:

What is:

- The probability that the project will finish in 61 days?
- The probability that the project will finish in 58 days?
- The probability that the project will take longer than 63 days?
- The probability that the project will finish at least 4 days earlier than expected?
- The completion date to finish with a 95% confidence level?

PERT - Introduction

- Incorporates uncertainty
- Assuming variability in activity durations
- Three estimates of duration for each activity:
  1. Most optimistic duration: t_a
  2. Most pessimistic duration: t_b
  3. Most likely duration: t_m

Expected duration

$$t_e = (t_a + 4t_m + t_b) / 6$$

- Why? Loosely based on statistical calculations
  → Normal Distribution

Normal Distribution

Normal Distribution: also "**Gaussian distribution**"

A family of Continuous Probability Distribution

$$X \sim N(\mu, \sigma^2)$$

real-valued random variable X is normally distributed with mean μ and variance σ^2

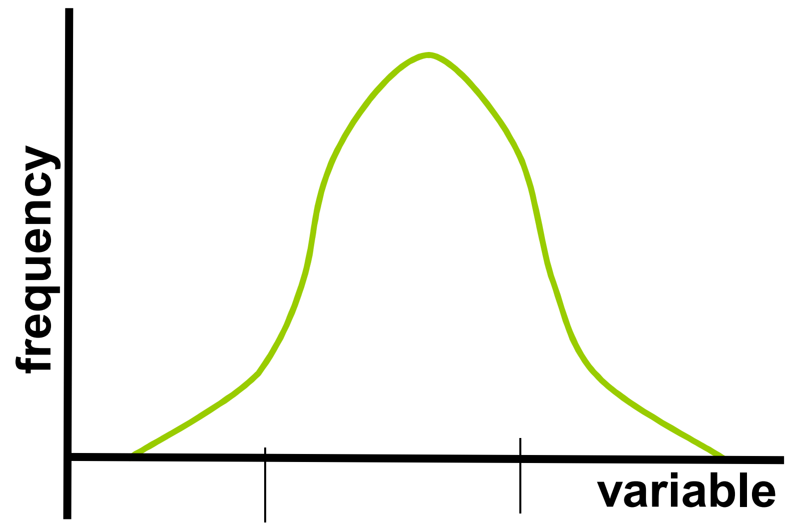
Probability Density Function for a continuous variable

Density: relative frequency of variables

Represented by a continuous curve

Also called **bell curve** →

shape of graph of PDF



Probability **D**ensity **F**unction

Reminders!

Standard deviation:

$$SD = \sigma = \frac{\sqrt{\sum (X_i - \overline{X}_i)^2}}{n}$$

deviation from mean

Variance:

$$\sigma^2 = \int (x-\mu)^2 f(x)\, dx$$

Median X_0 is median if $f(x_0) = 0.5$

Mode $f(x_m) = \max_x f(x)$

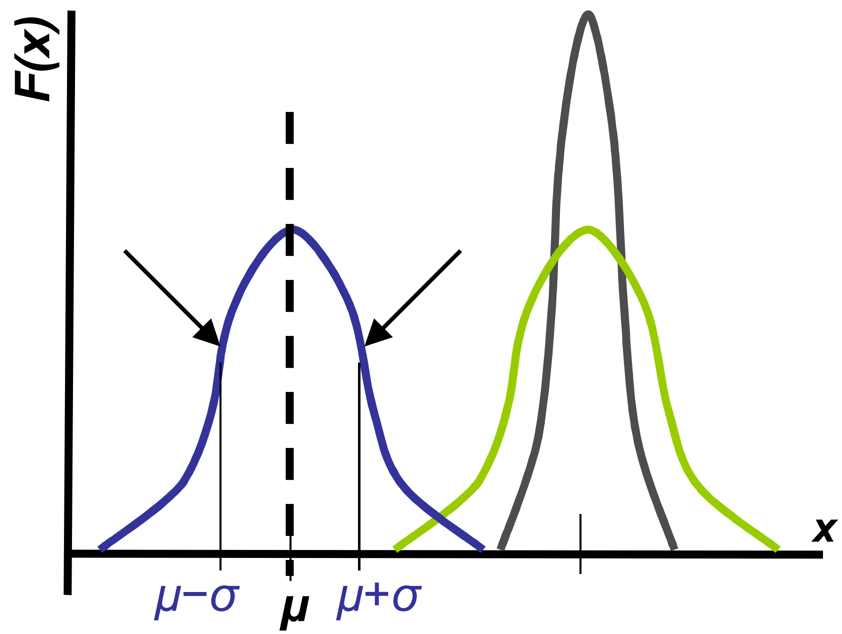
Characterization

$X \sim N(\mu, \sigma^2)$

Defined by two parameters: **location** & **scale**

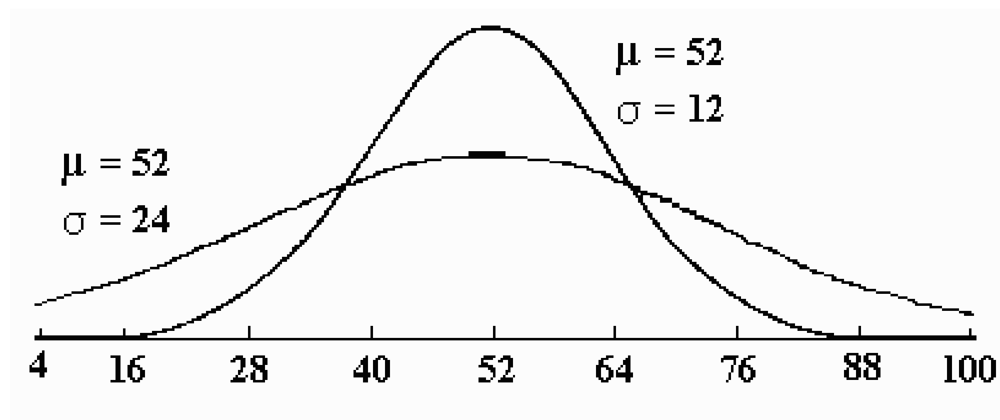
location → mean (μ) → midpoint

scale → SD (σ) → spread of scores



Properties of PDF:

- symmetry about its mean μ
- mode & median = mean μ
- inflection points → One SD from μ



Larger σ → more dispersed
Smaller σ → less dispersed

Probability Calculation

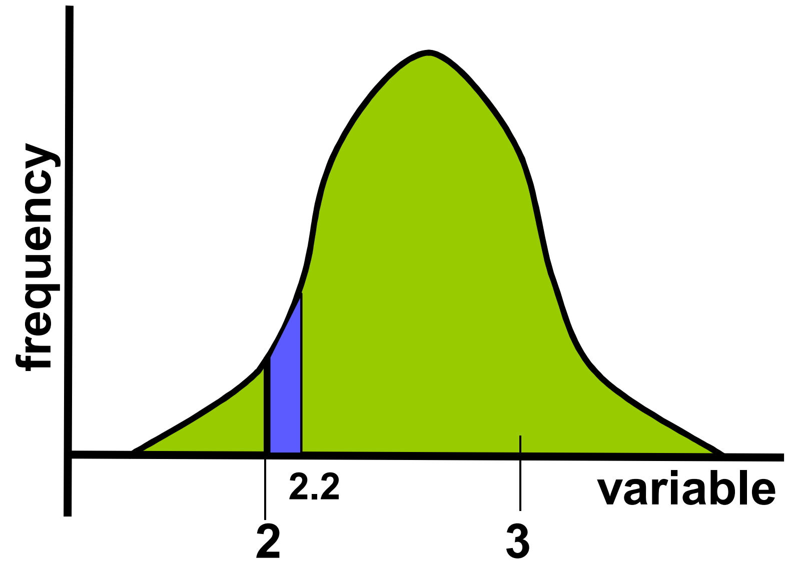
Area under curve = sum of expected frequencies

Not for probability of an exact value

e.g. probability of x=2

Estimate probability between 2 limits

e.g. probability falling between 2 & 2.2



standard deviation

mean

$$\varphi_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

constant 3.14159

base of natural log 2.718282

Standard Normal Distribution (Z)

Z Table: area under the *standard normal probability density* curve to the **left** of *z*

z	·00	·01	·02	·03	·04	·05	·06
·0	·5000	·5040	·5080	·5120	·5160	·5199	·5239
·1	·5398	·5438	·5478	·5517	·5557	·5596	·5636
·2	·5793	·5832	·5871	·5910	·5948	·5987	·6026
·3	·6179	·6217	·6255	·6293	·6331	·6368	·6406
·4	·6554	·6591	·6628	·6664	·6700	·6736	·6772
·5	·6915	·6950	·6985	·7019	·7054	·7088	·7123
·6	·7257	·7291	·7324	·7357	·7389	·7422	·7454
·7	·7580	·7611	·7642	·7673	·7703	·7734	·7764
·8	·7881	·7910	·7939	·7967	·7995	·8023	·8051
·9	·8159	·8186	·8212	·8238	·8264	·8289	·8315
1·0	·8413	·8438	·8461	·8485	·8508	·8531	·8554
1·1	·8643	·8665	·8686	·8708	·8729	·8749	·8770
1·2	·8849	·8869	·8888	·8907	·8925	·8944	·8962
1·3	·90320	·90490	·90658	·90824	·90988	·91149	·91309
1·4	·91924	·92073	·92220	·92364	·92507	·92647	·92785
1·5	·93319	·93448	·93574	·93699	·93822	·93943	·94062
1·6	·94520	·94630	·94738	·94845	·94950	·95053	·95154
1·7	·95543	·95637	·95728	·95818	·95907	·95994	·96080
1·8	·96407	·96485	·96562	·96638	·96712	·96784	·96856
1·9	·97128	·97193	·97257	·97320	·97381	·97441	·97500

$$\varphi_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

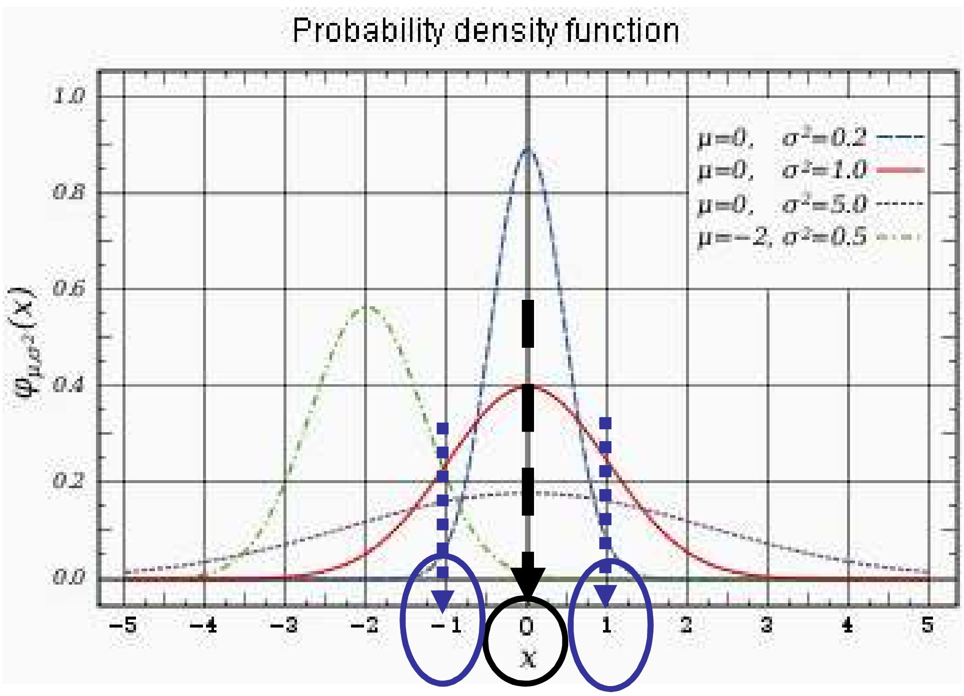
Standard Normal Distribution (Z)

