Polynomial Fuzzy Observer Designs: A Sum-of-Squares Approach

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***Abstract*—This paper presents a sum-of-squares (SOS) approach to polynomial fuzzy observer designs for three classes of polynomial fuzzy systems. The proposed SOS-based framework provides a number of innovations and improvements over the existing linear matrix inequality (LMI)-based approaches to Takagi–Sugeno (T–S) fuzzy controller and observer designs. First, we briefly summarize previous results with respect to a polynomial fuzzy system that is a more general representation of the well-known T–S fuzzy system. Next, we propose polynomial fuzzy observers to estimate states in three classes of polynomial fuzzy systems and derive SOS conditions to design polynomial fuzzy controllers and observers. A remarkable feature of the SOS design conditions for the first two classes (Classes I and II) is that they realize the so-called separation principle, i.e., the polynomial fuzzy controller and observer for each class can be separately designed without lack of guaranteeing the stability of the overall control system in addition to converging state-estimation error (via the observer) to zero. Although, for the last class (Class III), the separation principle does not hold, we propose an algorithm to design polynomial fuzzy controller and observer satisfying the stability of the overall control system in addition to converging state-estimation error (via the observer) to zero. All the design conditions in the proposed approach can be represented in terms of SOS and are symbolically and numerically solved via the recently developed SOSTOOLS and a semidefinite-program solver, respectively. To illustrate the validity and applicability of the proposed approach, three design examples are provided. The examples demonstrate the advantages of the SOS-based approaches for the existing LMI approaches to T–S fuzzy observer designs.**

***Index Terms*—Polynomial fuzzy observer, polynomial fuzzy system, separation principle, stability, sum of squares (SOS).**

I. Introduction

THE TAKAGI–SUGENO (T–S) fuzzy-model-based control methodology [1], [2] has received a great deal of attention after linear matrix inequality (LMI)-based designs have been discussed in [3] and [4]. The fuzzy-model-based control methodology provides a natural, simple, and effective design approach to complement other nonlinear control techniques (e.g., [5]) that require special and rather involved knowledge.

Recently, the authors have first presented a sum-of-squares (SOS) approach [6]–[11] to polynomial fuzzy control system designs. This is a completely different approach from the existing LMI approaches [2], [12]–[27]. Our SOS approach [6]–[11] provided more extensive results for the existing LMI approaches to T–S fuzzy model and control. However, to the best of our knowledge, there exists no literature on SOS-based observer designs for polynomial fuzzy systems.

This paper presents SOS-based observer designs to estimate the states of polynomial fuzzy systems. The proposed SOS-based framework for polynomial fuzzy systems provides a number of innovations and improvements over the existing LMI approaches to T–S fuzzy-observer-based control, e.g., [2], [12], and [13]. First, it is known that nonlinear systems with polynomial terms cannot be generally converted to globally exact T–S fuzzy models. Only local or semiglobal T–S fuzzy models can be constructed for such nonlinear systems [2]. Thus, resulting control design conditions guarantee global stabilization and global state-estimation convergence only for local or semiglobal models but not always guarantee global stabilization and global state-estimation convergence for original nonlinear systems. On the other hand, it is possible to convert even nonlinear systems with polynomial terms to globally exact polynomial fuzzy models. Hence, all the conditions derived here guarantee global stabilization and global state-estimation convergence for original nonlinear systems that are perfectly equivalent to polynomial fuzzy models. Second, even if local or semiglobal T–S fuzzy models are permitted to be used in practical sense, variables in polynomial terms appear in premise (part) variables of T–S fuzzy models. In polynomial fuzzy models, variables in polynomial terms do not appear in their premise parts and remain in system polynomial matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ in consequent parts of polynomial fuzzy models. The difference is quite large from fuzzy observer design points of view. In general, fuzzy observer designs are not permitted to have premise variables depending on the states to be estimated. Therefore, T–S fuzzy observer designs cannot be generally applied to nonlinear systems with polynomial terms. Conversely, the polynomial fuzzy observer designs proposed in this paper can be applied to even such systems. We will see these facts in the design examples later.

This paper presents three types of SOS-based observer designs according to three classes of polynomial fuzzy systems.

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First, we present an observer-based design for the polynomial fuzzy systems with the polynomial matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ being independent of the states $\boldsymbol{x}$ to be estimated (shortly name it as Class I). Second, we discuss an observer-based design for a wider class of polynomial fuzzy systems with the polynomial matrices $\boldsymbol{A}_i$ that are permitted to be dependent of the states $\boldsymbol{x}$ to be estimated (shortly name it as Class II). It should be emphasized that this paper realizes the so-called separation design for both of the classes. This paper also presents a polynomial fuzzy observer design for a more complicated class of polynomial fuzzy systems, i.e., the polynomial fuzzy systems with the polynomial matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ that are permitted to be dependent of the states $\boldsymbol{x}$ to be estimated (shortly name it as Class III). All the design conditions discussed here are represented in terms of SOS.

It is well known that stability conditions for the T–S fuzzy system reduce to LMIs, e.g., [2]. Hence, the stability conditions can be solved numerically and efficiently by interior point algorithms, e.g., by LMI solvers. On the other hand, some kinds of control design conditions [6]–[11] for polynomial fuzzy systems reduce to SOS problems. Clearly, the problems are never directly solved by LMI solvers and can be solved via the SOSTOOLS [28] and a semidefinite-program (SDP) solver. Thus, SOS can be regarded as an extensive representation of LMIs. The computational method used in this paper relies on the SOS decomposition of multivariate polynomials. A multivariate polynomial $f(\boldsymbol{x}(t))$ (where $\boldsymbol{x}(t) \in R^n$) is an SOS if there exist polynomials $f_1(\boldsymbol{x}(t)), \ldots, f_k(\boldsymbol{x}(t))$ such that $f(\boldsymbol{x}(t)) = \sum_{i=1}^{k} f_i^2(\boldsymbol{x}(t))$. It is clear that $f(\boldsymbol{x}(t))$ being an SOS naturally implies $f(\boldsymbol{x}(t)) \geq 0$ for all $\boldsymbol{x}(t) \in R^n$. For more details of SOS, see [28].

The rest of this paper is organized as follows. Section II recalls a polynomial fuzzy system defined in [6]–[11]. Sections III–V discuss SOS-based polynomial fuzzy controller and observer designs for Classes I, II, and III, respectively. In addition, each section entails a design example to demonstrate the viability of our SOS design approach.

In this paper, to save space, we employ the following short notations with respect to matrix representation:

$$\begin{aligned}\mathcal{L}\{\boldsymbol{M}\} &= \boldsymbol{M}^{\mathrm{T}} + \boldsymbol{M},\\ \boldsymbol{E}_1 &= \mathrm{diag}[\epsilon_{11}\ \epsilon_{12} \cdots \epsilon_{1s}]\\ \boldsymbol{E}_{2i}(\boldsymbol{x}) &= \mathrm{diag}\left[\epsilon_{2i1}(\boldsymbol{x})\ \epsilon_{2i2}(\boldsymbol{x}) \cdots\ \epsilon_{2is}(\boldsymbol{x})\right]\end{aligned}$$

where $\boldsymbol{M}$ is an arbitrary square matrix. ϵ_{1k} $(k = 1, 2, \ldots, s)$ has positive values, and $\epsilon_{2ik}(\boldsymbol{x})$ $(i = 1, 2, \ldots, r;\ k = 1, 2, \ldots, s)$ represents nonnegative polynomials such that $\epsilon_{2ik}(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq 0$. ϵ_{1k} and $\epsilon_{2ik}(\boldsymbol{x})$ ($\boldsymbol{E}_1$ and $\boldsymbol{E}_{2i}(\boldsymbol{x})$) will be used as slack variables (matrices) to keep positivity of SOS conditions derived in this paper. s is the matrix size of $\boldsymbol{E}_1$ and $\boldsymbol{E}_{2i}(\boldsymbol{x})$ that are assumed to have appropriate dimensions. r is the number of fuzzy model rules.

II. T–S Fuzzy Model and Polynomial Fuzzy Model

In this section, we recall the T–S fuzzy model. The T–S fuzzy model is described by fuzzy IF–THEN rules which represent local linear input–output relations of a nonlinear system. The main feature of this model is to express the local dynamics of each fuzzy implication (rule) by a linear system model. The overall fuzzy model of the system is achieved by fuzzy blending of the linear system models.

Consider the following nonlinear system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\left(\boldsymbol{x}(t), \boldsymbol{u}(t)\right) \tag{1}$$

where f is a smooth nonlinear function such that $f(0, 0) = 0$. $\boldsymbol{x}(t) = [x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^{\mathrm{T}}$ is the state vector, and $\boldsymbol{u}(t) = [u_1(t)\ u_2(t)\ \cdots\ u_m(t)]^{\mathrm{T}}$ is the input vector. Based on the sector nonlinearity concept [2], we can exactly represent (1) with the following T–S fuzzy model (globally or at least semiglobally):

Model Rule i:

$$\begin{aligned}&\text{If } z_1(t) \text{ is } M_{i1} \text{ and } \cdots \text{ and } z_p(t) \text{ is } M_{ip}\\ &\text{then } \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t), \qquad i = 1, 2, \ldots, r\end{aligned} \tag{2}$$

where $z_j(t) (j = 1, 2, \ldots, p)$ is the premise variable. The membership function associated with the ith *model rule* and the jth premise variable component is denoted by M_{ij}. r denotes the number of *model rules*. Note that $z_j(t)$ is assumed to be independent of the states $\boldsymbol{x}$ to be estimated. In other words, each $z_j(t)$ is a measurable time-varying quantity that may be states, measurable external variables, and/or time. The defuzzification process of the model (2) can be represented as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \frac{\sum_{i=1}^{r} w_i\left(z(t)\right)\left\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\right\}}{\sum_{i=1}^{r} w_i\left(z(t)\right)}\\ &= \sum_{i=1}^{r} h_i\left(z(t)\right)\left\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\right\}\end{aligned} \tag{3}$$

where

$$\begin{aligned}\boldsymbol{z}(t) &= [z_1(t) \cdots z_p(t)]\\ w_i\left(\boldsymbol{z}(t)\right) &= \prod_{j=1}^{p} M_{ij}\left(z_j(t)\right).\end{aligned}$$

It should be noted from the properties of membership functions that the following relations hold:

$$\sum_{i=1}^{r} w_i\left(\boldsymbol{z}(t)\right) > 0, \qquad w_i\left(\boldsymbol{z}(t)\right) \geq 0; \quad i = 1, 2, \ldots, r$$

Hence

$$h_i\left(z(t)\right) = \frac{w_i\left(\boldsymbol{z}(t)\right)}{\sum_{i=1}^{r} w_i\left(\boldsymbol{z}(t)\right)} \geq 0 \qquad \sum_{i=1}^{r} h_i\left(\boldsymbol{z}(t)\right) = 1.$$

In [6] and [9], we proposed a new type of fuzzy model with polynomial model consequence, i.e., a fuzzy model whose consequent parts are represented by polynomials. Using the sector nonlinearity concept [2], we exactly represent (1) with the following polynomial fuzzy model (4). The main difference between the T–S fuzzy model [29] and the polynomial fuzzy model is the consequent part representation. The following fuzzy model has a polynomial model consequence.

