Exercise Sheet 2

due 20.11.25

Exercise 1. (5 pt) Consider $\mathbb{R}^2$ as input space and let $P, N \subset \mathbb{R}^2$ be the following sets: $P = \{\mathbf{p}_1, \mathbf{p}_2\}$ and $N = \{\mathbf{n}_1\}$, with $\mathbf{p}_1 = [1, 2]$, $\mathbf{p}_2 = [-1, 1]$, $\mathbf{n}_1 = [-2, 0]$. Let $\mathbf{w}_0 = [1, 1, -1]$ be the (extended) weight vector given as input.

a. Are P and N absolutely linearly separable? If yes give an example of extended weight vector $\mathbf{w}_1^* \in \mathbb{R}^3$ absolutely linearly separating P and N, as well as the equation of the straight line associated to $\mathbf{w}_1^*$ in the input space, and the perceptron defined by $\mathbf{w}_1^*$ correctly classifying P and N. If not, explain why.

b. Are P and N linearly separable? If yes give an example of extended weight vector $\mathbf{w}_2^* \in \mathbb{R}^3$ linearly separating P and N, but not absolutely linearly separating them. If not, explain why.

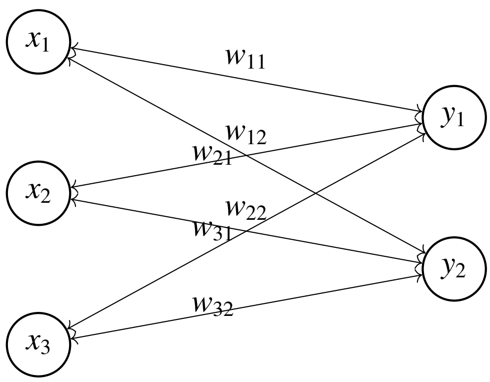
c. Write the coordinates of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{n}_1$ in the extended input space.

d. What is the equation of the straight line contained in the input space associated to $\mathbf{w}_0$?

e. What is the equation of the plane contained in the extended input space associated to $\mathbf{w}_0$?

f. Does the perceptron parametrized by $\mathbf{w}_0$ correctly classify all the elements in $P \cup N$? If yes write the perceptron. If not update $\mathbf{w}_0$ using the perceptron learning algorithm, until a solution $\mathbf{w}^* = \mathbf{w}_t$ is found that classifies them correctly. After how many iterations t do you find $\mathbf{w}^*$? Write the perceptron associated to $\mathbf{w}^*$.

Exercise 2. (5 pt) Consider the following BAM network:



a. Let the activation function be sgn and the transition matrix $\mathbf{W}$ be the following:

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}, \qquad \text{with } w_{i1} = 1,\ w_{i2} = -1,\ \text{for all } i = 1, 2, 3.$$

Let $\mathbf{x}_0 = [1, -1, 1]$ be the key. Write the vector pairs $(\mathbf{x}_i, \mathbf{y}_i)$, for $i = 0, 1, 2$ obtained through iterations (using synchronous updates for each layer) of the BAM. Does the network converge to a stable state $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$? If yes, specify the stable state and after how many forward and feedback steps the BAM converges to it.

b. Calculate the energy function E at each state $(\mathbf{x}_i, \mathbf{y}_i)$, $i = 0, 1, 2$.

c. Consider now a BAM with the same weight matrix $\mathbf{W}$ as in point (a.). Let the thresholds vectors be $\boldsymbol{\theta}$, for the left layer, and $\boldsymbol{\varphi}$, for the right layer, with coordinates:

$$\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3] = [1, 2, -1], \qquad \boldsymbol{\varphi} = [\varphi_1, \varphi_2] = [1, 3]$$

This means that θ_i is the threshold at the node x_i, $i = 1, 2, 3$ and φ_j is the threshold at the node y_j, $j = 1, 2$. Let the activation function at each node be f_{θ_i} or f_{φ_i}, defined as

$$f_{\theta_i}(x) = \begin{cases} 1 & \text{if } x \geq \theta_i \\ -1 & \text{if } x < \theta_i \end{cases} ,$$

Let $\mathbf{x}_0 = [1, -1, 1]$ be the key (the first given vector), find $\mathbf{y}_0$. Hint: since each node has a different threshold the coordinates of $\mathbf{y}_0$ cannot be calculated simultaneously.

d. Consider now the extended BAM of the BAM in point (c.). Let us recall that the extended BAM is obtained by adding one extra node in the left layer and one extra node in the right one, removing all thresholds and transforming them into weights. What is the weight matrix $\mathbf{W}'$ of this BAM? Let us call $\mathbf{X}_0$, respectively $\mathbf{Y}_0$, the extended vector associated to $\mathbf{x}_0$, respectively $\mathbf{y}_0$. Find the value of the energy function of the extended BAM at the state $(\mathbf{X}_0, \mathbf{Y}_0)$.