Unit Normal Distribution:

N (0, 1)

mean $\mu = 0$ SD $\sigma = 1$

→ Z tables



Probability Density Function

Standard Normal Distribution (Z)

Z Table: area under the *standard normal probability density* curve to the **left** of *z*

z	·00	·01	·02	·03	·04	·05	·06
·0	·5000	·5040	·5080	·5120	·5160	·5199	·5239
·1	·5398	·5438	·5478	·5517	·5557	·5596	·5636
·2	·5793	·5832	·5871	·5910	·5948	·5987	·6026
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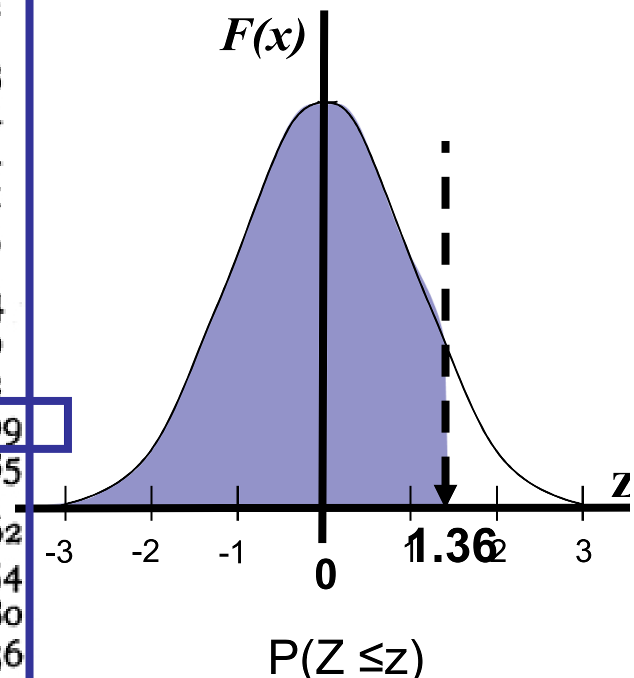
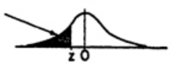
·6	·7257	·7291	·7324	·7357	·7389	·7422	·7454
·7	·7580	·7611	·7642	·7673	·7703	·7734	·7764
·8	·7881	·7910	·7939	·7967	·7995	·8023	·8051
·9	·8159	·8186	·8212	·8238	·8264	·8289	·8315
1·0	·8413	·8438	·8461	·8485	·8508	·8531	·8554
1·1	·8643	·8665	·8686	·8708	·8729	·8749	·8770
1·2	·8849	·8869	·8888	·8907	·8925	·8944	·8962
1·3	·90320	·90490	·90658	·90824	·90988	·91149	·91309
1·4	·91924	·92073	·92220	·92364	·92507	·92647	·92785
1·5	·93319	·93448	·93574	·93699	·93822	·93943	·94062
1·6	·94520	·94630	·94738	·94845	·94950	·95053	·95154
1·7	·95543	·95637	·95728	·95818	·95907	·95994	·96080
1·8	·96407	·96485	·96562	·96638	·96712	·96784	·96856
1·9	·97128	·97193	·97257	·97320	·97381	·97441	·97500

$$\varphi_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



$\Phi(z)=\int_{-\infty}^{z}\phi(t)\,dt$ for $(-\infty < z \le 0)$



z	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
−·0	·5000	·4960	·4920	·4880	·4840	·4801	·4761	·4721	·4681	·4641
−·1	·4602	·4562	·4522	·4483	·4443	·4404	·4364	·4325	·4286	·4247
−·2	·4207	·4168	·4129	·4090	·4052	·4013	·3974	·3936	·3897	·3859
−·3	·3821	·3783	·3745	·3707	·3669	·3632	·3594	·3557	·3520	·3483
−·4	·3446	·3409	·3372	·3336	·3300	·3264	·3228	·3192	·3156	·3121
−·5	·3085	·3050	·3015	·2981	·2946	·2912	·2877	·2843	·2810	·2776
−·6	·2743	·2709	·2676	·2643	·2611	·2578	·2546	·2514	·2483	·2451
−·7	·2420	·2389	·2358	·2327	·2297	·2266	·2236	·2206	·2177	·2148
−·8	·2119	·2090	·2061	·2033	·2005	·1977	·1949	·1922	·1894	·1867
−·9	·1841	·1814	·1788	·1762	·1736	·1711	·1685	·1660	·1635	·1611
−1·0	·1587	·1562	·1539	·1515	·1492	·1469	·1446	·1423	·1401	·1379
−1·1	·1357	·1335	·1314	·1292	·1271	·1251	·1230	·1210	·1190	·1170
−1·2	·1151	·1131	·1112	·1093	·1075	·1056	·1038	·1020	·1003	·09853
−1·3	·09680	·09510	·09342	·09176	·09012	·08851	·08691	·08534	·08379	·08226
−1·4	·08076	·07927	·07780	·07636	·07493	·07353	·07215	·07078	·06944	·06811
−1·5	·06681	·06552	·06426	·06301	·06178	·06057	·05938	·05821	·05705	·05592
−1·6	·05480	·05370	·05262	·05155	·05050	·04947	·04846	·04746	·04648	·04551
−1·7	·04457	·04363	·04272	·04182	·04093	·04006	·03920	·03836	·03754	·03673
−1·8	·03593	·03515	·03438	·03362	·03288	·03216	·03144	·03074	·03005	·02938
−1·9	·02872	·02807	·02743	·02680	·02619	·02559	·02500	·02442	·02385	·02330
−2·0	·02275	·02222	·02169	·02118	·02068	·02018	·01970	·01923	·01876	·01831
−2·1	·01786	·01743	·01700	·01659	·01618	·01578	·01539	·01500	·01463	·01426
−2·2	·01390	·01355	·01321	·01287	·01255	·01222	·01191	·01160	·01130	·01101
−2·3	·01072	·01044	·01017	·0^{2}9903	·0^{2}9642	·0^{2}9387	·0^{2}9137	·0^{2}8894	·0^{2}8656	·0^{2}8424
−2·4	·0^{2}8198	·0^{2}7976	·0^{2}7760	·0^{2}7549	·0^{2}7344	·0^{2}7143	·0^{2}6947	·0^{2}6756	·0^{2}6569	·0^{2}6387
−2·5	·0^{2}6210	·0^{2}6037	·0^{2}5868	·0^{2}5703	·0^{2}5543	·0^{2}5386	·0^{2}5234	·0^{2}5085	·0^{2}4940	·0^{2}4799
−2·6	·0^{2}4661	·0^{2}4527	·0^{2}4396	·0^{2}4269	·0^{2}4145	·0^{2}4025	·0^{2}3907	·0^{2}3793	·0^{2}3681	·0^{2}3573
−2·7	·0^{2}3467	·0^{2}3364	·0^{2}3264	·0^{2}3167	·0^{2}3072	·0^{2}2980	·0^{2}2890	·0^{2}2803	·0^{2}2718	·0^{2}2635
−2·8	·0^{2}2555	·0^{2}2477	·0^{2}2401	·0^{2}2327	·0^{2}2256	·0^{2}2186	·0^{2}2118	·0^{2}2052	·0^{2}1988	·0^{2}1926
−2·9	·0^{2}1866	·0^{2}1807	·0^{2}1750	·0^{2}1695	·0^{2}1641	·0^{2}1589	·0^{2}1538	·0^{2}1489	·0^{2}1441	·0^{2}1395
−3·0	·0^{2}1350	·0^{2}1306	·0^{2}1264	·0^{2}1223	·0^{2}1183	·0^{2}1144	·0^{2}1107	·0^{2}1070	·0^{2}1035	·0^{2}1001
−3·1	·0^{3}9676	·0^{3}9354	·0^{3}9043	·0^{3}8740	·0^{3}8447	·0^{3}8164	·0^{3}7888	·0^{3}7622	·0^{3}7364	·0^{3}7114
−3·2	·0^{3}6871	·0^{3}6637	·0^{3}6410	·0^{3}6190	·0^{3}5976	·0^{3}5770	·0^{3}5571	·0^{3}5377	·0^{3}5190	·0^{3}5009
−3·3	·0^{3}4834	·0^{3}4665	·0^{3}4501	·0^{3}4342	·0^{3}4189	·0^{3}4041	·0^{3}3897	·0^{3}3758	·0^{3}3624	·0^{3}3495
−3·4	·0^{3}3369	·0^{3}3248	·0^{3}3131	·0^{3}3018	·0^{3}2909	·0^{3}2803	·0^{3}2701	·0^{3}2602	·0^{3}2507	·0^{3}2415
−3·5	·0^{3}2326	·0^{3}2241	·0^{3}2158	·0^{3}2078	·0^{3}2001	·0^{3}1926	·0^{3}1854	·0^{3}1785	·0^{3}1718	·0^{3}1653
−3·6	·0^{3}1591	·0^{3}1531	·0^{3}1473	·0^{3}1417	·0^{3}1363	·0^{3}1311	·0^{3}1261	·0^{3}1213	·0^{3}1166	·0^{3}1121
−3·7	·0^{3}1078	·0^{3}1036	·0^{4}9961	·0^{4}9574	·0^{4}9201	·0^{4}8842	·0^{4}8496	·0^{4}8162	·0^{4}7841	·0^{4}7532
−3·8	·0^{4}7235	·0^{4}6948	·0^{4}6673	·0^{4}6407	·0^{4}6152	·0^{4}5906	·0^{4}5669	·0^{4}5442	·0^{4}5223	·0^{4}5012
−3·9	·0^{4}4810	·0^{4}4615	·0^{4}4427	·0^{4}4247	·0^{4}4074	·0^{4}3908	·0^{4}3747	·0^{4}3594	·0^{4}3446	·0^{4}3304
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−4·3	·0^{5}8540	·0^{5}8163	·0^{5}7801	·0^{5}7455	·0^{5}7124	·0^{5}6807	·0^{5}6503	·0^{5}6212	·0^{5}5934	·0^{5}5668
−4·4	·0^{5}5413	·0^{5}5169	·0^{5}4935	·0^{5}4712	·0^{5}4498	·0^{5}4294	·0^{5}4098	·0^{5}3911	·0^{5}3732	·0^{5}3561
−4·5	·0^{5}3398	·0^{5}3241	·0^{5}3092	·0^{5}2949	·0^{5}2813	·0^{5}2682	·0^{5}2558	·0^{5}2439	·0^{5}2325	·0^{5}2216
−4·6	·0^{5}2112	·0^{5}2013	·0^{5}1919	·0^{5}1828	·0^{5}1742	·0^{5}1660	·0^{5}1581	·0^{5}1506	·0^{5}1434	·0^{5}1366
−4·7	·0^{5}1301	·0^{5}1239	·0^{5}1179	·0^{5}1123	·0^{5}1069	·0^{5}1017	·0^{6}9680	·0^{6}9211	·0^{6}8765	·0^{6}8339
−4·8	·0^{6}7933	·0^{6}7547	·0^{6}7178	·0^{6}6827	·0^{6}6492	·0^{6}6173	·0^{6}5869	·0^{6}5580	·0^{6}5304	·0^{6}5042
−4·9	·0^{6}4792	·0^{6}4554	·0^{6}4327	·0^{6}4111	·0^{6}3906	·0^{6}3711	·0^{6}3525	·0^{6}3348	·0^{6}3179	·0^{6}3019

Example: $\Phi(-3{\cdot}57)$ = ·0^{3}1785 = 0·0001785.