Model Rule i:

$$\text{If } z_1(t) \text{ is } M_{i1} \text{ and } \cdots \text{ and } z_p(t) \text{ is } M_{ip}$$
$$\text{then } \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i(\boldsymbol{x}(t))\,\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{x}(t))\,\boldsymbol{u}(t) \tag{4}$$

where $i = 1, 2, \ldots, r$. r denotes the number of *model rules*. $\boldsymbol{A}_i(\boldsymbol{x}(t)) \in \boldsymbol{R}^{n\times n}$ and $\boldsymbol{B}_i(\boldsymbol{x}(t)) \in \boldsymbol{R}^{n\times m}$ are polynomial matrices in $\boldsymbol{x}(t)$. Therefore, $\boldsymbol{A}_i(\boldsymbol{x}(t))\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)$ is a polynomial vector. Thus, the polynomial fuzzy model (4) has a polynomial in each consequent part.

The defuzzification process of the model (4) can be represented as

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\left\{\boldsymbol{A}_i(\boldsymbol{x}(t))\,\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{x}(t))\,\boldsymbol{u}(t)\right\}. \tag{5}$$

Thus, the overall fuzzy model is achieved by fuzzy blending of the polynomial system models.

Remark 1: The polynomial fuzzy model is an extension of the T–S fuzzy model. Hence, the SOS conditions derived in this paper may be regarded as an extension of the previous LMI conditions for the T–S fuzzy model. However, it will be seen through the design examples in this paper that the polynomial fuzzy models are exact global models for the original nonlinear systems, although the T–S fuzzy models are not global models for the original nonlinear systems. In addition, the previous T–S fuzzy observer technique does not work completely for both of Classes II and III due to a premise variable restriction. For more details, we will mention it again in the design examples later.

As will be mentioned later, it is in general difficult to separately design a polynomial controller and a polynomial observer for (5) since $\boldsymbol{A}_i(\boldsymbol{x}(t))$ and $\boldsymbol{B}_i(\boldsymbol{x}(t))$ are dependent on the states $\boldsymbol{x}(t)$ to be estimated. Hence, as a first step, we introduce the following representation of polynomial fuzzy systems:

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{\rho}_A(t))\,\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{\rho}_B(t))\,\boldsymbol{u}(t)\} \tag{6}$$

where (6) reduces to (5) when $\boldsymbol{\rho}_A(t) = \boldsymbol{\rho}_B(t) = \boldsymbol{x}(t)$. In this paper, we discuss three types of polynomial-observer-based control according to three classes of polynomial fuzzy systems.

1) Class I: $\boldsymbol{\rho}_A(t) = \boldsymbol{\zeta}(t)$, and $\boldsymbol{\rho}_B(t) = \boldsymbol{\zeta}(t)$.
2) Class II: $\boldsymbol{\rho}_A(t) = \boldsymbol{x}(t)$, and $\boldsymbol{\rho}_B(t) = \boldsymbol{\zeta}(t)$.
3) Class III: $\boldsymbol{\rho}_A(t) = \boldsymbol{\rho}_B(t) = \boldsymbol{x}(t)$.

$\boldsymbol{\zeta}(t)$ is a measurable time-varying vector that may be measurable external variables, outputs, and/or time. In other words, $\boldsymbol{\zeta}(t)$ is assumed to be independent of the states $\boldsymbol{x}(t)$ to be estimated. As we can see, Class III is the most complicated class.

From now, to lighten the notation, we will drop the notation with respect to time t. For instance, we will employ $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ instead of $\boldsymbol{x}(t)$ and $\hat{\boldsymbol{x}}(t)$, respectively, where $\hat{\boldsymbol{x}}(t)$ denotes the state estimated by a polynomial fuzzy observer, as will be discussed later. Thus, we drop the notation with respect to time t, but it should be kept in mind that $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ means $\boldsymbol{x}(t)$ and $\hat{\boldsymbol{x}}(t)$, respectively.

Next, we define the outputs for the polynomial fuzzy model as

$$\boldsymbol{y} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\boldsymbol{x} \tag{7}$$

where $\boldsymbol{y} \in \mathbb{R}^q$ is the output.

III. Polynomial Controller and Observer Design (Class I)

Consider the following polynomial fuzzy system. The system matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ depend on the vector $\boldsymbol{\zeta}$

$$\begin{cases} \dot{\boldsymbol{x}} = \sum\limits_{i=1}^{r} h_i(\boldsymbol{z})\left\{\boldsymbol{A}_i(\boldsymbol{\zeta})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{u}\right\} \\ \boldsymbol{y} = \sum\limits_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\boldsymbol{x} \end{cases} \tag{8}$$

where $\boldsymbol{y} \in \mathbb{R}^q$ denotes the output.

We design a polynomial fuzzy observer to estimate the states of (8)

$$\begin{cases} \dot{\hat{\boldsymbol{x}}} = \sum\limits_{i=1}^{r} h_i(\boldsymbol{z})\left\{\boldsymbol{A}_i(\boldsymbol{\zeta})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{u} + \boldsymbol{L}_i(\boldsymbol{\zeta})(\boldsymbol{y} - \hat{\boldsymbol{y}})\right\} \\ \hat{\boldsymbol{y}} = \sum\limits_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\hat{\boldsymbol{x}} \end{cases} \tag{9}$$

where $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ is the sate vector estimated by the fuzzy observer and $\hat{\boldsymbol{y}} \in \mathbb{R}^q$ is estimated output calculated from $\hat{\boldsymbol{y}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\hat{\boldsymbol{x}}$.

To stabilize the system (8) and (9), we design a polynomial fuzzy controller with the state feedback estimated by the polynomial fuzzy observer

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{F}_i(\boldsymbol{\zeta})\hat{\boldsymbol{x}}. \tag{10}$$

Theorem 1 provides SOS conditions to separately design the polynomial fuzzy controller (10) and the polynomial fuzzy observer (9).

Theorem 1: If there exist positive definite matrices $\boldsymbol{X}_1 \in \mathbb{R}^{n\times n}$ and $\boldsymbol{X}_2 \in \mathbb{R}^{n\times n}$ and polynomial matrices $\boldsymbol{M}_i(\boldsymbol{\zeta}) \in \mathbb{R}^{p\times n}$ and $\boldsymbol{N}_i(\boldsymbol{\zeta}) \in \mathbb{R}^{n\times q}$ such that the following conditions are satisfied, the polynomial fuzzy controller (10) stabilizes the system (8), and the estimation error via the polynomial observer (9) tends to zero:

$$\boldsymbol{v}_1^{\mathrm{T}}(\boldsymbol{X}_1 - \boldsymbol{E}_1)\boldsymbol{v}_1 \text{ is SOS} \tag{11}$$

$$\boldsymbol{v}_2^{\mathrm{T}}(\boldsymbol{X}_2 - \boldsymbol{E}_2)\boldsymbol{v}_2 \text{ is SOS} \tag{12}$$

$$-\boldsymbol{v}_3^{\mathrm{T}}\left(\mathcal{L}\left\{\boldsymbol{A}_i(\boldsymbol{\zeta})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_i(\boldsymbol{\zeta})\right\} + \boldsymbol{E}_{3i}(\boldsymbol{\zeta})\right)\boldsymbol{v}_3 \text{ is SOS} \tag{13}$$

$$-\boldsymbol{v}_4^{\mathrm{T}}\left(\mathcal{L}\left\{\boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{N}_i(\boldsymbol{\zeta})\boldsymbol{C}_i\right\} + \boldsymbol{E}_{4i}(\boldsymbol{\zeta})\right)\boldsymbol{v}_4 \text{ is SOS} \tag{14}$$

$$-\boldsymbol{v}_5^{\mathrm{T}}\left(\mathcal{L}\left\{\boldsymbol{A}_i(\boldsymbol{\zeta})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_j(\boldsymbol{\zeta})\right\} + \mathcal{L}\left\{\boldsymbol{A}_j(\boldsymbol{\zeta})\boldsymbol{X}_1 - \boldsymbol{B}_j(\boldsymbol{\zeta})\boldsymbol{M}_i(\boldsymbol{\zeta})\right\}\right)\boldsymbol{v}_5 \text{ is SOS} \tag{15}$$

$$-\boldsymbol{v}_6^{\mathrm{T}}\left(\mathcal{L}\left\{\boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{N}_i(\boldsymbol{\zeta})\boldsymbol{C}_j\right\} + \mathcal{L}\left\{\boldsymbol{X}_2\boldsymbol{A}_j(\boldsymbol{\zeta}) - \boldsymbol{N}_j(\boldsymbol{\zeta})\boldsymbol{C}_i\right\}\right)\boldsymbol{v}_6 \text{ is SOS} \tag{16}$$

where $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_4$, $\boldsymbol{v}_5$, and $\boldsymbol{v}_6 \in \mathbb{R}^n$ denote the vectors that are independent of $\boldsymbol{x}$, $\hat{\boldsymbol{x}}$, and $\boldsymbol{\zeta}$. From the solutions $\boldsymbol{X}_1$ and $\boldsymbol{M}_i(\boldsymbol{\zeta})$, we obtain polynomial feedback gains $\boldsymbol{F}_i(\boldsymbol{\zeta})$ as $\boldsymbol{F}_i(\boldsymbol{\zeta}) = \boldsymbol{M}_i(\boldsymbol{\zeta})\boldsymbol{X}_1^{-1}$. From the solutions $\boldsymbol{X}_2$ and $\boldsymbol{N}_i(\boldsymbol{\zeta})$, we obtain polynomial observer gains $\boldsymbol{L}_i(\boldsymbol{\zeta})$ as $\boldsymbol{L}_i(\boldsymbol{\zeta}) = \boldsymbol{X}_2^{-1}\boldsymbol{N}_i(\boldsymbol{\zeta})$ as well.

Proof: We define the estimation error vector $\boldsymbol{e}$ as $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$. Then, the error dynamics can be described as

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\left\{\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{L}_i(\boldsymbol{\zeta})\boldsymbol{C}_j\right\}\boldsymbol{e}.$$

Next, using the augmented vector $\boldsymbol{x}_v = [\hat{\boldsymbol{x}}^{\mathrm{T}}\ \boldsymbol{e}^{\mathrm{T}}]^{\mathrm{T}}$, the augmented system consisting of the system, the polynomial fuzzy controller, and the observer can be represented as

$$\begin{aligned}\dot{\boldsymbol{x}}_v &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{G}_{ij}(\boldsymbol{\zeta})\boldsymbol{x}_v \\ &= \sum_{i=1}^{r} h_i^2(\boldsymbol{z})\boldsymbol{G}_{ii}(\boldsymbol{\zeta})\boldsymbol{x}_v \\ &\quad + \sum_{i=1}^{r}\sum_{i<j}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\left(\boldsymbol{G}_{ij}(\boldsymbol{\zeta}) + \boldsymbol{G}_{ji}(\boldsymbol{\zeta})\right)\boldsymbol{x}_v \end{aligned} \tag{17}$$

where

$$\begin{aligned}\boldsymbol{G}_{ij}(\boldsymbol{\zeta}) &= \begin{bmatrix}\boldsymbol{G}_{11_{ij}}(\boldsymbol{\zeta}) & \boldsymbol{G}_{12_{ij}}(\boldsymbol{\zeta}) \\ 0 & \boldsymbol{G}_{22_{ij}}(\boldsymbol{\zeta})\end{bmatrix} \\ \boldsymbol{G}_{11_{ij}}(\boldsymbol{\zeta}) &= \boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{F}_j(\boldsymbol{\zeta}) \\ \boldsymbol{G}_{12_{ij}}(\boldsymbol{\zeta}) &= \boldsymbol{L}_i(\boldsymbol{\zeta})\boldsymbol{C}_j \\ \boldsymbol{G}_{22_{ij}}(\boldsymbol{\zeta}) &= \boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{L}_i(\boldsymbol{\zeta})\boldsymbol{C}_j.\end{aligned}$$

Next, consider a candidate Lyapunov function

$$V(\boldsymbol{x}_v) = \boldsymbol{x}_v^{\mathrm{T}}\tilde{\boldsymbol{X}}\boldsymbol{x}_v \tag{18}$$

where

$$\tilde{\boldsymbol{X}} = \begin{bmatrix}\alpha\boldsymbol{X}_1^{-1} & 0 \\ 0 & \boldsymbol{X}_2\end{bmatrix}. \tag{19}$$

α has a positive value, and $\boldsymbol{X}_1^{-1} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{X}_2 \in \mathbb{R}^{n\times n}$ are positive definite matrices. Note that $V(\boldsymbol{x}_v) > 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$. It is clear from the Lyapunov theory that the overall control system (17) is stable if it is proved that $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$.

