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The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer

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Abstract

In this Letter, homotopy perturbation method (HPM), which does not need small parameters in the equations, is compared with the perturbation and numerical methods in the heat transfer field. The perturbation method depends on small parameter assumption, and the obtained results, in most cases, end up with a non-physical result, the numerical method leads to inaccurate results when the equation is intensively dependent on time, while He's homotopy perturbation method (HPM) overcomes completely the above shortcomings, revealing that the HPM is very convenient and effective. Comparing different methods shows that, when the effect of the nonlinear term is negligible, homotopy perturbation method and the common perturbation method have got nearly the same answers but when the nonlinear term in the heat equation is more effective, there will be a considerable difference between the results. As the homotopy perturbation method does not need a small parameter, the answer will be nearer to the exact solution and also to the numerical one.

Keywords: Fin radiation; Variable specific heat; Perturbation method; Homotopy perturbation method

1. Introduction

Most of engineering problems, especially some heat transfer equations are nonlinear, therefore some of them are solved using numerical solution and some are solved using the analytic perturbation method.

In the numerical method, stability and convergence should be considered, so as to avoid divergent or inappropriate results. In the analytical perturbation method, we should exert the small parameter in the equation. Finding the small parameter and exerting it into the equation are therefore the problems with this method.

Perturbation method is one of the well-known methods to solve the nonlinear equations which was studied by a large number of researchers such as Bellman [1], Cole [2], and O'Malley [24]. Actually, these scientists had paid more attention to the mathematical aspects of the subject which included a loss of physical verification. This loss in the physical verification of the subject was recovered by Nayfeh [23] and Van Dyke [25].

Since there are some limitations with the common perturbation method, and also because the basis of the common perturbation method was upon the existence of a small parameter, developing the method for different usage is very difficult. Therefore, many different new methods have recently introduced some ways to eliminate the small parameter such as artificial parameter method introduced by Liu [22], the variational iteration method by He [6–8,11,13,20], and others. One of the semi-exact methods is the homotopy perturbation method [9,10,12,14–17].

In this Letter, the basic idea of the HPM is introduced and then its application in some heat transfer equations is studied. The nonlinear equation of conduction heat transfer with the variable physical properties are solved through the two methods: homotopy perturbation method and the common perturbation method, and compare with the exact solution is also made.

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In addition, the heat radiation and conduction equations of a fin in the steady state and in the free space are solved using the two mentioned methods and compared with each other and also with the numerical solution.

2. Basic idea of homotopy perturbation method

In this Letter, we apply the homotopy perturbation method [9,10,12,14–17] to the discussed problem. To illustrate the basic ideas of the new method, we consider the following nonlinear differential equation,

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω.

The operator A can be generally divided into two parts L and N, where L is linear, whereas N is nonlinear. Therefore, Eq. (1) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

In case the nonlinear equation (1) has no "small parameter", we can construct the following homotopy,

$$\mathcal{H}(v, p) = L(v) - L(u_0) + pL(u_0) + p\big[N(v) - f(r)\big] = 0, \tag{4}$$

where p is called homotopy parameter.

According to the homotopy perturbation method, the approximation solution of Eq. (4) can be expressed as a series of the power of p, i.e.,

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots, \tag{5}$$

when, Eq. (4) correspond to Eq. (1), and Eq. (5) becomes the approximate solution of Eq. (1). Some interesting results have been attained using this method [3–5,18,19,21].

3. Applications

3.1. *Cooling of a lumped system with variable specific heat (the first case)*

Consider the cooling of a lumped system [26]. Let the system have volume V, surface area A, density ρ, specific heat c and initial temperature T_i. At time $t = 0$, the system is exposed to a convective environment at temperature T_a with convective heat transfer coefficient h. Assume that the specific heat c is a linear function temperature of the form:

$$c = c_a\big[1 + \beta(T - T_a)\big], \tag{6}$$

where c_a is the specific heat, at temperature T_a and β is a constant. The cooling equation and the initial condition are:

$$\rho V c \frac{dT}{dt} + hA(T - T_a) = 0, \quad T(0) = T_i, \tag{7}$$

introducing Eq. (6) and using the dimensionless parameters:

$$\theta = \frac{T - T_a}{T_i - T_a}, \qquad \tau = \frac{t}{\rho V c_a/(hA)}, \qquad \varepsilon = \beta(T - T_a), \tag{8}$$

transforms Eq. (7) to:

$$(1 + \varepsilon\theta)\frac{d\theta}{d\tau} + \theta = 0, \quad \theta(0) = 1. \tag{9}$$