Pay attention to the legend

$\Phi(z)=\int_{-\infty}^{z}\phi(t)\,dt$ for $(0 \le z < \infty)$

z	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
·0	·5000	·5040	·5080	·5120	·5160	·5199	·5239	·5279	·5319	·5359
·1	·5398	·5438	·5478	·5517	·5557	·5596	·5636	·5675	·5714	·5753
·2	·5793	·5832	·5871	·5910	·5948	·5987	·6026	·6064	·6103	·6141
·3	·6179	·6217	·6255	·6293	·6331	·6368	·6406	·6443	·6480	·6517
·4	·6554	·6591	·6628	·6664	·6700	·6736	·6772	·6808	·6844	·6879
·5	·6915	·6950	·6985	·7019	·7054	·7088	·7123	·7157	·7190	·7224
·6	·7257	·7291	·7324	·7357	·7389	·7422	·7454	·7486	·7517	·7549
·7	·7580	·7611	·7642	·7673	·7703	·7734	·7764	·7794	·7823	·7852
·8	·7881	·7910	·7939	·7967	·7995	·8023	·8051	·8078	·8106	·8133
·9	·8159	·8186	·8212	·8238	·8264	·8289	·8315	·8340	·8365	·8389
1·0	·8413	·8438	·8461	·8485	·8508	·8531	·8554	·8577	·8599	·8621
1·1	·8643	·8665	·8686	·8708	·8729	·8749	·8770	·8790	·8810	·8830
1·2	·8849	·8869	·8888	·8907	·8925	·8944	·8962	·8980	·8997	·90147
1·3	·90320	·90490	·90658	·90824	·90988	·91149	·91309	·91466	·91621	·91774
1·4	·91924	·92073	·92220	·92364	·92507	·92647	·92785	·92922	·93056	·93189
1·5	·93319	·93448	·93574	·93699	·93822	·93943	·94062	·94179	·94295	·94408
1·6	·94520	·94630	·94738	·94845	·94950	·95053	·95154	·95254	·95352	·95449
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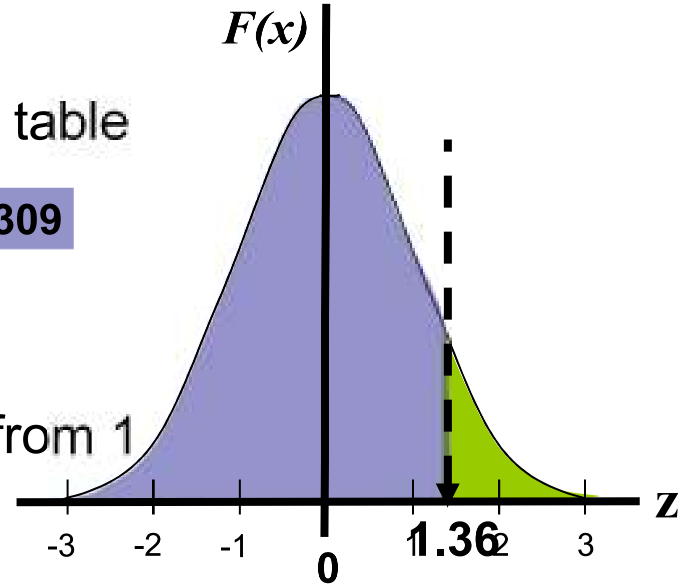
1·8	·96407	·96485	·96562	·96638	·96712	·96784	·96856	·96926	·96995	·97062
1·9	·97128	·97193	·97257	·97320	·97381	·97441	·97500	·97558	·97615	·97670
2·0	·97725	·97778	·97831	·97882	·97932	·97982	·98030	·98077	·98124	·98169
2·1	·98214	·98257	·98300	·98341	·98382	·98422	·98461	·98500	·98537	·98574
2·2	·98610	·98645	·98679	·98713	·98745	·98778	·98809	·98840	·98870	·98899
2·3	·98928	·98956	·98983	·9^{2}0097	·9^{2}0358	·9^{2}0613	·9^{2}0863	·9^{2}1106	·9^{2}1344	·9^{2}1576
2·4	·9^{2}1802	·9^{2}2024	·9^{2}2240	·9^{2}2451	·9^{2}2656	·9^{2}2857	·9^{2}3053	·9^{2}3244	·9^{2}3431	·9^{2}3613
2·5	·9^{2}3790	·9^{2}3963	·9^{2}4132	·9^{2}4297	·9^{2}4457	·9^{2}4614	·9^{2}4766	·9^{2}4915	·9^{2}5060	·9^{2}5201
2·6	·9^{2}5339	·9^{2}5473	·9^{2}5604	·9^{2}5731	·9^{2}5855	·9^{2}5975	·9^{2}6093	·9^{2}6207	·9^{2}6319	·9^{2}6427
2·7	·9^{2}6533	·9^{2}6636	·9^{2}6736	·9^{2}6833	·9^{2}6928	·9^{2}7020	·9^{2}7110	·9^{2}7197	·9^{2}7282	·9^{2}7365
2·8	·9^{2}7445	·9^{2}7523	·9^{2}7599	·9^{2}7673	·9^{2}7744	·9^{2}7814	·9^{2}7882	·9^{2}7948	·9^{2}8012	·9^{2}8074
2·9	·9^{2}8134	·9^{2}8193	·9^{2}8250	·9^{2}8305	·9^{2}8359	·9^{2}8411	·9^{2}8462	·9^{2}8511	·9^{2}8559	·9^{2}8605
3·0	·9^{2}8650	·9^{2}8694	·9^{2}8736	·9^{2}8777	·9^{2}8817	·9^{2}8856	·9^{2}8893	·9^{2}8930	·9^{2}8965	·9^{2}8999
3·1	·9^{3}0324	·9^{3}0646	·9^{3}0957	·9^{3}1260	·9^{3}1553	·9^{3}1836	·9^{3}2112	·9^{3}2378	·9^{3}2636	·9^{3}2886
3·2	·9^{3}3129	·9^{3}3363	·9^{3}3590	·9^{3}3810	·9^{3}4024	·9^{3}4230	·9^{3}4429	·9^{3}4623	·9^{3}4810	·9^{3}4991
3·3	·9^{3}5166	·9^{3}5335	·9^{3}5499	·9^{3}5658	·9^{3}5811	·9^{3}5959	·9^{3}6103	·9^{3}6242	·9^{3}6376	·9^{3}6505
3·4	·9^{3}6631	·9^{3}6752	·9^{3}6869	·9^{3}6982	·9^{3}7091	·9^{3}7197	·9^{3}7299	·9^{3}7398	·9^{3}7493	·9^{3}7585
3·5	·9^{3}7674	·9^{3}7759	·9^{3}7842	·9^{3}7922	·9^{3}7999	·9^{3}8074	·9^{3}8146	·9^{3}8215	·9^{3}8282	·9^{3}8347
3·6	·9^{3}8409	·9^{3}8469	·9^{3}8527	·9^{3}8583	·9^{3}8637	·9^{3}8689	·9^{3}8739	·9^{3}8787	·9^{3}8834	·9^{3}8879
3·7	·9^{3}8922	·9^{3}8964	·9^{4}0039	·9^{4}0426	·9^{4}0799	·9^{4}1158	·9^{4}1504	·9^{4}1838	·9^{4}2159	·9^{4}2468
3·8	·9^{4}2765	·9^{4}3052	·9^{4}3327	·9^{4}3593	·9^{4}3848	·9^{4}4094	·9^{4}4331	·9^{4}4558	·9^{4}4777	·9^{4}4988
3·9	·9^{4}5190	·9^{4}5385	·9^{4}5573	·9^{4}5753	·9^{4}5926	·9^{4}6092	·9^{4}6253	·9^{4}6406	·9^{4}6554	·9^{4}6696
4·0	·9^{4}6833	·9^{4}6964	·9^{4}7090	·9^{4}7211	·9^{4}7327	·9^{4}7439	·9^{4}7546	·9^{4}7649	·9^{4}7748	·9^{4}7843
4·1	·9^{4}7934	·9^{4}8022	·9^{4}8106	·9^{4}8186	·9^{4}8263	·9^{4}8338	·9^{4}8409	·9^{4}8477	·9^{4}8542	·9^{4}8605
4·2	·9^{4}8665	·9^{4}8723	·9^{4}8778	·9^{4}8832	·9^{4}8882	·9^{4}8931	·9^{4}8978	·9^{5}0226	·9^{5}0655	·9^{5}1066
4·3	·9^{5}1460	·9^{5}1837	·9^{5}2199	·9^{5}2545	·9^{5}2876	·9^{5}3193	·9^{5}3497	·9^{5}3788	·9^{5}4066	·9^{5}4332
4·4	·9^{5}4587	·9^{5}4831	·9^{5}5065	·9^{5}5288	·9^{5}5502	·9^{5}5706	·9^{5}5902	·9^{5}6089	·9^{5}6268	·9^{5}6439
4·5	·9^{5}6602	·9^{5}6759	·9^{5}6908	·9^{5}7051	·9^{5}7187	·9^{5}7318	·9^{5}7442	·9^{5}7561	·9^{5}7675	·9^{5}7784
4·6	·9^{5}7888	·9^{5}7987	·9^{5}8081	·9^{5}8172	·9^{5}8258	·9^{5}8340	·9^{5}8419	·9^{5}8494	·9^{5}8566	·9^{5}8634
4·7	·9^{5}8699	·9^{5}8761	·9^{5}8821	·9^{5}8877	·9^{5}8931	·9^{5}8983	·9^{6}0320	·9^{6}0789	·9^{6}1235	·9^{6}1661
4·8	·9^{6}2067	·9^{6}2453	·9^{6}2822	·9^{6}3173	·9^{6}3508	·9^{6}3827	·9^{6}4131	·9^{6}4420	·9^{6}4696	·9^{6}4958
4·9	·9^{6}5208	·9^{6}5446	·9^{6}5673	·9^{6}5889	·9^{6}6094	·9^{6}6289	·9^{6}6475	·9^{6}6652	·9^{6}6821	·9^{6}6981

Standard Normal Distribution (Z)

area to the **left** of *z*: P(Z ≤z) → directly from table

Probability of being *smaller* than 1.36

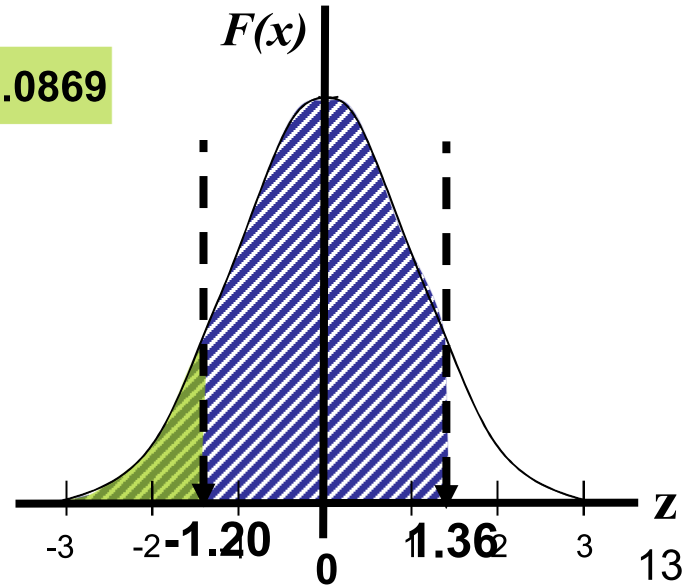
0.91309



area to the **right** of *z*: P(Z > z) → subtracted from 1

Probability of being *larger* than 1.36

1- 0.9131 = 0.0869



area between two values of *z*: P(a≤ Z ≤b)
→ subtract P(Z ≤a) from P(Z ≤b)