The time derivative of $V(\boldsymbol{x}_v)$ along the trajectory of the system is obtained as

$$\begin{aligned}\dot{V}(\boldsymbol{x}_v) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{x}_v^{\mathrm{T}}\mathcal{L}\left\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ij}(\boldsymbol{\zeta})\right\}\boldsymbol{x}_v \\ &= \sum_{i=1}^{r} h_i^2(\boldsymbol{z})\boldsymbol{x}_v^{\mathrm{T}}\mathcal{L}\left\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{\zeta})\right\}\boldsymbol{x}_v \\ &\quad + \sum_{i=1}^{r}\sum_{i<j}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z}) \\ &\quad \times \boldsymbol{x}_v^{\mathrm{T}}\mathcal{L}\left\{\tilde{\boldsymbol{X}}\left(\boldsymbol{G}_{ij}(\boldsymbol{\zeta}) + \boldsymbol{G}_{ji}(\boldsymbol{\zeta})\right)\right\}\boldsymbol{x}_v.\end{aligned}$$

If the following conditions are satisfied, $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$:

$$\mathcal{L}\left\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{\zeta})\right\} < 0 \tag{20}$$

$$\mathcal{L}\left\{\tilde{\boldsymbol{X}}\left(\boldsymbol{G}_{ij}(\boldsymbol{\zeta}) + \boldsymbol{G}_{ji}(\boldsymbol{\zeta})\right)\right\} \leq 0, \qquad i < j \leq r. \tag{21}$$

(20) can be rewritten as

$$\mathcal{L}\left\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{\zeta})\right\} = \begin{bmatrix}\alpha\boldsymbol{\Omega}_{11_{ii}}(\boldsymbol{\zeta}) & \alpha\boldsymbol{\Omega}_{12_{ii}}(\boldsymbol{\zeta}) \\ \alpha\boldsymbol{\Omega}_{12_{ii}}^{\mathrm{T}}(\boldsymbol{\zeta}) & \boldsymbol{\Omega}_{22_{ii}}(\boldsymbol{\zeta})\end{bmatrix} < 0 \tag{22}$$

where

$$\begin{aligned}\boldsymbol{\Omega}_{11_{ii}}(\boldsymbol{\zeta}) &= \mathcal{L}\left\{\boldsymbol{X}_1^{-1}\boldsymbol{G}_{11_{ii}}(\boldsymbol{\zeta})\right\} \\ \boldsymbol{\Omega}_{12_{ii}}(\boldsymbol{\zeta}) &= \boldsymbol{X}_1^{-1}\boldsymbol{G}_{12_{ii}}(\boldsymbol{\zeta}) \\ \boldsymbol{\Omega}_{22_{ii}}(\boldsymbol{\zeta}) &= \mathcal{L}\left\{\boldsymbol{X}_2\boldsymbol{G}_{22_{ii}}(\boldsymbol{\zeta})\right\}.\end{aligned}$$

From the Schur complement, (22) can be converted into

$$\boldsymbol{\Omega}_{22_{ii}}(\boldsymbol{\zeta}) < 0 \tag{23}$$

$$\boldsymbol{\Omega}_{11_{ii}}(\boldsymbol{\zeta}) - \alpha\boldsymbol{\Omega}_{12_{ii}}(\boldsymbol{\zeta})\left(\boldsymbol{\Omega}_{22_{ii}}(\boldsymbol{\zeta})\right)^{-1}\boldsymbol{\Omega}_{12_{ii}}^{\mathrm{T}}(\boldsymbol{\zeta}) < 0. \tag{24}$$

From (23) and (24), we have

$$\boldsymbol{\Omega}_{11_{ii}}(\boldsymbol{\zeta}) < \alpha\boldsymbol{\Omega}_{12_{ii}}(\boldsymbol{\zeta})\left(\boldsymbol{\Omega}_{22_{ii}}(\boldsymbol{\zeta})\right)^{-1}\boldsymbol{\Omega}_{12_{ii}}^{\mathrm{T}}(\boldsymbol{\zeta}) \leq 0.$$

Hence, if the following conditions hold, then (20) is satisfied:

$$\mathcal{L}\left\{\boldsymbol{X}_1^{-1}\left(\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{F}_i(\boldsymbol{\zeta})\right)\right\} < 0 \tag{25}$$

$$\mathcal{L}\left\{\boldsymbol{X}_2\left(\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{L}_i(\boldsymbol{\zeta})\boldsymbol{C}_i\right)\right\} < 0. \tag{26}$$

Multiplying both sides of (25) by $\boldsymbol{X}_1$ and defining a new variable $\boldsymbol{M}_i(\boldsymbol{\zeta}) = \boldsymbol{F}_i(\boldsymbol{\zeta})\boldsymbol{X}_1$, we obtain the following conditions:

$$\mathcal{L}\left\{\boldsymbol{A}_i(\boldsymbol{\zeta})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_i(\boldsymbol{\zeta})\right\} < 0. \tag{27}$$

Defining another new variable $\boldsymbol{N}_i(\boldsymbol{\zeta}) = \boldsymbol{X}_2\boldsymbol{L}_i(\boldsymbol{\zeta})$, (26) can be described as

$$\mathcal{L}\left\{\boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{N}_i(\boldsymbol{\zeta})\boldsymbol{C}_i\right\} < 0. \tag{28}$$

In the same way as above, (21) can be also represented as

$$\begin{aligned}&\mathcal{L}\{\boldsymbol{A}_i(\boldsymbol{\zeta})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_j(\boldsymbol{\zeta}) \\ &\quad + \boldsymbol{A}_j(\boldsymbol{\zeta})\boldsymbol{X}_1 - \boldsymbol{B}_j(\boldsymbol{\zeta})\boldsymbol{M}_i(\boldsymbol{\zeta})\} \leq 0 \end{aligned} \tag{29}$$

$$\begin{aligned}&\mathcal{L}\{\boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{\zeta}) - \boldsymbol{N}_i(\boldsymbol{\zeta})\boldsymbol{C}_j \\ &\quad + \boldsymbol{X}_2\boldsymbol{A}_j(\boldsymbol{\zeta}) - \boldsymbol{N}_j(\boldsymbol{\zeta})\boldsymbol{C}_i\} \leq 0 \end{aligned} \tag{30}$$

for $i < j \leq r$. It is clear from the inequality conditions (27)–(30) that $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$ if the SOS conditions (11)–(16) hold. ■

Remark 2: The conditions (11), (13), and (15) are for SOS conditions of polynomial fuzzy controller design. The conditions (12), (14), and (16) are for SOS conditions of polynomial fuzzy observer design. Thus, Theorem 1 provides SOS design conditions to separately design polynomial fuzzy controllers and observers.

Remark 3: If $\boldsymbol{A}_i(\boldsymbol{\zeta})$, $\boldsymbol{B}_i(\boldsymbol{\zeta})$, $\boldsymbol{L}_i(\boldsymbol{\zeta})$, and $\boldsymbol{F}_i(\boldsymbol{\zeta})$ reduce to constant matrices in (8)–(10), they reduce to the ordinary T–S fuzzy model and the T–S fuzzy controller and observer, respectively. In addition, Theorem 1 reduces to the existing LMI design conditions, e.g., [13], for the T–S fuzzy controller and observer. Hence, Theorem 1 provides more general results.

Remark 4: Currently, SOS programs (SOSPs) are solved by reformulating them as SDPs, which, in turn, are solved efficiently, e.g., using interior point methods. Several commercial, as well as noncommercial, software packages are available for solving SDPs. While the conversion from SOSPs to SDPs can be manually performed for small-size instances or tailored for specific problem classes, such a conversion can be quite cumbersome to perform in general. It is therefore desirable to have a computational aid that automatically performs this conversion for general SOSPs. This is exactly where SOSTOOLS comes to play. SOSTOOLS automates the conversion from SOSP to SDP, called the SDP solver, and converts the SDP solution back to the solution of the original SOSP. At present, it uses other free MATLAB add-ons such as SeDuMi [30] or SDPT3 [31] as the SDP solver. It should be noted that we can numerically find the SOS variables (matrices) $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{M}_i(\boldsymbol{\zeta})$, and $\boldsymbol{N}_i(\boldsymbol{\zeta})$ satisfying the SOS conditions in Theorem 1 via SeDuMi in addition to SOSTOOLS because Theorem 1 provides the SOS conditions that are convex with respect to the SOS variables (matrices) $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{M}_i(\boldsymbol{\zeta})$, and $\boldsymbol{N}_i(\boldsymbol{\zeta})$. If nonconvex terms exist in SOS conditions, they cannot be numerically solved in general even via SOSTOOLS and SeDuMi. All the SOS conditions derived in this paper are convex with respect to SOS variables. Thus, our SOS-based designs proposed in this paper become numerically feasible problems. For more details of how to solve the SDPs using SeDuMi, see [28] and [30].

Remark 5: To obtain more reliable solutions for SOS conditions, we perform the following double checking throughout this paper. We first carefully check whether the command "sossolve" finds a solution without any error messages, i.e., $pinf = 0$, $dinf = 0$, and $numerr = 0$, or not. If any error messages exist, we judge it as "infeasible." After getting the feasible solutions using the command "sossolve," the "findsos" command is employed to check the feasibility of SOS conditions by substituting solutions into SOS conditions. We also carefully check whether the command "findsos" provides a feasibility solution or not. If the command "findsos" returns an infeasible result, we also judge it as "infeasible." This double checking is important to have reliable solutions in the use of SOSTOOLS [28] and SeDuMi [30].

Remark 6: The conditions $\epsilon_{1k} > 0$, $\epsilon_{2k} > 0$, $\epsilon_{3ik}(\boldsymbol{\zeta}) > 0$, and $\epsilon_{4ik}(\boldsymbol{\zeta}) > 0$ for $\boldsymbol{\zeta} \neq 0$ can be accommodated by SOS optimization in a similar way as in [32].

A) Design Example I: Consider the following nonlinear system:

$$\begin{cases} \dot{x}_1 = 0.1x_1^3 - x_2 + u \\ \dot{x}_2 = \sin x_1 - x_1^2 x_2 \end{cases} \tag{31}$$

This system has polynomial terms $0.1x_1^3$ and $x_1^2 x_2$. To obtain a T–S fuzzy model using the well-known sector nonlinearity [2], we need to assume the range of x_1, i.e., $x_1 \in [-d\ d]$, where d has a positive value. For x_1 outside the range, i.e., $x_1 < -d$ or $x_1 > d$, the T–S fuzzy model dynamics never agree with the original system dynamics. Thus, the T–S fuzzy model constructed for (31) is a local model. This means that the T–S fuzzy model stabilization and state-estimation convergence are not guaranteed for x_1 outside the range. Conversely, the polynomial fuzzy model constructed in this example can exactly and globally represent the dynamics of the original system.