3.1.1. *Homotopy method*

We can construct the homotopy $v(r, p): \Omega \times [0, 1] \to \mathfrak{R}$ which satisfies

$$\left(\frac{d\theta(\tau)}{d\tau} + \theta(\tau)\right) - \left(\frac{dy_0(\tau)}{d\tau} + y_0(\tau)\right) + p\left(\frac{dy_0(\tau)}{d\tau} + y_0(\tau)\right) + p\left[\varepsilon\theta(\tau)\frac{d\theta(\tau)}{d\tau}\right] = 0. \tag{10}$$

With initial approximation $y_0 = e^{-\tau}$, suppose the solution of Eq. (9) has the form:

$$\theta = Y_0 + pY_1 + p^2 Y_2 + \cdots. \tag{11}$$

Then, substituting Eq. (11) into Eq. (10), and equating the terms with identical powers of p,

$$p^0: \quad \frac{dY_0(\tau)}{d\tau} - \frac{dy_0(\tau)}{d\tau} - y_0(\tau) + Y_0(\tau) = 0, \tag{12}$$

$$p^1: \quad Y_1(\tau) + y_0(\tau) + \varepsilon Y_0(\tau)\frac{dY_0(\tau)}{d\tau} + \frac{dY_1(\tau)}{d\tau} + \frac{dy_0(\tau)}{d\tau} = 0, \tag{13}$$

$$p^2: \quad \frac{dY_2(\tau)}{d\tau} + \varepsilon Y_0(\tau)\frac{dY_1(\tau)}{d\tau} + Y_2(\tau) + \varepsilon Y_1(\tau)\frac{dY_0(\tau)}{d\tau} = 0, \tag{14}$$

with solving Eqs. (12)–(14),

$$Y_0 = e^{-\tau}, \quad Y_0(0) = 1, \tag{15}$$

$$Y_1 = \varepsilon\big(e^{-\tau} - e^{-2\tau}\big), \quad Y_1(0) = 0, \tag{16}$$

$$Y_2 = \varepsilon^2\left(0.5e^{-\tau} - 2e^{-2\tau} + \frac{3}{2}e^{-3\tau}\right), \quad Y_2(0) = 0, \tag{17}$$

according to Eq. (11) and the assumption $p = 1$, we get:

$$\theta = e^{-\tau} + \varepsilon\big(e^{-\tau} - e^{-2\tau}\big) + \varepsilon^2\left(\frac{1}{2}e^{-\tau} - 2e^{-2\tau} + \frac{3}{2}e^{-3\tau}\right). \tag{18}$$

3.1.2. *Perturbation method*

For very small ε, let us assume a regular perturbation expansion and calculate the first three terms [23], thus we assume:

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \cdots. \tag{19}$$

Substituting Eq. (19) into Eq. (9) and collecting terms with powers of ε as $0, 1, 2, \ldots$ gives:

$$\frac{d\theta_0}{d\tau} + \theta_0 + \varepsilon\left(\frac{d\theta_1}{d\tau} + \theta_1 + \theta_0\frac{d\theta_0}{d\tau}\right) + \varepsilon^2\left(\frac{d\theta_2}{d\tau} + \theta_2 + \theta_1\frac{d\theta_0}{d\tau} + \theta_0\frac{d\theta_1}{d\tau}\right) = 0. \tag{20}$$

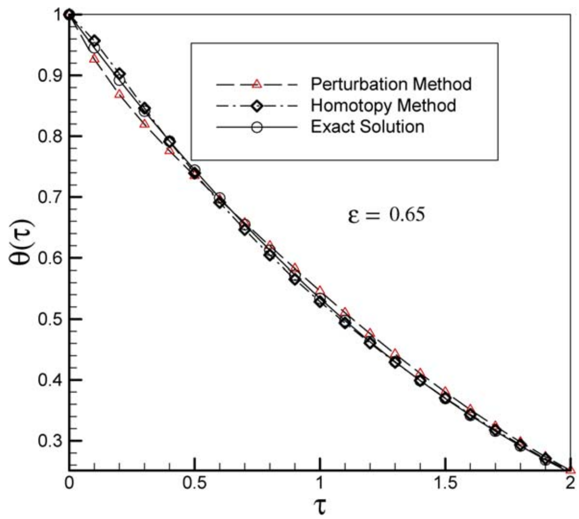

Fig. 1. Comparison of the three different methods for the first case in $\varepsilon = 0.65$.