Probability of -1.20 ≤ Z ≤ 1.36

0.9131 – 0.1151 = 0.7980

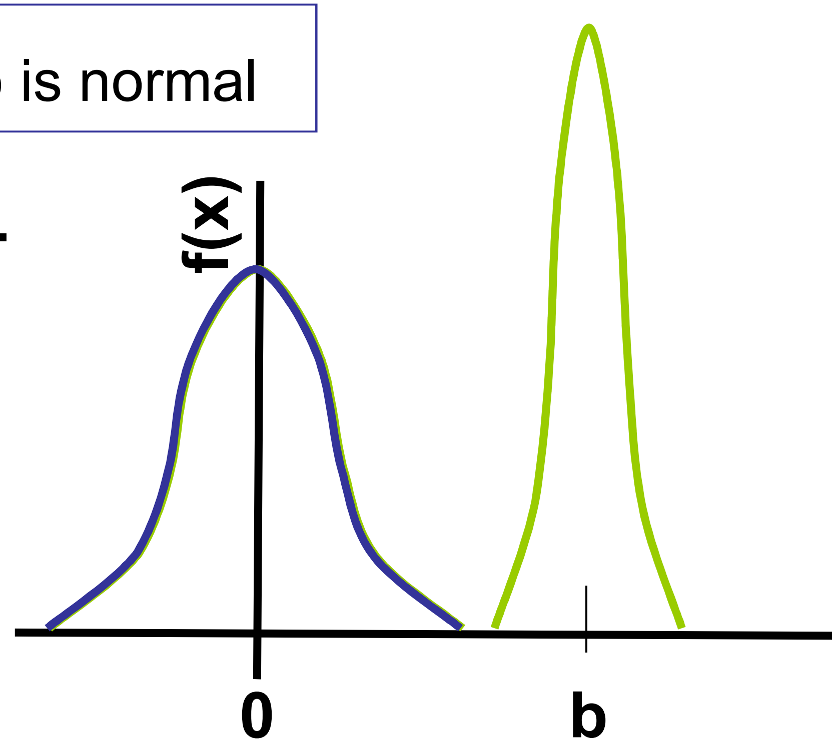
Standardization (Coding)

What about normal distributions that are not Standard (z)?

If **Z** is normal *then* **X= aZ + b** is normal

$$Z = \frac{X - b}{a}$$

$$\mathbf{Z} = \frac{\mathbf{X_i - \mu}}{\sigma}$$



Any **X ~ N (μ, σ²)** → transformed to Z

Standardization (Coding)

The duration of a project has a normal distribution with **mean of 65 days** and **SD of 5 days**.

What is the probability that the project will finish **between 55 & 72.5 days**?

$$Z = \frac{X_i - \mu}{\sigma}$$

$P(55 < \mathbf{X} < 72.5) =$

$P\left(\frac{55 - 65}{5} < \mathbf{Z} < \frac{72.5 - 65}{5}\right) =$

$P(-2 < \mathbf{Z} < 1.5) =$

$0.93319 - 0.02275 = 0.91044$

PERT

- Three estimates of duration for each activity:
  1. Most optimistic duration: t_a
  2. Most pessimistic duration: t_b
  3. Most likely duration: t_m

Expected duration = Mean (μ)

$$t_e = (t_a + 4t_m + t_b) / 6$$

Variance

$$\sigma^2 = [(t_b - t_a) / 6]^2$$

PERT

1 → 2 → 3 → 4 → 5 → 6 → 7 → 8 → 9

$$t_e = (t_a + 4t_m + t_b) / 6$$

$$\sigma^2 = [(t_b - t_a) / 6]^2$$

Optimistic (t_a) — Most Probable (t_m) — Pessimistic (t_b)

Activity	t_a	t_m	t_b	Mean μ (t_e)	Variance σ^2
1	1	3	5		
2	3	6	9		
3	10	13	19		
4	3	9	12		
5	1	3	8		
6	8	9	16		
7	4	7	13		
8	3	6	9		
9	1	3	8		
Sum					

Central Limit Theorem

x_1 to x_n :random variables with normal distributions

t_1 to t_n : mean of durations v_1 to v_n :variances

If $X = x_1 + x_2 + \ldots + x_n$

Then $T_x = t_1 + t_2 + \ldots + t_n$

$V_x = v_1 + v_2 + \ldots + v_n$

<u>In other words:</u>

The mean of the sum is the sum of means

The variance of the sum is the sum of the variances

<u>Also:</u>

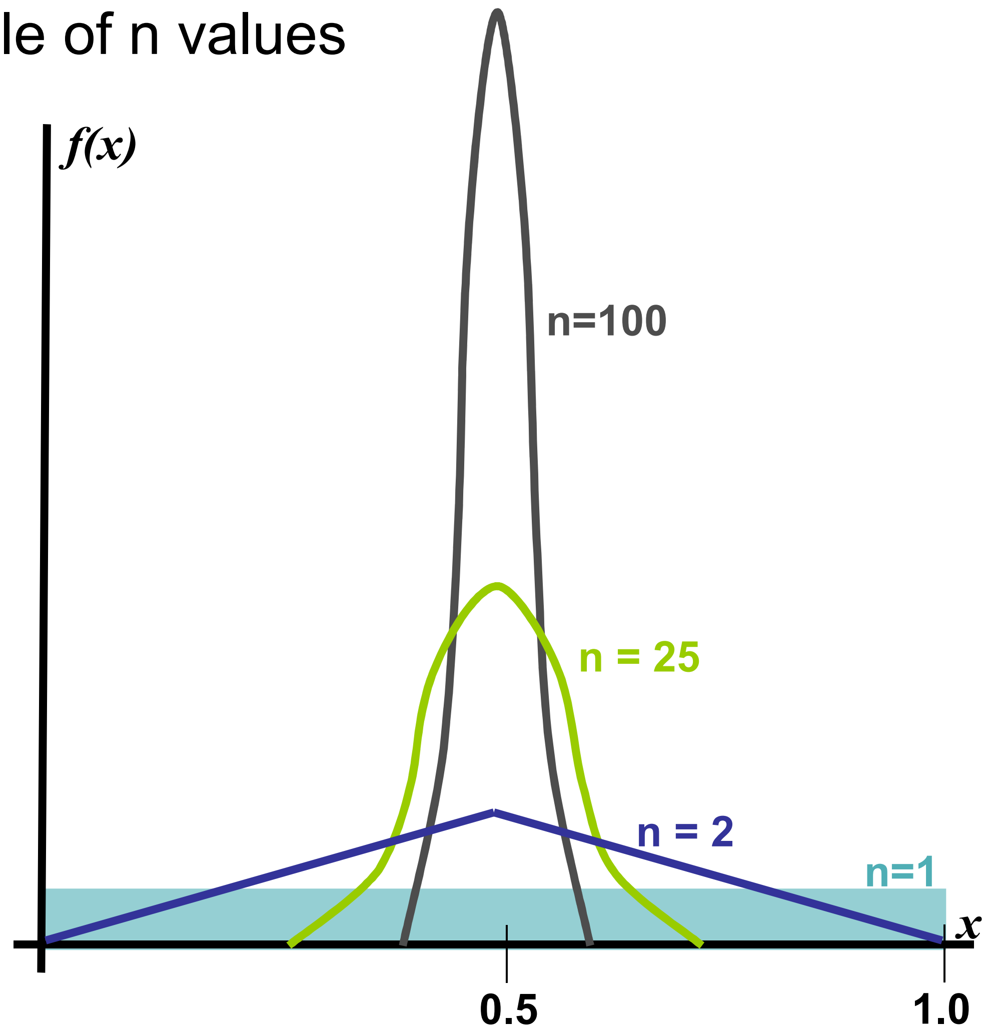
The distribution of the sum of distributions will be normal; regardless of the shape of each distribution

Central Limit Theorem

X rectangular distribution R(0,1)

$\bar{X}$ is the mean of random sample of n values

Shape of $\bar{X}$ for different n's:



CLT

If **X** random variable,

has a mean & variance,

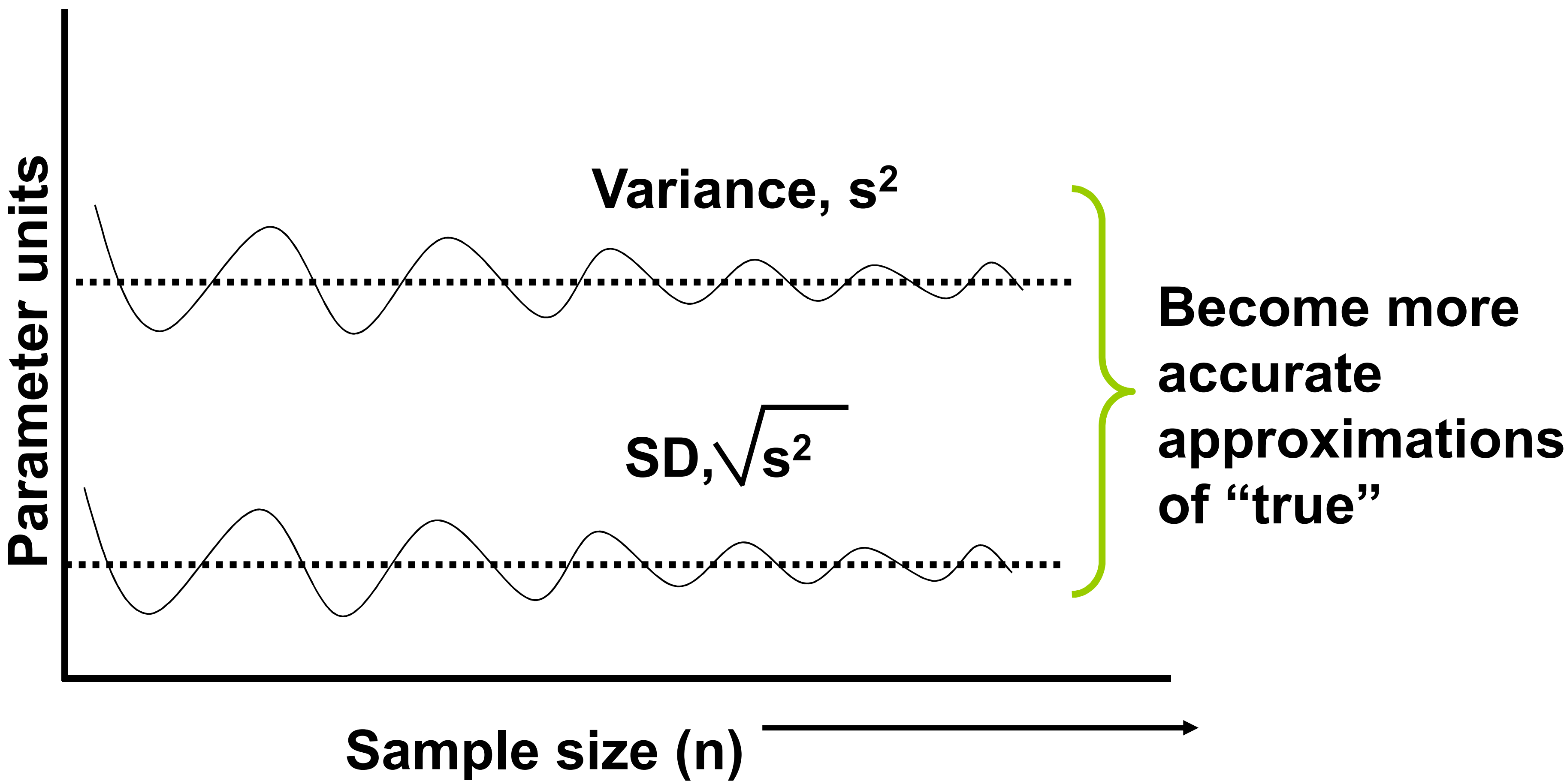
then

for sufficiently **large n**, **> 30**

$\bar{\mathbf{X}}$ has approximately

a **Normal Distribution**

Changes in s^2 & SD with increasing N



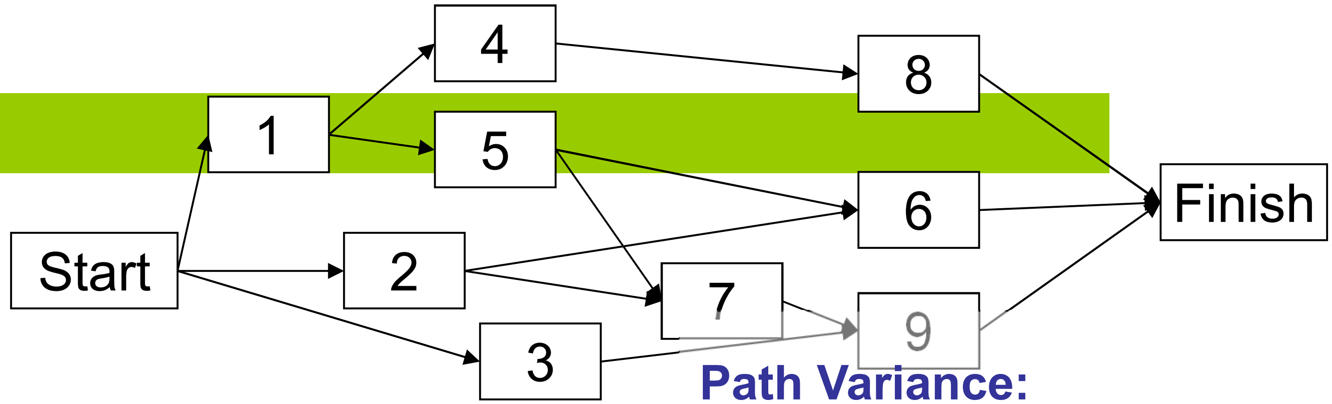
PERT

$$t_e = (t_a + 4t_m + t_b) / 6$$

$$\sigma^2 = [\,(t_b - t_a) / 6\,]^2$$

Activity	t_a	t_m	t_b	Mean μ (t_e)	Variance σ^2
1	1	3	5	3	0.44
2	3	6	9	6	1.00
3	10	13	19	13.5	2.25
4	3	9	12	8.5	2.25
5	1	3	8	3.5	1.36
6	8	9	16	10	1.77
7	4	7	13	7.5	2.25
8	3	6	9	6	1.00
9	1	3	8	3.5	1.36
Sum				**61.5**	**13.69**

PERT



Duration?

Critical Path?

Path Variance:

Activity	t_a	t_m	t_b	Mean μ (t_e)	Variance σ^2
1	1	3	5	3	0.44
2	3	6	9	6	1.00
3	10	13	19	13.5	2.25
4	3	9	12	8.5	2.25
5	1	3	8	3.5	1.36
6	8	9	16	10	1.23
7	4	7	13	7.5	2.25
8	3	6	9	6	1.00
9	1	3	8	3.5	1.36
Sum				**61.5**	**13.14**

?

?

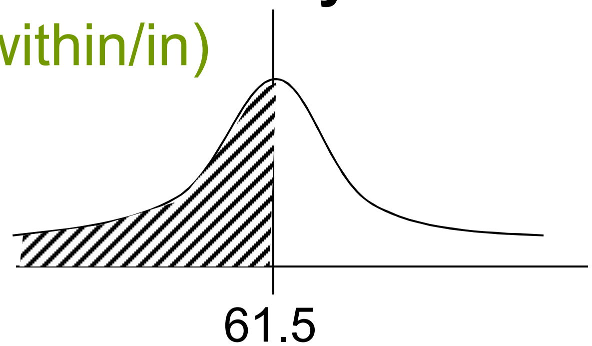
PERT - Example

t_e = 61.5 days ; σ^2 = 13.14 days ; σ = 3.5 days

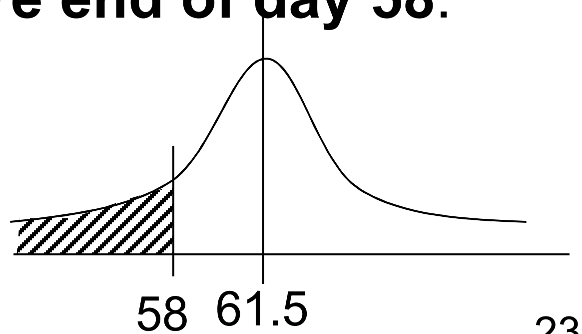
Z = (X – Mean) / σ → Z-Table to determine *P*

a) The probability that the project will finish before **61.5 days**.

(within/in)



b) The probability that the project will finish **before end of day 58**.



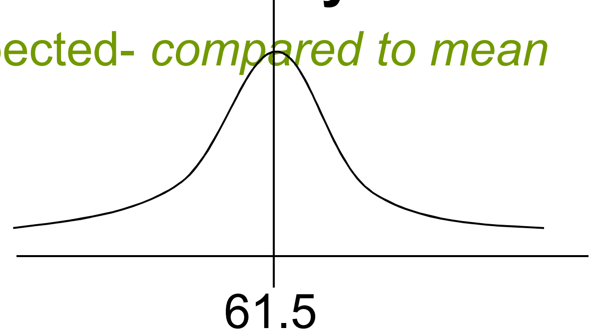
PERT - Example

t_e = 61.5 days ; σ^2 = 13.14 days ; σ = 3.5 days

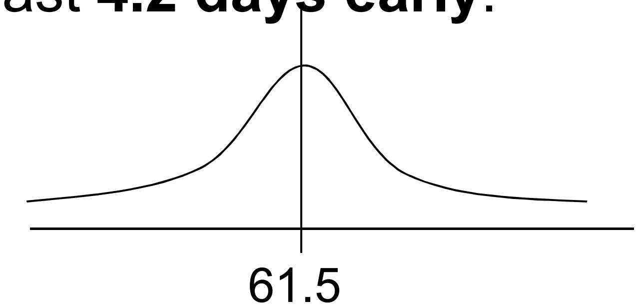
Z = (X – Mean) / σ → Z-Table to determine %

c) The probability that the project will finish more than **2.8 days late**.

later than expected- *compared to mean*



d) The probability that the project will finish at least **4.2 days early**.



e) Completion date to finish with at least a **95% confidence** level.

Approximate Probabilities

How many SD from Mean?

$P(\mu - 1\sigma < \mathbf{X} < \mu + 1\sigma) \sim 0.68$

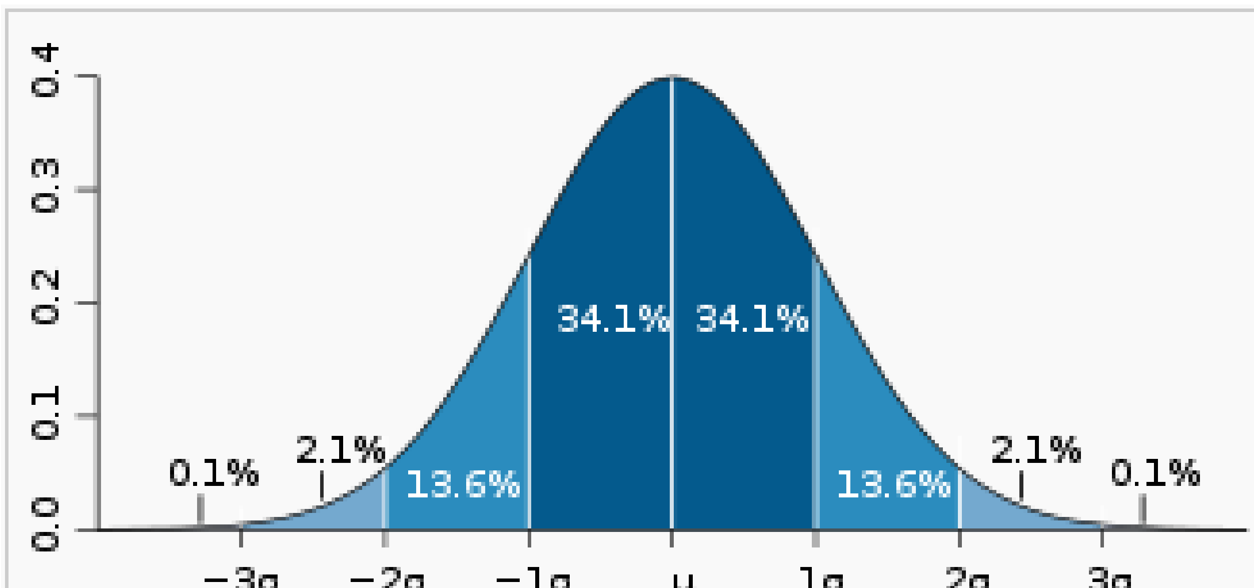
$P(\mu - 2\sigma < \mathbf{X} < \mu + 2\sigma) \sim 0.95$

$P(\mu - 3\sigma < \mathbf{X} < \mu + 3\sigma) \sim 0.99$

~ 2/3 of cases lie w/in **1 SD** of μ

~ 95% of cases lie w/in **2 SD** of μ

~ ALL cases lie w/in **3 SD** of μ



Approximately: 0 2% 15% 50% 85% 98% 100%

Example:

$P(\mu - 1\sigma < \mathbf{X} < \mu + 1\sigma) \sim 0.68$

$P(\mu - 2\sigma < \mathbf{X} < \mu + 2\sigma) \sim 0.95$

$P(\mu - 3\sigma < \mathbf{X} < \mu + 3\sigma) \sim 0.99$

~ 2/3 of cases lie w/in **1 SD** of μ

~ 95% of cases lie w/in **2 SD** of μ

~ ALL cases lie w/in **3 SD** of μ

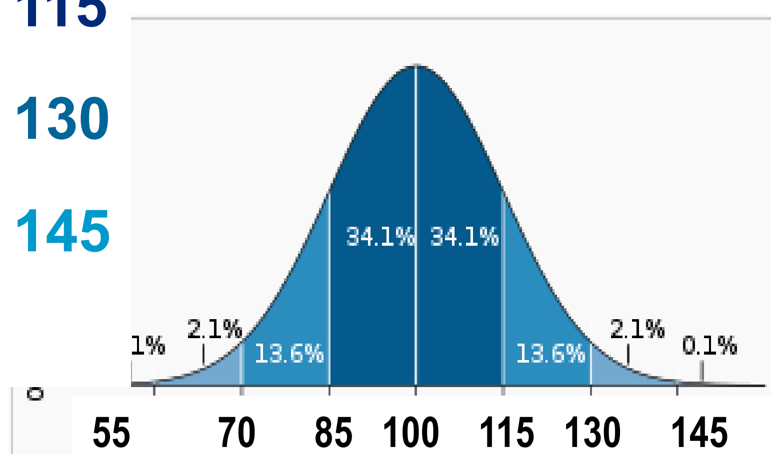
The duration of a certain project has a **mean of 100** and **SD of 15**.