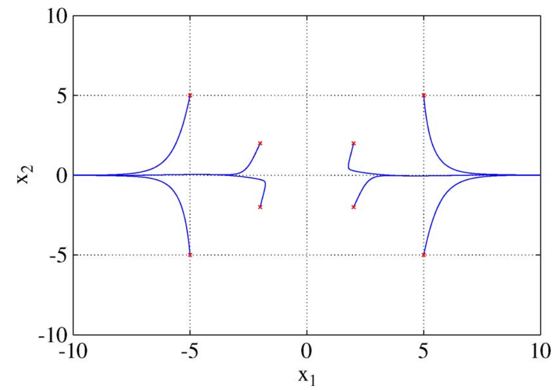

Fig. 1. System behavior without input.

Assume that x_1 is measurable and $y = x_1$. Fig. 1 shows the behavior of this system without input. It can be seen that the system is unstable.

2) Existing LMI Design Approach Based on T–S Fuzzy Systems: The existing LMI design approach for T–S fuzzy models can be applied only to Class I. First, we construct the following T–S fuzzy model for the nonlinear dynamics using the sector nonlinearity idea [2]:

$$\begin{cases} \dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \{\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{B}_i \boldsymbol{u}\} \\ \boldsymbol{y} = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \boldsymbol{C}_i \boldsymbol{x} \end{cases} \tag{32}$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} 0.1d^2 & -1 \\ 1 & -d^2 \end{bmatrix} \qquad \boldsymbol{A}_2 = \begin{bmatrix} 0.1d^2 & -1 \\ -0.217 & -d^2 \end{bmatrix}$$

$$\boldsymbol{A}_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{A}_4 = \begin{bmatrix} 0 & -1 \\ -0.217 & 0 \end{bmatrix}$$

$$\boldsymbol{B}_1 = \boldsymbol{B}_2 = \boldsymbol{B}_3 = \boldsymbol{B}_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{C}_1 = \boldsymbol{C}_2 = \boldsymbol{C}_3 = \boldsymbol{C}_4 = [1 \quad 0]$$

$$h_1(\boldsymbol{z}) = \frac{x_1^2}{d^2} \frac{\sin x_1 + 0.217x_1}{1.217x_1}$$

$$h_2(\boldsymbol{z}) = \frac{x_1^2}{d^2} \frac{x_1 - \sin x_1}{1.217x_1}$$

$$h_3(\boldsymbol{z}) = \frac{d^2 - x_1^2}{d^2} \frac{\sin x_1 + 0.217x_1}{1.217x_1}$$

$$h_4(\boldsymbol{z}) = \frac{d^2 - x_1^2}{d^2} \frac{x_1 - \sin x_1}{1.217x_1}.$$

As mentioned just before, to obtain the T–S fuzzy model, we need to assume the modeling range of x_1, i.e., $-d < x_1 < d$, where $d > 0$, since the original nonlinear system has polynomial terms. This means that the constructed fuzzy model is a semiglobal model even if we select a larger value of d. We can see in Section III-A2 that the polynomial fuzzy model becomes a global model that is equivalent to the nonlinear dynamics of (31) for any x_1. This is an advantage point using the polynomial fuzzy model and our SOS-based designs. In addition, it should be noted that the existing LMI design approach for T–S fuzzy models cannot be applied to more complicated classes, i.e., Classes II and III.

The LMI design conditions [2], [13] based on T–S fuzzy systems are derived as

$$\boldsymbol{P}_1, \boldsymbol{P}_2 > 0 \tag{33}$$
$$\boldsymbol{P}_1\boldsymbol{A}_i^{\mathrm{T}} - \boldsymbol{M}_{1i}^{\mathrm{T}}\boldsymbol{B}_i^{\mathrm{T}} + \boldsymbol{A}_i\boldsymbol{P}_1 - \boldsymbol{B}_i\boldsymbol{M}_{1i} < 0 \tag{34}$$
$$\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{P}_2 - \boldsymbol{C}_i^{\mathrm{T}}\boldsymbol{N}_{2i}^{\mathrm{T}} + \boldsymbol{P}_2\boldsymbol{A}_i - \boldsymbol{N}_{2i}\boldsymbol{C}_i < 0 \tag{35}$$
$$\boldsymbol{P}_1\boldsymbol{A}_i^{\mathrm{T}} - \boldsymbol{M}_{1j}^{\mathrm{T}}\boldsymbol{B}_i^{\mathrm{T}} + \boldsymbol{A}_i\boldsymbol{P}_1 - \boldsymbol{B}_i\boldsymbol{M}_{1j} + \boldsymbol{P}_1\boldsymbol{A}_j^{\mathrm{T}} - \boldsymbol{M}_{1i}^{\mathrm{T}}\boldsymbol{B}_j^{\mathrm{T}} + \boldsymbol{A}_j\boldsymbol{P}_1 - \boldsymbol{B}_j\boldsymbol{M}_{1i} < 0, \quad i < j \tag{36}$$
$$\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{P}_2 - \boldsymbol{C}_j^{\mathrm{T}}\boldsymbol{N}_{2i}^{\mathrm{T}} + \boldsymbol{P}_2\boldsymbol{A}_i - \boldsymbol{N}_{2i}\boldsymbol{C}_j + \boldsymbol{A}_j^{\mathrm{T}}\boldsymbol{P}_2 - \boldsymbol{C}_i^{\mathrm{T}}\boldsymbol{N}_{2j}^{\mathrm{T}} + \boldsymbol{P}_2\boldsymbol{A}_j - \boldsymbol{N}_{2j}\boldsymbol{C}_i < 0, \quad i < j. \tag{37}$$

For all the ranges from a smaller d $(d = 10^{-3})$ to a larger d $(d = 10^9)$, the LMI conditions (33)–(37) are infeasible. This means that the T–S fuzzy controller and observer for the nonlinear system cannot be designed using the existing approach. Conversely, we will see in Section III-A2 that the SOS design approach based on the polynomial fuzzy systems realizes that the polynomial fuzzy controller stabilizes the system and the estimation error via the polynomial fuzzy observer tends to zero.

3) SOS Design Approach Based on Polynomial Fuzzy Systems: The dynamics of the nonlinear system (31) can be exactly represented as the polynomial fuzzy system (8), where $r = 2$, $\boldsymbol{z} = \boldsymbol{\zeta} = y$

$$\boldsymbol{A}_1(\boldsymbol{\zeta}) = \begin{bmatrix} 0.1y^2 & -1 \\ 1 & -y^2 \end{bmatrix} \qquad \boldsymbol{A}_2(\boldsymbol{\zeta}) = \begin{bmatrix} 0.1y^2 & -1 \\ -0.2172 & -y^2 \end{bmatrix}$$
$$\boldsymbol{B}_1(\boldsymbol{\zeta}) = \boldsymbol{B}_2(\boldsymbol{\zeta}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{C}_1 = \boldsymbol{C}_2 = [1 \quad 0]$$
$$h_1(\boldsymbol{z}) = \frac{\sin y + 0.2172y}{1.2172y} \qquad h_2(\boldsymbol{z}) = \frac{y - \sin y}{1.2172y}.$$

By solving the SOS conditions in Theorem 1, we have $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{M}_i(\boldsymbol{\zeta})$, and $\boldsymbol{N}_i(\boldsymbol{\zeta})$, where the orders of $\boldsymbol{M}_i(\boldsymbol{\zeta})$ and $\boldsymbol{N}_i(\boldsymbol{\zeta})$ are two. e^{-10} and e^{-2} mean 10^{-10} and 10^{-2}, respectively

$$\boldsymbol{X}_1 = \begin{bmatrix} 0.61825 & -0.5326e^{-10} \\ -0.5326e^{-10} & 0.42137 \end{bmatrix}$$
$$\boldsymbol{X}_2 = \begin{bmatrix} 0.68214 & 0.27426 \\ 0.27426 & 0.46738 \end{bmatrix}$$
$$\boldsymbol{M}_1(\boldsymbol{\zeta}) = [\,0.14778 + 0.41613y^2 \quad 0.19687 - 0.53405e^{-2}y^2\,]$$
$$\boldsymbol{M}_2(\boldsymbol{\zeta}) = [\,0.44549 + 0.41613y^2 \quad -0.55566 - 0.53404e^{-2}y^2\,]$$
$$\boldsymbol{N}_1(\boldsymbol{\zeta}) = \begin{bmatrix} 0.61756 + 0.42283y^2 \\ -0.20621 - 0.21828y^2 \end{bmatrix}$$
$$\boldsymbol{N}_2(\boldsymbol{\zeta}) = \begin{bmatrix} 0.30425 + 0.42283y^2 \\ -0.72299 - 0.21828y^2 \end{bmatrix}.$$

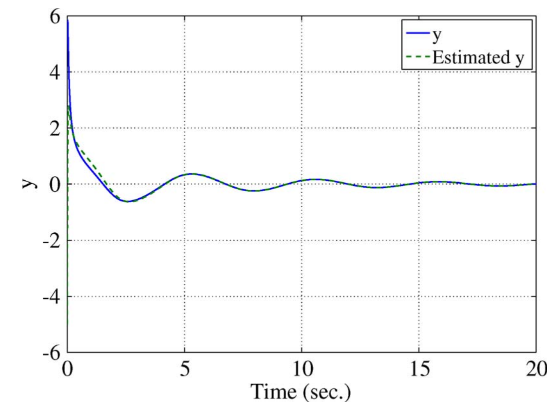

Fig. 2. Control and estimation results.

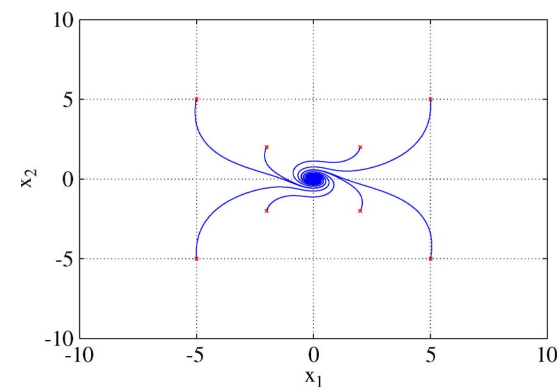

Fig. 3. Control trajectory for same initial states as in Fig. 1.

From the solutions $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{M}_i(\boldsymbol{\zeta})$, and $\boldsymbol{N}_i(\boldsymbol{\zeta})$, the polynomial feedback gains $\boldsymbol{F}_i(\boldsymbol{\zeta})$ and observer gains $\boldsymbol{L}_i(\boldsymbol{\zeta})$ are given as

$$\boldsymbol{F}_1(\boldsymbol{\zeta}) = [\,0.23903 + 0.67308y^2 \quad 0.46721 - 0.12674e^{-1}y^2\,]$$
$$\boldsymbol{F}_2(\boldsymbol{\zeta}) = [\,0.72057 + 0.67308y^2 \quad -1.31870 - 0.12674e^{-1}y^2\,]$$
$$\boldsymbol{L}_1(\boldsymbol{\zeta}) = \begin{bmatrix} 1.41704 + 1.05701y^2 \\ -1.27273 - 1.08729y^2 \end{bmatrix}$$
$$\boldsymbol{L}_2(\boldsymbol{\zeta}) = [\,1.39773 + 1.05701y^2 \quad -2.36709 - 1.08729y^2\,].$$

Fig. 2 shows the control and estimation results by the designed polynomial fuzzy controller and observer with their gains $\boldsymbol{F}_i(\boldsymbol{\zeta})$ and $\boldsymbol{L}_i(\boldsymbol{\zeta})$, where the initial states are $\boldsymbol{x}(0) = [5\ 5]$ and $\hat{\boldsymbol{x}}(0) = [-5\ \ -5]$. Fig. 3 shows the phase plot of the control results for the same initial states as in Fig. 1. It can be seen from these figures that the polynomial fuzzy controller stabilizes the system and the estimation error via the polynomial observer tends to zero.

IV. Polynomial Controller and Observer Design (Class II)

In Section III, we discussed an observer design for the polynomial fuzzy system (8) with $\boldsymbol{A}_i(\zeta)$ and $\boldsymbol{B}_i(\zeta)$ matrices. This section presents a more complicated class, i.e., $\boldsymbol{A}_i$ depends on the state $\boldsymbol{x}$ instead of the vector $\boldsymbol{\zeta}$. Although the separation design for Class II is difficult, we derive SOS conditions to achieve it in this section. The reason will be mentioned in Remark 7. Consider the following polynomial fuzzy system:

$$\begin{cases} \dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \left\{ \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{u} \right\} \\ \boldsymbol{y} = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \boldsymbol{C}_i \boldsymbol{x}. \end{cases} \tag{38}$$

We design a polynomial fuzzy observer to estimate the states of (38)

$$\begin{cases} \dot{\hat{\boldsymbol{x}}} = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \left\{ \boldsymbol{A}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{u} + \boldsymbol{L}_i(\hat{\boldsymbol{x}})(\boldsymbol{y} - \hat{\boldsymbol{y}}) \right\} \\ \hat{\boldsymbol{y}} = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \boldsymbol{C}_i \hat{\boldsymbol{x}}. \end{cases} \tag{39}$$

To stabilize the system, we design a polynomial fuzzy controller with the state feedback estimated by the polynomial observer

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z}) \boldsymbol{F}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}. \tag{40}$$

The difference between (40) and (10) is that (40) has the polynomial feedback gains in $\hat{\boldsymbol{x}}$ instead of those in $\boldsymbol{\zeta}$ in (10). Theorem 2 provides SOS conditions to separately design the polynomial fuzzy controller (40) and the polynomial fuzzy observer (39).

Theorem 2: If there exist positive definite matrices $\boldsymbol{X}_1 \in \mathbb{R}^{n\times n}$ and $\boldsymbol{X}_2 \in \mathbb{R}^{n\times n}$ and polynomial matrices $\boldsymbol{M}_i(\hat{\boldsymbol{x}}) \in \mathbb{R}^{p\times n}$ and $\boldsymbol{N}_i(\hat{\boldsymbol{x}}) \in \mathbb{R}^{n\times q}$ satisfying the following conditions, the polynomial fuzzy controller (40) stabilizes the system (38) and the estimation error via the polynomial fuzzy observer (39) tends to zero

$$\boldsymbol{v}_1^{\mathrm{T}}(\boldsymbol{X}_1 - \boldsymbol{E}_1)\boldsymbol{v}_1 \text{ is SOS} \tag{41}$$

$$\boldsymbol{v}_2^{\mathrm{T}}(\boldsymbol{X}_2 - \boldsymbol{E}_2)\boldsymbol{v}_2 \text{ is SOS} \tag{42}$$

$$-\boldsymbol{v}_3^{\mathrm{T}} \left(\mathcal{L}\left\{ \boldsymbol{A}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_i(\hat{\boldsymbol{x}}) \right\} + \boldsymbol{E}_{3i}(\boldsymbol{\zeta}, \hat{\boldsymbol{x}}) \right) \boldsymbol{v}_3 \text{ is SOS} \tag{43}$$

$$-\boldsymbol{v}_4^{\mathrm{T}} \left(\mathcal{L}\left\{ \boldsymbol{X}_2\bar{\boldsymbol{A}}_i(\boldsymbol{x}, \hat{\boldsymbol{x}}) - \boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_i \right\} + \boldsymbol{E}_{4i}(\boldsymbol{x}, \hat{\boldsymbol{x}}) \right) \boldsymbol{v}_4 \text{ is SOS} \tag{44}$$

$$-\boldsymbol{v}_5^{\mathrm{T}} \left(\mathcal{L}\left\{ \boldsymbol{A}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_j(\hat{\boldsymbol{x}}) \right\} + \mathcal{L}\left\{ \boldsymbol{A}_j(\hat{\boldsymbol{x}})\boldsymbol{X}_1 - \boldsymbol{B}_j(\boldsymbol{\zeta})\boldsymbol{M}_i(\hat{\boldsymbol{x}}) \right\} \right) \boldsymbol{v}_5 \text{ is SOS}, \quad i < j \le r \tag{45}$$

$$-\boldsymbol{v}_6^{\mathrm{T}} \left(\mathcal{L}\left\{ \boldsymbol{X}_2\bar{\boldsymbol{A}}_i(\boldsymbol{x}, \hat{\boldsymbol{x}}) - \boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j \right\} + \mathcal{L}\left\{ \boldsymbol{X}_2\bar{\boldsymbol{A}}_j(\boldsymbol{x}, \hat{\boldsymbol{x}}) - \boldsymbol{N}_j(\hat{\boldsymbol{x}})\boldsymbol{C}_i \right\} \right) \boldsymbol{v}_6 \text{ is SOS}, \quad i < j \le r \tag{46}$$

where $\bar{\boldsymbol{A}}_i(\boldsymbol{x}, \hat{\boldsymbol{x}})\boldsymbol{e} = \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}$. $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6 \in \mathbb{R}^n$ denote the vectors that are independent of $\boldsymbol{x}$, $\hat{\boldsymbol{x}}$, and $\boldsymbol{\zeta}$. From the solutions $\boldsymbol{X}_1$ and $\boldsymbol{M}_i(\hat{\boldsymbol{x}})$, we obtain polynomial feedback gains $\boldsymbol{F}_i(\hat{\boldsymbol{x}})$ as $\boldsymbol{F}_i(\hat{\boldsymbol{x}}) = \boldsymbol{M}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1^{-1}$. From the solutions $\boldsymbol{X}_2$ and $\boldsymbol{N}_i(\hat{\boldsymbol{x}})$, we obtain polynomial observer gains $\boldsymbol{L}_i(\hat{\boldsymbol{x}})$ as $\boldsymbol{L}_i(\hat{\boldsymbol{x}}) = \boldsymbol{X}_2^{-1}\boldsymbol{N}_i(\hat{\boldsymbol{x}})$ as well.

Proof: Consider the estimation error $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ by the observer. Then, the error system with respect to $\boldsymbol{e}$ can be represented as

$$\begin{aligned} \dot{\boldsymbol{e}} &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z}) \left\{ \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j\boldsymbol{e} \right\} \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z}) \left\{ \bar{\boldsymbol{A}}_i(\boldsymbol{x}, \hat{\boldsymbol{x}}) - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j \right\} \boldsymbol{e} \end{aligned}$$

where $\bar{\boldsymbol{A}}(\boldsymbol{x}, \hat{\boldsymbol{x}})\boldsymbol{e} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}$. The augmented system with the augmented vector $\boldsymbol{x}_v = [\hat{\boldsymbol{x}}^{\mathrm{T}}\ \boldsymbol{e}^{\mathrm{T}}]^{\mathrm{T}}$ is given as

$$\begin{aligned} \dot{\boldsymbol{x}}_v &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z}) \times \begin{bmatrix} \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{F}_j(\hat{\boldsymbol{x}}) & \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j \\ 0 & \bar{\boldsymbol{A}}_i(\boldsymbol{x}, \hat{\boldsymbol{x}}) - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j \end{bmatrix} \boldsymbol{x}_v \\ &= \sum_{i=1}^{r} h_i^2(\boldsymbol{z}) \boldsymbol{G}_{ii}(\boldsymbol{x}, \boldsymbol{\zeta}, \hat{\boldsymbol{x}})\boldsymbol{x}_v \\ &\quad + \sum_{i=1}^{r}\sum_{i<j} h_i(\boldsymbol{z})h_j(\boldsymbol{z}) \left(\boldsymbol{G}_{ij}(\boldsymbol{x}, \boldsymbol{\zeta}, \hat{\boldsymbol{x}}) + \boldsymbol{G}_{ji}(\boldsymbol{x}, \boldsymbol{\zeta}, \hat{\boldsymbol{x}}) \right) \boldsymbol{x}_v \end{aligned} \tag{47}$$

where

$$\begin{aligned} \boldsymbol{G}_{ij}(\boldsymbol{x}, \boldsymbol{\zeta}, \hat{\boldsymbol{x}}) &= \begin{bmatrix} \boldsymbol{G}_{11_{ij}}(\boldsymbol{\zeta}, \hat{\boldsymbol{x}}) & \boldsymbol{G}_{12_{ij}}(\hat{\boldsymbol{x}}) \\ 0 & \boldsymbol{G}_{22_{ij}}(\boldsymbol{x}, \hat{\boldsymbol{x}}) \end{bmatrix} \\ \boldsymbol{G}_{11_{ij}}(\boldsymbol{\zeta}, \hat{\boldsymbol{x}}) &= \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{F}_j(\hat{\boldsymbol{x}}) \\ \boldsymbol{G}_{12_{ij}}(\hat{\boldsymbol{x}}) &= \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j \\ \boldsymbol{G}_{22_{ij}}(\boldsymbol{x}, \hat{\boldsymbol{x}}) &= \bar{\boldsymbol{A}}_i(\boldsymbol{x}, \hat{\boldsymbol{x}}) - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j. \end{aligned}$$

Now, consider a candidate of Lyapunov function

$$V(\boldsymbol{x}_v) = \boldsymbol{x}_v^{\mathrm{T}} \tilde{\boldsymbol{X}} \boldsymbol{x}_v \tag{48}$$

where

$$\tilde{\boldsymbol{X}} = \begin{bmatrix} \alpha \boldsymbol{X}_1^{-1} & 0 \\ 0 & \boldsymbol{X}_2 \end{bmatrix}. \tag{49}$$

α has a positive value, and $\boldsymbol{X}_1^{-1} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{X}_2 \in \mathbb{R}^{n\times n}$ are positive definite matrices. Note that $V(\boldsymbol{x}_v) > 0$ at $\boldsymbol{x}_v \ne \boldsymbol{0}$. It is clear from the Lyapunov theory that the overall control system (47) is stable if it is proved that $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \ne \boldsymbol{0}$.