~ **68%** p project will finish between **85 & 115**

~ **95%** p project will finish between **70 & 130**

~ **99%** p project will finish between **55 & 145**

~ **2%** p project will take longer than **130**



Approximately: 0 2% 15% 50% 85% 98% 100%

PERT

t_e = 61.5 days ; σ^2 = 13 days ; σ = 3.5 days

Z = (X – Mean) / σ → Z-Table to determine %

What is:

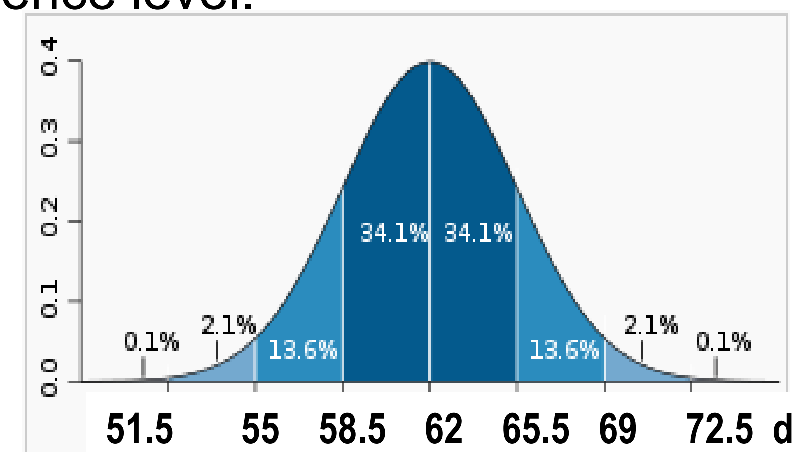
a) The probability that the project will finish **within 61.5 days**.

b) The probability that the project will finish **before day 58**.

c) The probability that the project will finish more than **3.5 days late**.

d) The probability that the project will finish at least **7 days early**.

e) The completion date with 95% confidence level.



PERT

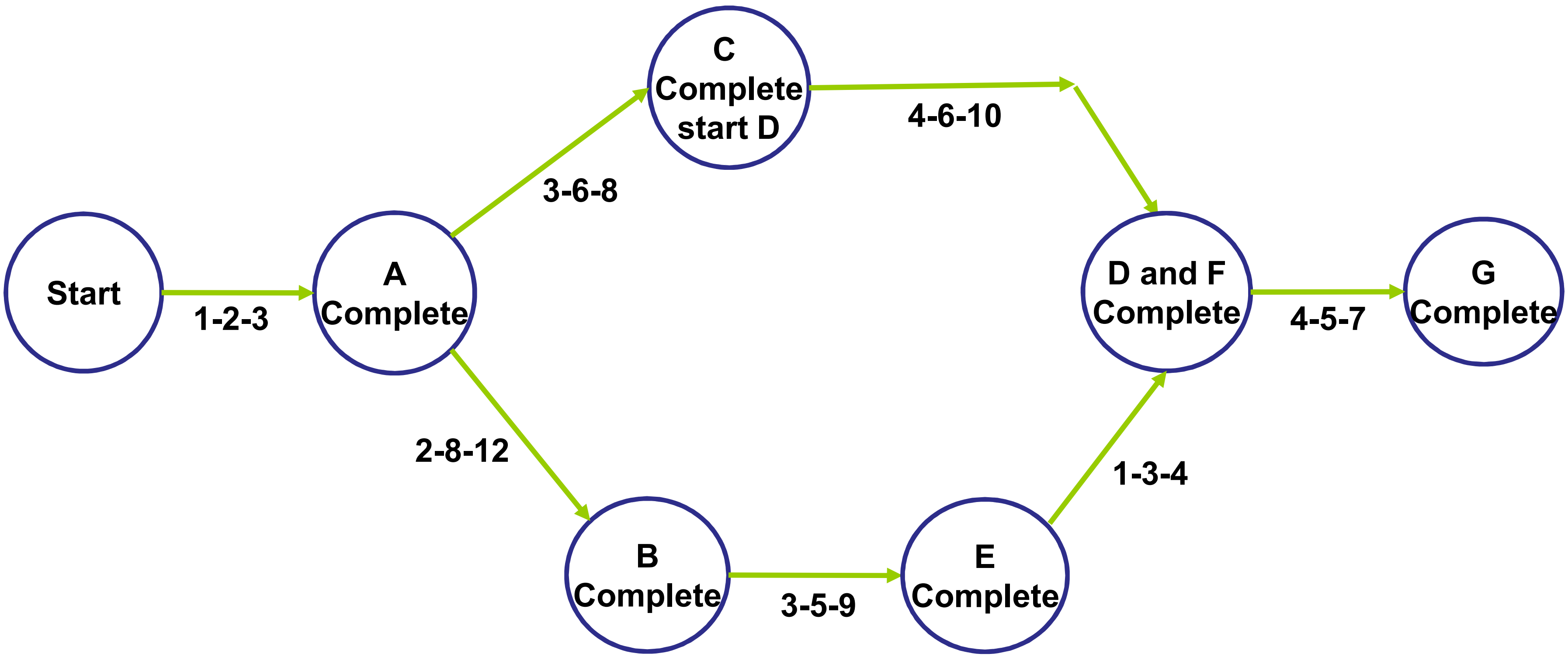
Program Evaluation & Review Technique

- Expected duration $t_e = (t_a + 4t_m + t_b) / 6$
- Variance $\sigma^2 = [\ (t_b - t_a) / 6\ ]^2$
- Standard deviation σ
- Probability of completion with certain days

  $Z = (\ X – \text{Mean}\ ) / \sigma$ → Z-Table to determine P

PERT - Notation

ADM: Activity Diagram Method commonly used for PERT



What is the probability of finishing this project within 25 days?

Floats in PERT

In PERT floats are commonly referred to as "Slack"

Probability of having Slack of x or *less*

Event Slack
(Late Time – Early Time)

$$Z = \frac{x - ES}{\sqrt{\sum SD^2{}_{TE} + \sum SD^2{}_{TL}}}$$

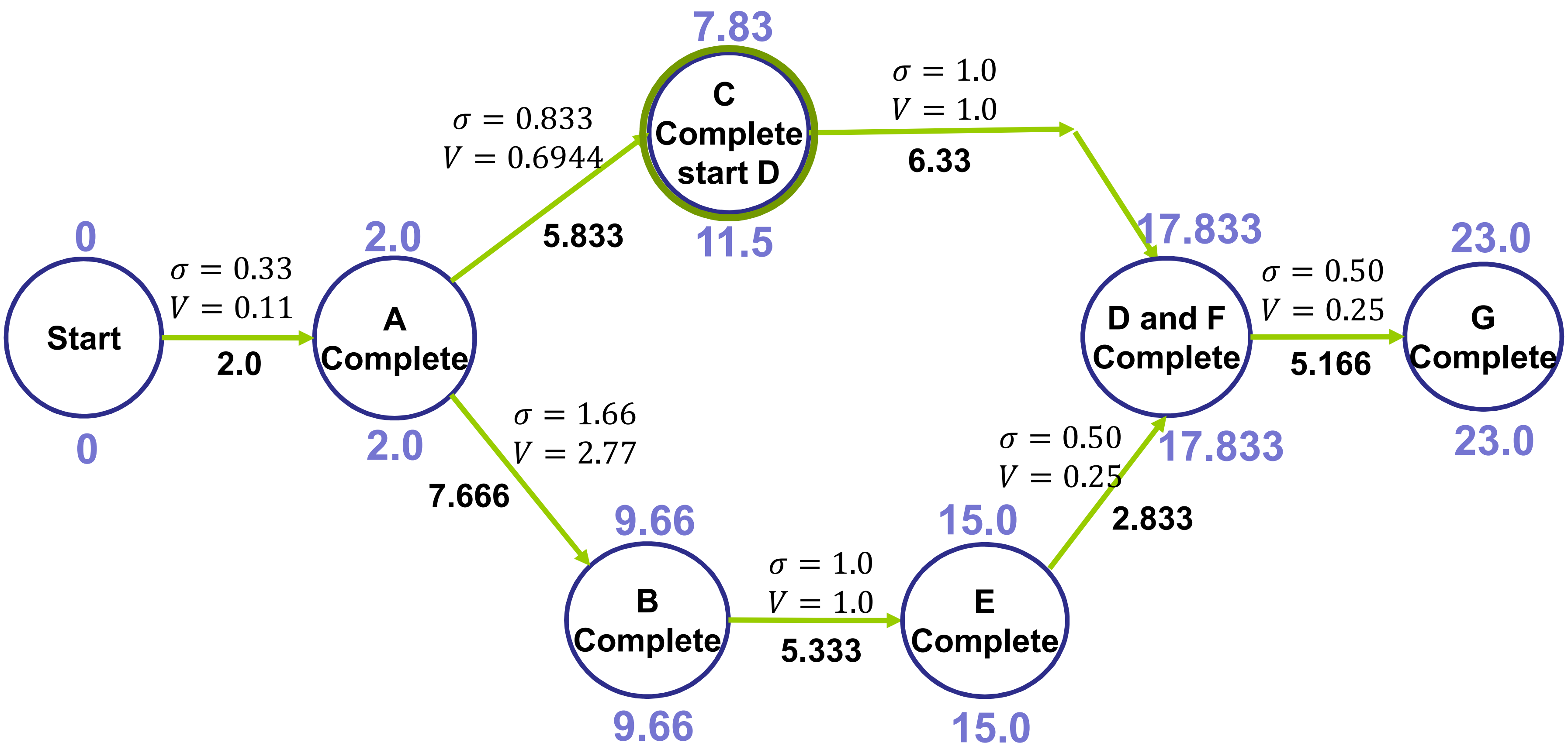
Variance of the **path** that created **TE**
(Time Early on the Node/Event)

Variance of the **path** that created **TL**
(Time Late on the Node/Event)

→ Z-Table to determine P

Float (Slack) in PERT

Example: What is the probability of "C complete Start D" slack > 0?



Closing Notes:

- Shortcoming of PERT:

  Does not consider all scenarios
  Focus on CP
  e.g. subcritical paths

Monte Carlo Simulation

- A class of computational algorithms
- Rely on **repeated random sampling**
- Projects associated with high degree of uncertainty

  →unpredictable nature of events
- Application example: Probabilistic Scheduling

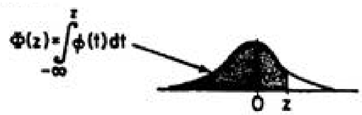
History: Invented in1940's - nuclear weapon projects
Manhattan Project: US, UK, Canada
1st atomic bomb World War II
Named after gambling uncle!

PERT

Program Evaluation & Review Technique

- Expected duration $t_e = (t_a + 4t_m + t_b) / 6$
- Variance $\sigma^2 = [\ (t_b - t_a) / 6\ ]^2$
- Standard deviation σ
- Probability of completion with certain days