The time derivative of $V(\boldsymbol{x}_v)$ along the trajectory of the system is obtained as

$$\begin{aligned} \dot{V}(\boldsymbol{x}_v) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z}) \boldsymbol{x}_v^{\mathrm{T}} \mathcal{L}\left\{ \tilde{\boldsymbol{X}} \boldsymbol{G}_{ij}(\boldsymbol{x}, \boldsymbol{\zeta}, \hat{\boldsymbol{x}}) \right\} \boldsymbol{x}_v \\ &= \sum_{i=1}^{r} h_i^2(\boldsymbol{z}) \boldsymbol{x}_v^{\mathrm{T}} \mathcal{L}\left\{ \tilde{\boldsymbol{X}} \boldsymbol{G}_{ii}(\boldsymbol{x}, \boldsymbol{\zeta}, \hat{\boldsymbol{x}}) \right\} \boldsymbol{x}_v \\ &\quad + \sum_{i=1}^{r}\sum_{i<j} h_i(\boldsymbol{z})h_j(\boldsymbol{z}) \\ &\quad \times \boldsymbol{x}_v^{\mathrm{T}} \mathcal{L}\left\{ \tilde{\boldsymbol{X}} \left(\boldsymbol{G}_{ij}(\boldsymbol{x}, \boldsymbol{\zeta}, \hat{\boldsymbol{x}}) + \boldsymbol{G}_{ji}(\boldsymbol{x}, \boldsymbol{\zeta}, \hat{\boldsymbol{x}}) \right) \right\} \boldsymbol{x}_v. \end{aligned}$$

If the following conditions are satisfied, $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$:

$$\mathcal{L}\left\{\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})\right\}<0 \tag{50}$$

$$\mathcal{L}\left\{\tilde{\boldsymbol{X}}\left(\boldsymbol{G}_{ij}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})+\boldsymbol{G}_{ji}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})\right)\right\}\leq 0, \qquad i<j\leq r. \tag{51}$$

As well as in Theorem 1, (50) can be separately rewritten as

$$\mathcal{L}\left\{\boldsymbol{X}_1^{-1}\left(\boldsymbol{A}_i(\hat{\boldsymbol{x}})-\boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{F}_i(\hat{\boldsymbol{x}})\right)\right\}<0 \tag{52}$$

$$\mathcal{L}\left\{\boldsymbol{X}_2\left(\bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}})-\boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_i\right)\right\}<0. \tag{53}$$

Multiplying both sides of (52) by $\boldsymbol{X}_1$ and defining a new variable $\boldsymbol{M}_i(\hat{\boldsymbol{x}}) = \boldsymbol{F}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1$, we obtain the following conditions:

$$\mathcal{L}\left\{\boldsymbol{A}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1-\boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_i(\hat{\boldsymbol{x}})\right\}<0. \tag{54}$$

Defining another new variable $\boldsymbol{N}_i(\hat{\boldsymbol{x}}) = \boldsymbol{X}_2\boldsymbol{L}_i(\hat{\boldsymbol{x}})$, the inequality (53) can be described as

$$\mathcal{L}\left\{\boldsymbol{X}_2\bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}})-\boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_i\right\}<0. \tag{55}$$

In the same way as before, (51) can be also represented as

$$\begin{aligned}&\mathcal{L}\left\{\boldsymbol{A}_i(\hat{\boldsymbol{x}})\boldsymbol{X}_1-\boldsymbol{B}_i(\boldsymbol{\zeta})\boldsymbol{M}_j(\hat{\boldsymbol{x}})\right\}\\&\quad+\mathcal{L}\left\{\boldsymbol{A}_j(\hat{\boldsymbol{x}})\boldsymbol{X}_1-\boldsymbol{B}_j(\boldsymbol{\zeta})\boldsymbol{M}_i(\hat{\boldsymbol{x}})\right\}\leq 0\end{aligned} \tag{56}$$

$$\begin{aligned}&\mathcal{L}\left\{\boldsymbol{X}_2\bar{\boldsymbol{A}}_i(\boldsymbol{x},\hat{\boldsymbol{x}})-\boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j\right\}\\&\quad+\mathcal{L}\left\{\boldsymbol{X}_2\bar{\boldsymbol{A}}_j(\boldsymbol{x},\hat{\boldsymbol{x}})-\boldsymbol{N}_j(\hat{\boldsymbol{x}})\boldsymbol{C}_i\right\}\leq 0\end{aligned} \tag{57}$$

for $i < j \leq r$. It is clear from the inequality conditions (54)–(57) that $\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq \boldsymbol{0}$ if the SOS conditions (41)–(46) hold. ■

Remark 7: As we can see, Theorems 1 and 2 show that the so-called separation principle is realized, i.e., the fuzzy polynomial controller and observer can be separately designed without lack of guaranteeing the stability of the overall control system in addition to converging state-estimation error (via the observer) to zero. This is a very important point in our fuzzy polynomial controller and observer design. In particular, in Theorem 2, a key feature of realizing the separation design is that, by introducing the transformation $\bar{\boldsymbol{A}}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{e} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}$, the (2, 1) element in $\boldsymbol{G}_{ij}(\boldsymbol{x},\boldsymbol{\zeta},\hat{\boldsymbol{x}})$ becomes zero element (matrix). This transformation idea leads to the successful separation design.

A. Design Example II

Consider the following nonlinear system, where x_1 is measurable and $y = x_1$:

$$\begin{cases}\dot{x}_1 = \sin x_1 - 0.3x_2 + \left(x_1^2+1\right)u\\ \dot{x}_2 = -1.5x_1 - 2x_2 - x_2^3.\end{cases} \tag{58}$$

This system has polynomial terms $(x_1^2 + 1)u$ and x_2^3. To obtain a T–S fuzzy model, we need to assume the ranges of x_1 and x_2. Thus, as well as in Example I, the T–S fuzzy model is a local model. This means that the T–S fuzzy model stabilization and state-estimation convergence are not guaranteed for x_1 and x_2 outside the ranges. The polynomial fuzzy model constructed in this example can exactly and globally represent the dynamics of the original system. Even if a local or semiglobal T–S fuzzy model is permitted to be used in practical sense, the premise variable vector $\boldsymbol{z}$ contains x_2 to be estimated. Hence, the previous LMI conditions mentioned in Section III-A1 cannot be applied to the nonlinear system. On the other hand, the premise variable vector $\boldsymbol{z}$ in the polynomial fuzzy model does not contain x_2, and x_2 appears in polynomial system matrices $\boldsymbol{A}_i$ in consequent parts of the polynomial fuzzy models. Since the Class II design permits one to have unmeasurable states in $\boldsymbol{A}_i$ matrices, it is possible to design a polynomial fuzzy observer in this example.

The dynamics of the nonlinear system can be exactly represented as the polynomial fuzzy system (38), where $r = 2$, $\boldsymbol{z} = \boldsymbol{\zeta} = y$

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix}1 & -0.3x_2\\ -1.5 & -2-x_2^2\end{bmatrix}$$

$$\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix}-0.2172 & -0.3x_2\\ -1.5 & -2-x_2^2\end{bmatrix}$$

$$\boldsymbol{B}_1(\boldsymbol{\zeta}) = \boldsymbol{B}_2(\boldsymbol{\zeta}) = \begin{bmatrix}y^2+1\\ 0\end{bmatrix} \qquad \boldsymbol{C}_1 = \boldsymbol{C}_2 = [1 \quad 0]$$

$$h_1(\boldsymbol{z}) = \frac{\sin y + 0.2172y}{1.2172y} \qquad h_2(\boldsymbol{z}) = \frac{y - \sin y}{1.2172y}.$$

In this example, note that

$$\begin{aligned}\bar{\boldsymbol{A}}_1(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{e} &= \boldsymbol{A}_1(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_1(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}\\ &= \begin{bmatrix}1 & -0.3(x_2+\hat{x}_2)\\ -1.5 & -2-x_2^2-x_2\hat{x}_2-\hat{x}_2^2\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}\end{aligned} \tag{59}$$

$$\begin{aligned}\bar{\boldsymbol{A}}_2(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{e} &= \boldsymbol{A}_2(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_2(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}\\ &= \begin{bmatrix}-0.2172 & -0.3(x_2+\hat{x}_2)\\ -1.5 & -2-x_2^2-x_2\hat{x}_2-\hat{x}_2^2\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}.\end{aligned} \tag{60}$$

By solving the SOS conditions in Theorem 2, we obtain the polynomial feedback and observer gains (shown at the bottom of the page), where the orders of $\boldsymbol{M}_i(\hat{\boldsymbol{x}})$ and $\boldsymbol{N}_i(\hat{\boldsymbol{x}})$ are two. Fig. 4 shows the control and estimation results by the designed polynomial fuzzy controller and observer, where the initial states are $\boldsymbol{x}(0) = [1 \ 1]$ and $\hat{\boldsymbol{x}}(0) = [0 \ 0]$. It can be seen that the designed controller stabilizes the nonlinear system, and the estimation error via the polynomial fuzzy observer tends to zero.

$$\boldsymbol{F}_1(\hat{\boldsymbol{x}}) = \left[\,2.17028 + 0.31476e^{-17}\hat{x}_2^2 \quad 0.35016e^{-5} - 0.37934e^{-11}\hat{x}_2^2\,\right]$$

$$\boldsymbol{F}_2(\hat{\boldsymbol{x}}) = \left[\,1.38495 + 0.31482e^{-17}\hat{x}_2^2 \quad 0.34413e^{-5} - 0.37942e^{-11}\hat{x}_2^2\,\right]$$

$$\boldsymbol{L}_1(\boldsymbol{y},\hat{\boldsymbol{x}}) = \begin{bmatrix}1.75626 + 0.650097e^{-11}\hat{x}_2^2\\ -1.46221 - 0.52724e^{-5}\hat{x}_2^2\end{bmatrix}$$

$$\boldsymbol{L}_2(\boldsymbol{y},\hat{\boldsymbol{x}}) = \begin{bmatrix}0.64328 + 0.65012e^{-11}\hat{x}_2^2\\ -1.41280 - 0.52725e^{-5}\hat{x}_2^2\end{bmatrix}$$

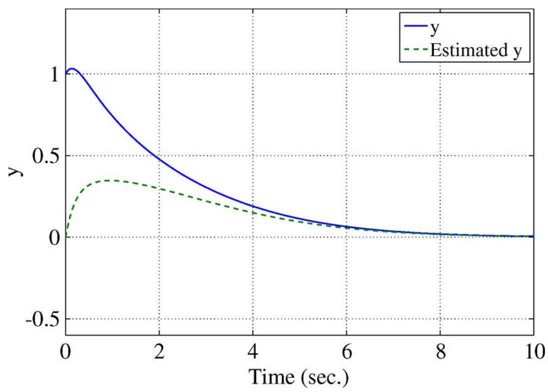

Fig. 4. Control and estimation results.

Remark 8: Since $\boldsymbol{A}_1(\boldsymbol{x})$ and $\boldsymbol{A}_2(\boldsymbol{x})$ have unmeasurable x_2 in this design example, the Class I SOS-based observer design (Theorem 1) cannot be applied to this design example. The previous LMI conditions mentioned in Section III-A1 cannot be also applied to the nonlinear system. On the other hand, since the Class II design (Theorem 2) permits one to have unmeasurable states in $\boldsymbol{A}_i$ matrices, it is possible to design a polynomial fuzzy observer in this example.

V. Polynomial Controller and Observer Design (Class III)

In this section, we consider a more complicated class, i.e., $\boldsymbol{A}_i(\boldsymbol{x})$ and $\boldsymbol{B}_i(\boldsymbol{x})$ case. Class III design deals with the polynomial fuzzy system (7) and (61)

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \left\{ \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u} \right\}. \tag{61}$$

For the system (7) and (61), we design the following polynomial fuzzy observer:

$$\dot{\hat{\boldsymbol{x}}} = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \{\boldsymbol{A}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\hat{\boldsymbol{x}})\boldsymbol{u} + \boldsymbol{L}_i(\hat{\boldsymbol{x}})(\boldsymbol{y} - \hat{\boldsymbol{y}}) \tag{62}$$

$$\hat{\boldsymbol{y}} = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \boldsymbol{C}_i \hat{\boldsymbol{x}} \tag{63}$$

where $\boldsymbol{L}_i(\hat{\boldsymbol{x}})$ for all i represents the polynomial observer gain matrices in $\hat{\boldsymbol{x}}$.

It is known that it is extremely difficult to separately design polynomial fuzzy controller and observer in Class III. In fact, to the best of our knowledge, there exist no literatures on achieving the separation design in this class of polynomial fuzzy systems. To overcome the difficulty, we propose a practical algorithm to design polynomial fuzzy controller and observer satisfying the stability of the overall augmented system in addition to converging state-estimation error (via the observer) to zero.

The algorithm mainly consists of three steps.

Step 1) By assuming that all the states are measurable, we design the following controller:

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z}) \boldsymbol{F}_i(\boldsymbol{x})\boldsymbol{x}. \tag{64}$$

The SOS conditions (see Theorem 3) derived in [7] and [9] are applied to determine the polynomial feedback gains $\boldsymbol{F}_i(\boldsymbol{x})$.

Step 2) We replace the controller designed in Step 1) with

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z}) \boldsymbol{F}_i(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} \tag{65}$$

where $\boldsymbol{x}$ is replaced with $\hat{\boldsymbol{x}}$.

Step 3) Note that the $\boldsymbol{F}_i(\hat{\boldsymbol{x}})$ and $\boldsymbol{X}_1$ (see Theorem 3) obtained in Step 2) are known polynomial matrices in $\hat{\boldsymbol{x}}$ and a positive definite matrix, respectively. We determine the polynomial observer gains $\boldsymbol{L}_i(\hat{\boldsymbol{x}})$ by solving new SOS design conditions (see Theorem 4).

We present the previous SOS conditions [7], [9] (Theorem 3) to determine the polynomial feedback gains $\boldsymbol{F}_i(\boldsymbol{x})$ and new SOS design conditions (Theorem 4) to determine the polynomial observer gains that are newly derived in this paper.

Theorem 3 [7], [9]: The system (7) and (61) can be stabilized by the controller (64) if there exist a positive definite matrix $\boldsymbol{X}_1 \in \mathbb{R}^{n\times n}$ and polynomial matrices $\boldsymbol{M}_i(\boldsymbol{x}) \in \mathbb{R}^{p\times n}$ satisfying the following SOS conditions:

$$\boldsymbol{v}_1^{\mathrm{T}} \left(\boldsymbol{X}_1 - \boldsymbol{E}_1^{\mathrm{reg}}\right) \boldsymbol{v}_1 \text{ is SOS} \tag{66}$$

$$-\boldsymbol{v}_2^{\mathrm{T}} \left(\mathcal{L}\left\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{M}_i(\boldsymbol{x})\right\} + \boldsymbol{E}_{2i}^{\mathrm{reg}}(\boldsymbol{x})\right) \boldsymbol{v}_2 \text{ is SOS} \tag{67}$$

$$-\boldsymbol{v}_3^{\mathrm{T}} \left(\mathcal{L}\left\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{X}_1 - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{M}_j(\boldsymbol{x})\right\} + \mathcal{L}\left\{\boldsymbol{A}_j(\boldsymbol{x})\boldsymbol{X}_1 - \boldsymbol{B}_j(\boldsymbol{x})\boldsymbol{M}_i(\boldsymbol{x})\right\}\right) \boldsymbol{v}_3 \text{ is SOS}, \quad i < j \le r \tag{68}$$

where $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3 \in \mathbb{R}^n$ denote the vectors that are independent of $\boldsymbol{x}$. From the solutions $\boldsymbol{X}_1$ and $\boldsymbol{M}_i(\boldsymbol{x})$, the feedback gain can be obtained as $\boldsymbol{F}_i(\boldsymbol{x}) = \boldsymbol{M}_i(\boldsymbol{x})\boldsymbol{X}_1^{-1}$.

Theorem 4: The system (7) and (61) can be stabilized by the polynomial fuzzy controller (65) and the estimation error via the polynomial fuzzy observer (62) and (63) tends to zero if there exist a positive definite matrix $\boldsymbol{X}_2 \in \mathbb{R}^{n\times n}$ and polynomial matrices $\boldsymbol{N}_i(\hat{\boldsymbol{x}}) \in \mathbb{R}^{n\times q}$ satisfying the following SOS conditions, where $\boldsymbol{X}_1$ and $\boldsymbol{F}_j(\hat{\boldsymbol{x}})$ are solutions satisfying the SOS conditions in Theorem 3 and are given (known) matrices in Theorem 4:

$$\boldsymbol{x}_v^{\mathrm{T}} \left(\begin{bmatrix} \boldsymbol{X}_1^{-1}\boldsymbol{X}_2 & 0 \\ 0 & \boldsymbol{X}_2 \end{bmatrix} - \boldsymbol{E}_1^{\mathrm{obs}} \right) \boldsymbol{x}_v \text{ is SOS} \tag{69}$$

$$-\boldsymbol{x}_v^{\mathrm{T}} \left(\boldsymbol{\Omega}_{ii}(\boldsymbol{x},\hat{\boldsymbol{x}}) + \boldsymbol{E}_{2i}^{\mathrm{obs}}(\boldsymbol{x},\hat{\boldsymbol{x}})\right) \boldsymbol{x}_v \text{ is SOS} \tag{70}$$

$$-\boldsymbol{x}_v^{\mathrm{T}} \left(\boldsymbol{\Omega}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) + \boldsymbol{\Omega}_{ji}(\boldsymbol{x},\hat{\boldsymbol{x}})\right) \boldsymbol{x}_v \text{ is SOS}, \quad i < j \le r \tag{71}$$

where

$$\boldsymbol{\Omega}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \begin{bmatrix} \boldsymbol{\Omega}_{ij}^{11}(\hat{\boldsymbol{x}}) & \boldsymbol{\Omega}_{ij}^{12}(\hat{\boldsymbol{x}}) \\ \boldsymbol{\Omega}_{ij}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) & \boldsymbol{\Omega}_{ij}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}}) \end{bmatrix}$$

$$\boldsymbol{\Omega}_{ij}^{11}(\hat{\boldsymbol{x}}) = \boldsymbol{X}_1^{-1}\boldsymbol{X}_2 \left(\boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}})\boldsymbol{F}_j(\hat{\boldsymbol{x}})\right)$$

$$\boldsymbol{\Omega}_{ij}^{12}(\hat{\boldsymbol{x}}) = \boldsymbol{X}_1^{-1}\boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j$$

$$\boldsymbol{\Omega}_{ij}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \boldsymbol{X}_2 \left(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \left(\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}})\right)\boldsymbol{F}_j(\hat{\boldsymbol{x}})\right)$$

$$\boldsymbol{\Omega}_{ij}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}}) = \boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j$$

$\boldsymbol{x}_v = [\hat{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{e}^{\mathrm{T}}]^{\mathrm{T}}$, and $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$. From the solutions $\boldsymbol{X}_2$ and $\boldsymbol{N}_i(\hat{\boldsymbol{x}})$, we can obtain observer gain matrices as $\boldsymbol{L}_i(\hat{\boldsymbol{x}}) = \boldsymbol{X}_2^{-1}\boldsymbol{N}_i(\hat{\boldsymbol{x}})$.

Proof: Define the estimation error via the observer as $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$. Then, the error dynamics are represented as

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z}) \times \{(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}}))\boldsymbol{F}_j(\hat{\boldsymbol{x}}))\hat{\boldsymbol{x}} + (\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j)\boldsymbol{e}\}.$$

We obtain the following augmented system:

$$\dot{\boldsymbol{x}}_v = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v$$

where

$$\begin{aligned}
\boldsymbol{x}_v &= [\,\hat{\boldsymbol{x}}^{\mathrm{T}} \quad \boldsymbol{e}^{\mathrm{T}}\,]^{\mathrm{T}}\\
\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) &= \begin{bmatrix} \boldsymbol{G}_{ij}^{11}(\hat{\boldsymbol{x}}) & \boldsymbol{G}_{ij}^{12}(\hat{\boldsymbol{x}}) \\ \boldsymbol{G}_{ij}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) & \boldsymbol{G}_{ij}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}}) \end{bmatrix}\\
\boldsymbol{G}_{ij}^{11}(\hat{\boldsymbol{x}}) &= \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}})\boldsymbol{F}_j(\hat{\boldsymbol{x}})\\
\boldsymbol{G}_{ij}^{12}(\hat{\boldsymbol{x}}) &= \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j\\
\boldsymbol{G}_{ij}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) &= \boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}}))\boldsymbol{F}_j(\hat{\boldsymbol{x}})\\
\boldsymbol{G}_{ij}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}}) &= \boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{L}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j.
\end{aligned}$$

Now, consider the following candidate of Lyapunov functions:

$$V(\boldsymbol{x}_v) = \boldsymbol{x}_v^{\mathrm{T}}\tilde{\boldsymbol{X}}\boldsymbol{x}_v \tag{72}$$

where

$$\tilde{\boldsymbol{X}} = \begin{bmatrix} \boldsymbol{X}_1^{-1}\boldsymbol{X}_2 & 0 \\ 0 & \boldsymbol{X}_2 \end{bmatrix} > 0. \tag{73}$$

The time derivative of $V(\boldsymbol{x}_v)$ along the system trajectories is

$$\dot{V}(\boldsymbol{x}_v) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{x}_v^{\mathrm{T}}\left(\boldsymbol{G}_{ij}^{\mathrm{T}}(\boldsymbol{x},\hat{\boldsymbol{x}})\tilde{\boldsymbol{X}} + \tilde{\boldsymbol{X}}\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}})\right)\boldsymbol{x}_v.$$

Since $\boldsymbol{x}_v^{\mathrm{T}}\boldsymbol{H}\boldsymbol{x}_v = \boldsymbol{x}_v^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{x}_v$ for any square matrix $\boldsymbol{H}$, we have

$$\begin{aligned}
\dot{V}(\boldsymbol{x}_v) &= 2\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{x}_v^{\mathrm{T}}\tilde{\boldsymbol{X}}\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v\\
&= 2\sum_{i=1}^{r} h_i^2(\boldsymbol{z})\boldsymbol{x}_v^{\mathrm{T}}\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v\\
&\quad + 2\sum_{i=1}^{r}\sum_{i<j}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\\
&\quad \times \boldsymbol{x}_v^{\mathrm{T}}\tilde{\boldsymbol{X}}\left(\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) + \boldsymbol{G}_{ji}(\boldsymbol{x},\hat{\boldsymbol{x}})\right)\boldsymbol{x}_v.
\end{aligned} \tag{74}$$

$\dot{V}(\boldsymbol{x}_v) < 0$ at $\boldsymbol{x}_v \neq 0$ if the following conditions hold:

$$-\boldsymbol{x}_v^{\mathrm{T}}\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v > 0 \tag{75}$$

$$-\boldsymbol{x}_v^{\mathrm{T}}\tilde{\boldsymbol{X}}\left(\boldsymbol{G}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) + \boldsymbol{G}_{ji}(\boldsymbol{x},\hat{\boldsymbol{x}})\right)\boldsymbol{x}_v \geq 0, \qquad i < j \leq r. \tag{76}$$