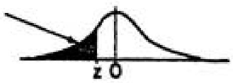
  $Z = (\ X – \text{Mean}\ ) / \sigma \rightarrow$ Z-Table to determine %



for $(0 \leq z < \infty)$

z	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
·0	·5000	·5040	·5080	·5120	·5160	·5199	·5239	·5279	·5319	·5359
·1	·5398	·5438	·5478	·5517	·5557	·5596	·5636	·5675	·5714	·5753
·2	·5793	·5832	·5871	·5910	·5948	·5987	·6026	·6064	·6103	·6141
·3	·6179	·6217	·6255	·6293	·6331	·6368	·6406	·6443	·6480	·6517
·4	·6554	·6591	·6628	·6664	·6700	·6736	·6772	·6808	·6844	·6879
·5	·6915	·6950	·6985	·7019	·7054	·7088	·7123	·7157	·7190	·7224
·6	·7257	·7291	·7324	·7357	·7389	·7422	·7454	·7486	·7517	·7549
·7	·7580	·7611	·7642	·7673	·7703	·7734	·7764	·7794	·7823	·7852
·8	·7881	·7910	·7939	·7967	·7995	·8023	·8051	·8078	·8106	·8133
·9	·8159	·8186	·8212	·8238	·8264	·8289	·8315	·8340	·8365	·8389
1·0	·8413	·8438	·8461	·8485	·8508	·8531	·8554	·8577	·8599	·8621
1·1	·8643	·8665	·8686	·8708	·8729	·8749	·8770	·8790	·8810	·8830
1·2	·8849	·8869	·8888	·8907	·8925	·8944	·8962	·8980	·8997	·90147
1·3	·90320	·90490	·90658	·90824	·90988	·91149	·91309	·91466	·91621	·91774
1·4	·91924	·92073	·92220	·92364	·92507	·92647	·92785	·92922	·93056	·93189
1·5	·93319	·93448	·93574	·93699	·93822	·93943	·94062	·94179	·94295	·94408
1·6	·94520	·94630	·94738	·94845	·94950	·95053	·95154	·95254	·95352	·95449
1·7	·95543	·95637	·95728	·95818	·95907	·95994	·96080	·96164	·96246	·96327
1·8	·96407	·96485	·96562	·96638	·96712	·96784	·96856	·96926	·96995	·97062
1·9	·97128	·97193	·97257	·97320	·97381	·97441	·97500	·97558	·97615	·97670
2·0	·97725	·97778	·97831	·97882	·97932	·97982	·98030	·98077	·98124	·98169
2·1	·98214	·98257	·98300	·98341	·98382	·98422	·98461	·98500	·98537	·98574
2·2	·98610	·98645	·98679	·98713	·98745	·98778	·98809	·98840	·98870	·98899
2·3	·98928	·98956	·98983	$\cdot 9^{2}0097$	$\cdot 9^{2}0358$	$\cdot 9^{2}0613$	$\cdot 9^{2}0863$	$\cdot 9^{2}1106$	$\cdot 9^{2}1344$	$\cdot 9^{2}1576$
2·4	$\cdot 9^{2}1802$	$\cdot 9^{2}2024$	$\cdot 9^{2}2240$	$\cdot 9^{2}2451$	$\cdot 9^{2}2656$	$\cdot 9^{2}2857$	$\cdot 9^{2}3053$	$\cdot 9^{2}3244$	$\cdot 9^{2}3431$	$\cdot 9^{2}3613$
2·5	$\cdot 9^{2}3790$	$\cdot 9^{2}3963$	$\cdot 9^{2}4132$	$\cdot 9^{2}4297$	$\cdot 9^{2}4457$	$\cdot 9^{2}4614$	$\cdot 9^{2}4766$	$\cdot 9^{2}4915$	$\cdot 9^{2}5060$	$\cdot 9^{2}5201$
2·6	$\cdot 9^{2}5339$	$\cdot 9^{2}5473$	$\cdot 9^{2}5604$	$\cdot 9^{2}5731$	$\cdot 9^{2}5855$	$\cdot 9^{2}5975$	$\cdot 9^{2}6093$	$\cdot 9^{2}6207$	$\cdot 9^{2}6319$	$\cdot 9^{2}6427$
2·7	$\cdot 9^{2}6533$	$\cdot 9^{2}6636$	$\cdot 9^{2}6736$	$\cdot 9^{2}6833$	$\cdot 9^{2}6928$	$\cdot 9^{2}7020$	$\cdot 9^{2}7110$	$\cdot 9^{2}7197$	$\cdot 9^{2}7282$	$\cdot 9^{2}7365$
2·8	$\cdot 9^{2}7445$	$\cdot 9^{2}7523$	$\cdot 9^{2}7599$	$\cdot 9^{2}7673$	$\cdot 9^{2}7744$	$\cdot 9^{2}7814$	$\cdot 9^{2}7882$	$\cdot 9^{2}7948$	$\cdot 9^{2}8012$	$\cdot 9^{2}8074$
2·9	$\cdot 9^{2}8134$	$\cdot 9^{2}8193$	$\cdot 9^{2}8250$	$\cdot 9^{2}8305$	$\cdot 9^{2}8359$	$\cdot 9^{2}8411$	$\cdot 9^{2}8462$	$\cdot 9^{2}8511$	$\cdot 9^{2}8559$	$\cdot 9^{2}8605$
3·0	$\cdot 9^{2}8650$	$\cdot 9^{2}8694$	$\cdot 9^{2}8736$	$\cdot 9^{2}8777$	$\cdot 9^{2}8817$	$\cdot 9^{2}8856$	$\cdot 9^{2}8893$	$\cdot 9^{2}8930$	$\cdot 9^{2}8965$	$\cdot 9^{2}8999$
3·1	$\cdot 9^{3}0324$	$\cdot 9^{3}0646$	$\cdot 9^{3}0957$	$\cdot 9^{3}1260$	$\cdot 9^{3}1553$	$\cdot 9^{3}1836$	$\cdot 9^{3}2112$	$\cdot 9^{3}2378$	$\cdot 9^{3}2636$	$\cdot 9^{3}2886$
3·2	$\cdot 9^{3}3129$	$\cdot 9^{3}3363$	$\cdot 9^{3}3590$	$\cdot 9^{3}3810$	$\cdot 9^{3}4024$	$\cdot 9^{3}4230$	$\cdot 9^{3}4429$	$\cdot 9^{3}4623$	$\cdot 9^{3}4810$	$\cdot 9^{3}4991$
3·3	$\cdot 9^{3}5166$	$\cdot 9^{3}5335$	$\cdot 9^{3}5499$	$\cdot 9^{3}5658$	$\cdot 9^{3}5811$	$\cdot 9^{3}5959$	$\cdot 9^{3}6103$	$\cdot 9^{3}6242$	$\cdot 9^{3}6376$	$\cdot 9^{3}6505$
3·4	$\cdot 9^{3}6631$	$\cdot 9^{3}6752$	$\cdot 9^{3}6869$	$\cdot 9^{3}6982$	$\cdot 9^{3}7091$	$\cdot 9^{3}7197$	$\cdot 9^{3}7299$	$\cdot 9^{3}7398$	$\cdot 9^{3}7493$	$\cdot 9^{3}7585$
3·5	$\cdot 9^{3}7674$	$\cdot 9^{3}7759$	$\cdot 9^{3}7842$	$\cdot 9^{3}7922$	$\cdot 9^{3}7999$	$\cdot 9^{3}8074$	$\cdot 9^{3}8146$	$\cdot 9^{3}8215$	$\cdot 9^{3}8282$	$\cdot 9^{3}8347$
3·6	$\cdot 9^{3}8409$	$\cdot 9^{3}8469$	$\cdot 9^{3}8527$	$\cdot 9^{3}8583$	$\cdot 9^{3}8637$	$\cdot 9^{3}8689$	$\cdot 9^{3}8739$	$\cdot 9^{3}8787$	$\cdot 9^{3}8834$	$\cdot 9^{3}8879$
3·7	$\cdot 9^{3}8922$	$\cdot 9^{3}8964$	$\cdot 9^{4}0039$	$\cdot 9^{4}0426$	$\cdot 9^{4}0799$	$\cdot 9^{4}1158$	$\cdot 9^{4}1504$	$\cdot 9^{4}1838$	$\cdot 9^{4}2159$	$\cdot 9^{4}2468$
3·8	$\cdot 9^{4}2765$	$\cdot 9^{4}3052$	$\cdot 9^{4}3327$	$\cdot 9^{4}3593$	$\cdot 9^{4}3848$	$\cdot 9^{4}4094$	$\cdot 9^{4}4331$	$\cdot 9^{4}4558$	$\cdot 9^{4}4777$	$\cdot 9^{4}4988$
3·9	$\cdot 9^{4}5190$	$\cdot 9^{4}5385$	$\cdot 9^{4}5573$	$\cdot 9^{4}5753$	$\cdot 9^{4}5926$	$\cdot 9^{4}6092$	$\cdot 9^{4}6253$	$\cdot 9^{4}6406$	$\cdot 9^{4}6554$	$\cdot 9^{4}6696$
4·0	$\cdot 9^{4}6833$	$\cdot 9^{4}6964$	$\cdot 9^{4}7090$	$\cdot 9^{4}7211$	$\cdot 9^{4}7327$	$\cdot 9^{4}7439$	$\cdot 9^{4}7546$	$\cdot 9^{4}7649$	$\cdot 9^{4}7748$	$\cdot 9^{4}7843$
4·1	$\cdot 9^{4}7934$	$\cdot 9^{4}8022$	$\cdot 9^{4}8106$	$\cdot 9^{4}8186$	$\cdot 9^{4}8263$	$\cdot 9^{4}8338$	$\cdot 9^{4}8409$	$\cdot 9^{4}8477$	$\cdot 9^{4}8542$	$\cdot 9^{4}8605$
4·2	$\cdot 9^{4}8665$	$\cdot 9^{4}8723$	$\cdot 9^{4}8778$	$\cdot 9^{4}8832$	$\cdot 9^{4}8882$	$\cdot 9^{4}8931$	$\cdot 9^{4}8978$	$\cdot 9^{5}0226$	$\cdot 9^{5}0655$	$\cdot 9^{5}1066$
4·3	$\cdot 9^{5}1460$	$\cdot 9^{5}1837$	$\cdot 9^{5}2199$	$\cdot 9^{5}2545$	$\cdot 9^{5}2876$	$\cdot 9^{5}3193$	$\cdot 9^{5}3497$	$\cdot 9^{5}3788$	$\cdot 9^{5}4066$	$\cdot 9^{5}4332$
4·4	$\cdot 9^{5}4587$	$\cdot 9^{5}4831$	$\cdot 9^{5}5065$	$\cdot 9^{5}5288$	$\cdot 9^{5}5502$	$\cdot 9^{5}5706$	$\cdot 9^{5}5902$	$\cdot 9^{5}6089$	$\cdot 9^{5}6268$	$\cdot 9^{5}6439$
4·5	$\cdot 9^{5}6602$	$\cdot 9^{5}6759$	$\cdot 9^{5}6908$	$\cdot 9^{5}7051$	$\cdot 9^{5}7187$	$\cdot 9^{5}7318$	$\cdot 9^{5}7442$	$\cdot 9^{5}7561$	$\cdot 9^{5}7675$	$\cdot 9^{5}7784$
4·6	$\cdot 9^{5}7888$	$\cdot 9^{5}7987$	$\cdot 9^{5}8081$	$\cdot 9^{5}8172$	$\cdot 9^{5}8258$	$\cdot 9^{5}8340$	$\cdot 9^{5}8419$	$\cdot 9^{5}8494$	$\cdot 9^{5}8566$	$\cdot 9^{5}8634$
4·7	$\cdot 9^{5}8699$	$\cdot 9^{5}8761$	$\cdot 9^{5}8821$	$\cdot 9^{5}8877$	$\cdot 9^{5}8931$	$\cdot 9^{5}8983$	$\cdot 9^{6}0320$	$\cdot 9^{6}0789$	$\cdot 9^{6}1235$	$\cdot 9^{6}1661$
4·8	$\cdot 9^{6}2067$	$\cdot 9^{6}2453$	$\cdot 9^{6}2822$	$\cdot 9^{6}3173$	$\cdot 9^{6}3508$	$\cdot 9^{6}3827$	$\cdot 9^{6}4131$	$\cdot 9^{6}4420$	$\cdot 9^{6}4696$	$\cdot 9^{6}4958$
4·9	$\cdot 9^{6}5208$	$\cdot 9^{6}5446$	$\cdot 9^{6}5673$	$\cdot 9^{6}5889$	$\cdot 9^{6}6094$	$\cdot 9^{6}6289$	$\cdot 9^{6}6475$	$\cdot 9^{6}6652$	$\cdot 9^{6}6821$	$\cdot 9^{6}6981$

Example: $\Phi(3\cdot57) = \cdot 9^{3}8215 = 0\cdot9998215$.