By defining as $\boldsymbol{N}_i(\hat{\boldsymbol{x}}) = \boldsymbol{X}_2\boldsymbol{L}_i(\hat{\boldsymbol{x}})$, (75) can be rewritten as

$$\begin{aligned}
-\boldsymbol{x}_v^{\mathrm{T}}\tilde{\boldsymbol{X}}\boldsymbol{G}_{ii}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v &= -\boldsymbol{x}_v^{\mathrm{T}}\begin{bmatrix} \boldsymbol{\Omega}_{ii}^{11}(\hat{\boldsymbol{x}}) & \boldsymbol{\Omega}_{ii}^{12}(\hat{\boldsymbol{x}}) \\ \boldsymbol{\Omega}_{ii}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) & \boldsymbol{\Omega}_{ii}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}}) \end{bmatrix}\boldsymbol{x}_v\\
&= -\boldsymbol{x}_v^{\mathrm{T}}\boldsymbol{\Omega}_{ii}(\boldsymbol{x},\hat{\boldsymbol{x}})\boldsymbol{x}_v > 0
\end{aligned} \tag{77}$$

where

$$\begin{aligned}
\boldsymbol{\Omega}_{ii}^{11}(\hat{\boldsymbol{x}}) &= \boldsymbol{X}_1^{-1}\boldsymbol{X}_2\left(\boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}})\boldsymbol{F}_i(\hat{\boldsymbol{x}})\right)\\
\boldsymbol{\Omega}_{ii}^{12}(\hat{\boldsymbol{x}}) &= \boldsymbol{X}_1^{-1}\boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_i\\
\boldsymbol{\Omega}_{ii}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) &= \boldsymbol{X}_2\left(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}}))\boldsymbol{F}_i(\hat{\boldsymbol{x}})\right)\\
\boldsymbol{\Omega}_{ii}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}}) &= \boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_i.
\end{aligned}$$

Also, (76) can be rewritten as

$$-\boldsymbol{x}_v^{\mathrm{T}}\left(\boldsymbol{\Omega}_{ij}(\boldsymbol{x},\hat{\boldsymbol{x}}) + \boldsymbol{\Omega}_{ji}(\boldsymbol{x},\hat{\boldsymbol{x}})\right)\boldsymbol{x}_v \geq 0, \qquad i < j \leq r \tag{78}$$

where

$$\begin{aligned}
\boldsymbol{\Omega}_{ij}^{11}(\hat{\boldsymbol{x}}) &= \boldsymbol{X}_1^{-1}\boldsymbol{X}_2\left(\boldsymbol{A}_i(\hat{\boldsymbol{x}}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}})\boldsymbol{F}_j(\hat{\boldsymbol{x}})\right)\\
\boldsymbol{\Omega}_{ij}^{12}(\hat{\boldsymbol{x}}) &= \boldsymbol{X}_1^{-1}\boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j\\
\boldsymbol{\Omega}_{ij}^{21}(\boldsymbol{x},\hat{\boldsymbol{x}}) &= \boldsymbol{X}_2\left(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_i(\hat{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\hat{\boldsymbol{x}}))\boldsymbol{F}_j(\hat{\boldsymbol{x}})\right)\\
\boldsymbol{\Omega}_{ij}^{22}(\boldsymbol{x},\hat{\boldsymbol{x}}) &= \boldsymbol{X}_2\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{N}_i(\hat{\boldsymbol{x}})\boldsymbol{C}_j.
\end{aligned}$$

Now, we arrive at the SOSPs (69)–(71). ■

Clearly, the overall control system consisting of (7), (61), (62), and (63), (65) is asymptotically and globally stable, and the estimation error tends to zero.

Remark 9: Note that (73) is different from (19) and (49). Equation (73) is needed to have SOS conditions with respect to variables $\boldsymbol{X}_2$ and $\boldsymbol{N}_i(\hat{\boldsymbol{x}})$. If we use (19) or (49) instead of (73), the derived conditions have $\boldsymbol{X}_2$, $\boldsymbol{N}_i(\hat{\boldsymbol{x}})$, and $\boldsymbol{L}_i(\hat{\boldsymbol{x}})$. In this case, due to the constraint $\boldsymbol{N}_i(\hat{\boldsymbol{x}}) = \boldsymbol{X}_2\boldsymbol{L}_i(\hat{\boldsymbol{x}})$, they cannot be generally solved by SOSTOOLS and SeDuMi.

A. *Design Example III*

Consider the following nonlinear system:

$$\begin{cases} \dot{x}_1 = \sin x_1 - 5x_2 + \left(x_2^2 + 5\right)u \\ \dot{x}_2 = -x_1 - x_2^3. \end{cases} \tag{79}$$

This system has polynomial terms $(x_2^2+5)u$ and x_2^3. As well as in Examples I and II, the polynomial fuzzy model constructed in this example can exactly and globally represent

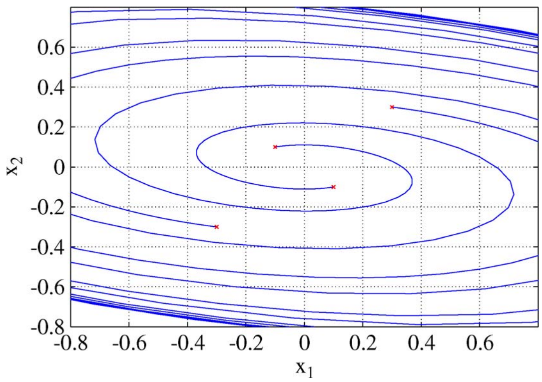

Fig. 5. System behavior without input.

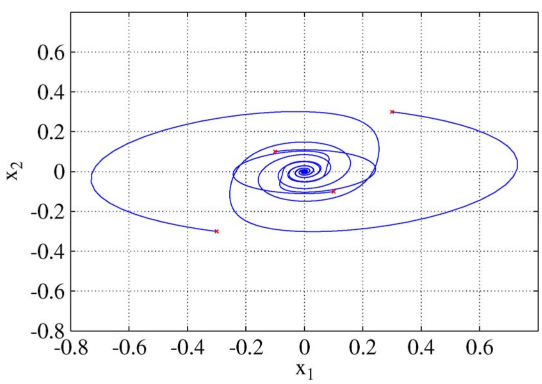

Fig. 6. Control trajectories for the same initial states as in Fig. 5.

the dynamics of the original system, although the T–S fuzzy model for (79) is a local model. In addition, the previous LMI conditions in Section III-A1 cannot be applied to the nonlinear system. Conversely, the Class III design can be applied to designing a polynomial fuzzy observer in this example.

Assume that x_1 is measurable and $y = x_1$. Fig. 5 shows the behavior of the nonlinear system without input for several initial states. It is found from the figure that this system is unstable.

The system (79) can be exactly converted into the polynomial fuzzy system (7) and (61) using the sector nonlinearity [2], where $r = 2$, $\boldsymbol{z} = y$

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} 1 & 5 \\ -1 & -x_2^2 \end{bmatrix} \qquad \boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} -0.2172 & 5 \\ -1 & -x_2^2 \end{bmatrix}$$

$$\boldsymbol{B}_1(\boldsymbol{x}) = \begin{bmatrix} x_2^2 + 5 \\ 0 \end{bmatrix} \qquad \boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} x_2^2 + 5 \\ 0 \end{bmatrix}$$

$$\boldsymbol{C}_1 = \boldsymbol{C}_2 = [1 \quad 0]$$

$$h_1(\boldsymbol{z}) = \frac{\sin y + 0.2172y}{1.2172y} \qquad h_2(\boldsymbol{z}) = \frac{y - \sin y}{1.2172y}.$$

Fig. 6 shows control result (for the same initial states as Fig. 5) by the polynomial fuzzy controller and observer designed using Theorem 3 and Theorem 4, where the orders of $\boldsymbol{M}_i(\hat{\boldsymbol{x}})$ and $\boldsymbol{N}_i(\hat{\boldsymbol{x}})$ are two. Fig. 7 shows the control and

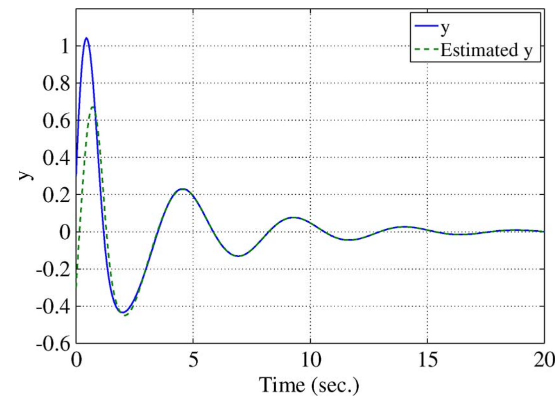

Fig. 7. Control and estimation results.

estimation results starting from one of the initial states, where $\boldsymbol{x}(0) = [0.3 \ 0.3]$ and $\hat{\boldsymbol{x}}(0) = [-0.3 \ -0.3]$. The polynomial feedback and observer gains are obtained as follows:

$$\boldsymbol{F}_1(\hat{\boldsymbol{x}}) = \left[0.29008 + 0.20778\hat{x}_2^2 \ \ 0.63772 - 0.22047e^{-1}\hat{x}_2^2\right]$$

$$\boldsymbol{F}_2(\hat{\boldsymbol{x}}) = \left[0.46829e^{-1} + 0.22751\hat{x}_2^2 \ \ 0.64532 - 0.24141e^{-1}\hat{x}_2^2\right]$$

$$\boldsymbol{L}_1(\hat{\boldsymbol{x}}) = \begin{bmatrix} 2.65691 + 17.71908\hat{x}_2^2 \\ 1.08259 + 1.76675\hat{x}_2^2 \end{bmatrix}$$

$$\boldsymbol{L}_2(\hat{\boldsymbol{x}}) = \begin{bmatrix} 3.68595 + 18.01543\hat{x}_2^2 \\ 1.52432 + 1.70592\hat{x}_2^2 \end{bmatrix}.$$

It can be found from the control results that the designed polynomial fuzzy controller stabilizes the system and the estimation error via the polynomial fuzzy observer tends to zero.

Remark 10: Since $\boldsymbol{A}_1(\boldsymbol{x})$, $\boldsymbol{A}_2(\boldsymbol{x})$, $\boldsymbol{B}_1(\boldsymbol{x})$, and $\boldsymbol{B}_2(\boldsymbol{x})$ have unmeasurable x_2 in this design example, the previous SOS-based observer designs (Classes I and II) cannot be applied to this design example. Even if the sector nonlinearity concept is applied to construct a T–S fuzzy model for the nonlinear system, the premise variables z contain x_2. Hence, the previous LMI conditions mentioned in Section III-A1 cannot be applied to the nonlinear system. On the other hand, since the Class III design permits one to have unmeasurable states in both of $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ matrices, it is possible to design a polynomial fuzzy observer in this example.

VI. Conclusion

This paper has presented an SOS approach for three classes of polynomial fuzzy controllers and observers. To illustrate the validity and applicability of the proposed approach, three design examples have been provided. The examples have demonstrated the advantages of the SOS-based approaches for the existing LMI approaches to T–S fuzzy observer designs.

Our next subjects are to derive SOS observer design conditions to realize the separation design even for Class III and to apply our observer designs to helicopter control [11].

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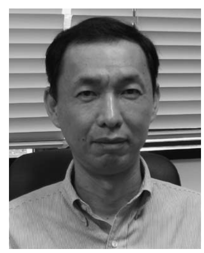

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