$\Phi(z) = \int_{-\infty}^{z} \phi(t)\,dt$ 

for $(-\infty < z \leq 0)$

Z	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
−·0	·5000	·4960	·4920	·4880	·4840	·4801	·4761	·4721	·4681	·4641
−·1	·4602	·4562	·4522	·4483	·4443	·4404	·4364	·4325	·4286	·4247
−·2	·4207	·4168	·4129	·4090	·4052	·4013	·3974	·3936	·3897	·3859
−·3	·3821	·3783	·3745	·3707	·3669	·3632	·3594	·3557	·3520	·3483
−·4	·3446	·3409	·3372	·3336	·3300	·3264	·3228	·3192	·3156	·3121
−·5	·3085	·3050	·3015	·2981	·2946	·2912	·2877	·2843	·2810	·2776
−·6	·2743	·2709	·2676	·2643	·2611	·2578	·2546	·2514	·2483	·2451
−·7	·2420	·2389	·2358	·2327	·2297	·2266	·2236	·2206	·2177	·2148
−·8	·2119	·2090	·2061	·2033	·2005	·1977	·1949	·1922	·1894	·1867
−·9	·1841	·1814	·1788	·1762	·1736	·1711	·1685	·1660	·1635	·1611
−1·0	·1587	·1562	·1539	·1515	·1492	·1469	·1446	·1423	·1401	·1379
−1·1	·1357	·1335	·1314	·1292	·1271	·1251	·1230	·1210	·1190	·1170
−1·2	·1151	·1131	·1112	·1093	·1075	·1056	·1038	·1020	·1003	·09853
−1·3	·09680	·09510	·09342	·09176	·09012	·08851	·08691	·08534	·08379	·08226
−1·4	·08076	·07927	·07780	·07636	·07493	·07353	·07215	·07078	·06944	·06811
−1·5	·06681	·06552	·06426	·06301	·06178	·06057	·05938	·05821	·05705	·05592
−1·6	·05480	·05370	·05262	·05155	·05050	·04947	·04846	·04746	·04648	·04551
−1·7	·04457	·04363	·04272	·04182	·04093	·04006	·03920	·03836	·03754	·03673
−1·8	·03593	·03515	·03438	·03362	·03288	·03216	·03144	·03074	·03005	·02938
−1·9	·02872	·02807	·02743	·02680	·02619	·02559	·02500	·02442	·02385	·02330
−2·0	·02275	·02222	·02169	·02118	·02068	·02018	·01970	·01923	·01876	·01831
−2·1	·01786	·01743	·01700	·01659	·01618	·01578	·01539	·01500	·01463	·01426
−2·2	·01390	·01355	·01321	·01287	·01255	·01222	·01191	·01160	·01130	·01101
−2·3	·01072	·01044	·01017	$\cdot0^{2}9903$	$\cdot0^{2}9642$	$\cdot0^{2}9387$	$\cdot0^{2}9137$	$\cdot0^{2}8894$	$\cdot0^{2}8656$	$\cdot0^{2}8424$
−2·4	$\cdot0^{2}8198$	$\cdot0^{2}7976$	$\cdot0^{2}7760$	$\cdot0^{2}7549$	$\cdot0^{2}7344$	$\cdot0^{2}7143$	$\cdot0^{2}6947$	$\cdot0^{2}6756$	$\cdot0^{2}6569$	$\cdot0^{2}6387$
−2·5	$\cdot0^{2}6210$	$\cdot0^{2}6037$	$\cdot0^{2}5868$	$\cdot0^{2}5703$	$\cdot0^{2}5543$	$\cdot0^{2}5386$	$\cdot0^{2}5234$	$\cdot0^{2}5085$	$\cdot0^{2}4940$	$\cdot0^{2}4799$
−2·6	$\cdot0^{2}4661$	$\cdot0^{2}4527$	$\cdot0^{2}4396$	$\cdot0^{2}4269$	$\cdot0^{2}4145$	$\cdot0^{2}4025$	$\cdot0^{2}3907$	$\cdot0^{2}3793$	$\cdot0^{2}3681$	$\cdot0^{2}3573$
−2·7	$\cdot0^{2}3467$	$\cdot0^{2}3364$	$\cdot0^{2}3264$	$\cdot0^{2}3167$	$\cdot0^{2}3072$	$\cdot0^{2}2980$	$\cdot0^{2}2890$	$\cdot0^{2}2803$	$\cdot0^{2}2718$	$\cdot0^{2}2635$
−2·8	$\cdot0^{2}2555$	$\cdot0^{2}2477$	$\cdot0^{2}2401$	$\cdot0^{2}2327$	$\cdot0^{2}2256$	$\cdot0^{2}2186$	$\cdot0^{2}2118$	$\cdot0^{2}2052$	$\cdot0^{2}1988$	$\cdot0^{2}1926$
−2·9	$\cdot0^{2}1866$	$\cdot0^{2}1807$	$\cdot0^{2}1750$	$\cdot0^{2}1695$	$\cdot0^{2}1641$	$\cdot0^{2}1589$	$\cdot0^{2}1538$	$\cdot0^{2}1489$	$\cdot0^{2}1441$	$\cdot0^{2}1395$
−3·0	$\cdot0^{2}1350$	$\cdot0^{2}1306$	$\cdot0^{2}1264$	$\cdot0^{2}1223$	$\cdot0^{2}1183$	$\cdot0^{2}1144$	$\cdot0^{2}1107$	$\cdot0^{2}1070$	$\cdot0^{2}1035$	$\cdot0^{2}1001$
−3·1	$\cdot0^{3}9676$	$\cdot0^{3}9354$	$\cdot0^{3}9043$	$\cdot0^{3}8740$	$\cdot0^{3}8447$	$\cdot0^{3}8164$	$\cdot0^{3}7888$	$\cdot0^{3}7622$	$\cdot0^{3}7364$	$\cdot0^{3}7114$
−3·2	$\cdot0^{3}6871$	$\cdot0^{3}6637$	$\cdot0^{3}6410$	$\cdot0^{3}6190$	$\cdot0^{3}5976$	$\cdot0^{3}5770$	$\cdot0^{3}5571$	$\cdot0^{3}5377$	$\cdot0^{3}5190$	$\cdot0^{3}5009$
−3·3	$\cdot0^{3}4834$	$\cdot0^{3}4665$	$\cdot0^{3}4501$	$\cdot0^{3}4342$	$\cdot0^{3}4189$	$\cdot0^{3}4041$	$\cdot0^{3}3897$	$\cdot0^{3}3758$	$\cdot0^{3}3624$	$\cdot0^{3}3495$
−3·4	$\cdot0^{3}3369$	$\cdot0^{3}3248$	$\cdot0^{3}3131$	$\cdot0^{3}3018$	$\cdot0^{3}2909$	$\cdot0^{3}2803$	$\cdot0^{3}2701$	$\cdot0^{3}2602$	$\cdot0^{3}2507$	$\cdot0^{3}2415$
−3·5	$\cdot0^{3}2326$	$\cdot0^{3}2241$	$\cdot0^{3}2158$	$\cdot0^{3}2078$	$\cdot0^{3}2001$	$\cdot0^{3}1926$	$\cdot0^{3}1854$	$\cdot0^{3}1785$	$\cdot0^{3}1718$	$\cdot0^{3}1653$
−3·6	$\cdot0^{3}1591$	$\cdot0^{3}1531$	$\cdot0^{3}1473$	$\cdot0^{3}1417$	$\cdot0^{3}1363$	$\cdot0^{3}1311$	$\cdot0^{3}1261$	$\cdot0^{3}1213$	$\cdot0^{3}1166$	$\cdot0^{3}1121$
−3·7	$\cdot0^{3}1078$	$\cdot0^{3}1036$	$\cdot0^{4}9961$	$\cdot0^{4}9574$	$\cdot0^{4}9201$	$\cdot0^{4}8842$	$\cdot0^{4}8496$	$\cdot0^{4}8162$	$\cdot0^{4}7841$	$\cdot0^{4}7532$
−3·8	$\cdot0^{4}7235$	$\cdot0^{4}6948$	$\cdot0^{4}6673$	$\cdot0^{4}6407$	$\cdot0^{4}6152$	$\cdot0^{4}5906$	$\cdot0^{4}5669$	$\cdot0^{4}5442$	$\cdot0^{4}5223$	$\cdot0^{4}5012$
−3·9	$\cdot0^{4}4810$	$\cdot0^{4}4615$	$\cdot0^{4}4427$	$\cdot0^{4}4247$	$\cdot0^{4}4074$	$\cdot0^{4}3908$	$\cdot0^{4}3747$	$\cdot0^{4}3594$	$\cdot0^{4}3446$	$\cdot0^{4}3304$
−4·0	$\cdot0^{4}3167$	$\cdot0^{4}3036$	$\cdot0^{4}2910$	$\cdot0^{4}2789$	$\cdot0^{4}2673$	$\cdot0^{4}2561$	$\cdot0^{4}2454$	$\cdot0^{4}2351$	$\cdot0^{4}2252$	$\cdot0^{4}2157$
−4·1	$\cdot0^{4}2066$	$\cdot0^{4}1978$	$\cdot0^{4}1894$	$\cdot0^{4}1814$	$\cdot0^{4}1737$	$\cdot0^{4}1662$	$\cdot0^{4}1591$	$\cdot0^{4}1523$	$\cdot0^{4}1458$	$\cdot0^{4}1395$
−4·2	$\cdot0^{4}1335$	$\cdot0^{4}1277$	$\cdot0^{4}1222$	$\cdot0^{4}1168$	$\cdot0^{4}1118$	$\cdot0^{4}1069$	$\cdot0^{4}1022$	$\cdot0^{5}9774$	$\cdot0^{5}9345$	$\cdot0^{5}8934$
−4·3	$\cdot0^{5}8540$	$\cdot0^{5}8163$	$\cdot0^{5}7801$	$\cdot0^{5}7455$	$\cdot0^{5}7124$	$\cdot0^{5}6807$	$\cdot0^{5}6503$	$\cdot0^{5}6212$	$\cdot0^{5}5934$	$\cdot0^{5}5668$
−4·4	$\cdot0^{5}5413$	$\cdot0^{5}5169$	$\cdot0^{5}4935$	$\cdot0^{5}4712$	$\cdot0^{5}4498$	$\cdot0^{5}4294$	$\cdot0^{5}4098$	$\cdot0^{5}3911$	$\cdot0^{5}3732$	$\cdot0^{5}3561$
−4·5	$\cdot0^{5}3398$	$\cdot0^{5}3241$	$\cdot0^{5}3092$	$\cdot0^{5}2949$	$\cdot0^{5}2813$	$\cdot0^{5}2682$	$\cdot0^{5}2558$	$\cdot0^{5}2439$	$\cdot0^{5}2325$	$\cdot0^{5}2216$
−4·6	$\cdot0^{5}2112$	$\cdot0^{5}2013$	$\cdot0^{5}1919$	$\cdot0^{5}1828$	$\cdot0^{5}1742$	$\cdot0^{5}1660$	$\cdot0^{5}1581$	$\cdot0^{5}1506$	$\cdot0^{5}1434$	$\cdot0^{5}1366$
−4·7	$\cdot0^{5}1301$	$\cdot0^{5}1239$	$\cdot0^{5}1179$	$\cdot0^{5}1123$	$\cdot0^{5}1069$	$\cdot0^{5}1017$	$\cdot0^{6}9680$	$\cdot0^{6}9211$	$\cdot0^{6}8765$	$\cdot0^{6}8339$
−4·8	$\cdot0^{6}7933$	$\cdot0^{6}7547$	$\cdot0^{6}7178$	$\cdot0^{6}6827$	$\cdot0^{6}6492$	$\cdot0^{6}6173$	$\cdot0^{6}5869$	$\cdot0^{6}5580$	$\cdot0^{6}5304$	$\cdot0^{6}5042$
−4·9	$\cdot0^{6}4792$	$\cdot0^{6}4554$	$\cdot0^{6}4327$	$\cdot0^{6}4111$	$\cdot0^{6}3906$	$\cdot0^{6}3711$	$\cdot0^{6}3525$	$\cdot0^{6}3348$	$\cdot0^{6}3179$	$\cdot0^{6}3019$

Example: $\Phi(-3{\cdot}57) = \cdot0^{3}1785 = 0{\cdot}0001785$.