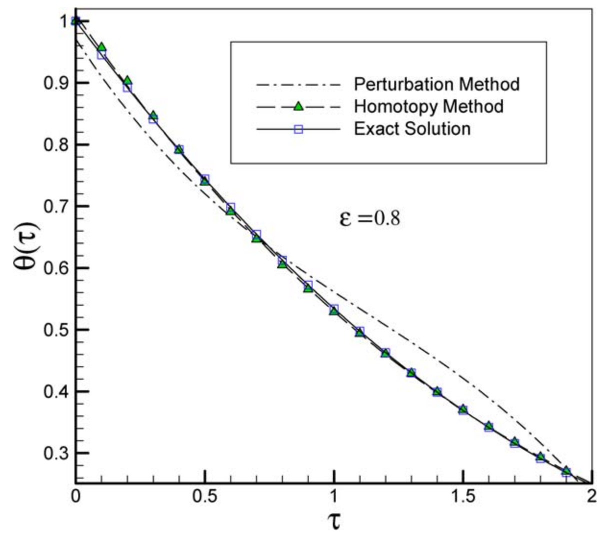

Fig. 2. Comparison of the three different methods for the first case in $\varepsilon = 0.8$.

Equating coefficients of each power of ε on both sides of Eq. (20) gives:

$$\varepsilon^0: \quad \frac{d\theta_0}{d\tau} + \theta_0 = 0, \quad \theta_0(0) = 1, \tag{21}$$

$$\varepsilon^1: \quad \frac{d\theta_1}{d\tau} + \theta_1 + \theta_0\frac{d\theta_0}{d\tau} = 0, \quad \theta_1(0) = 0, \tag{22}$$

$$\varepsilon^2: \quad \frac{d\theta_2}{d\tau} + \theta_2 + \theta_1\frac{d\theta_0}{d\tau} + \theta_0\frac{d\theta_1}{d\tau} = 0, \quad \theta_2(\tau = 0) = 0. \tag{23}$$

The solutions of Eqs. (21), (22) and (23) are:

$$\theta_0 = e^{-\tau}, \qquad \theta_1 = e^{-\tau} - e^{-2\tau},$$
$$\theta_2 = e^{-\tau} - 2e^{-2\tau} + \frac{3}{2}e^{-3\tau}. \tag{24}$$

The three-term expansion in Eq. (19) now becomes:

$$\theta = e^{-\tau} + \varepsilon\left(e^{-\tau} - e^{-2\tau}\right) + \varepsilon^2\left(e^{-\tau} - 2e^{-2\tau} + \frac{3}{2}e^{-3\tau}\right). \tag{25}$$

3.1.3. Exact solution

By separating the variables in Eq. (9) and carrying out the integration, the exact solution can be obtained as

$$\frac{d\theta}{\theta\,d\tau} + \varepsilon\frac{\theta\,d\theta}{\theta\,d\tau} + \frac{\theta}{\theta} = 0$$
$$\Rightarrow \int\frac{d\theta}{\theta} + \int\varepsilon\,d\theta + \int d\tau = \int 0\,d\tau$$
$$\Rightarrow \ln\theta + \varepsilon\theta + \tau = c \overset{\theta(0)=1}{\rightarrow} c = \varepsilon$$
$$\therefore\ \ln\theta + \varepsilon(\theta - 1) + \tau = 0. \tag{26}$$

As you can see in $\varepsilon = 0$, this two diagrams are identical and in $\varepsilon = 0.03$ and also $\varepsilon = 0.09$ the homotopy has a high accuracy.

The results of the three methods are illustrated in Figs. 1, 2, 3 and 4 for the first case.

3.2. The temperature distribution equation in a thick rectangular fin radiation to free space (the second case) [26]

Now we will consider a nonlinear equation, the temperature distribution equation in a uniformly thick rectangular fin radia-

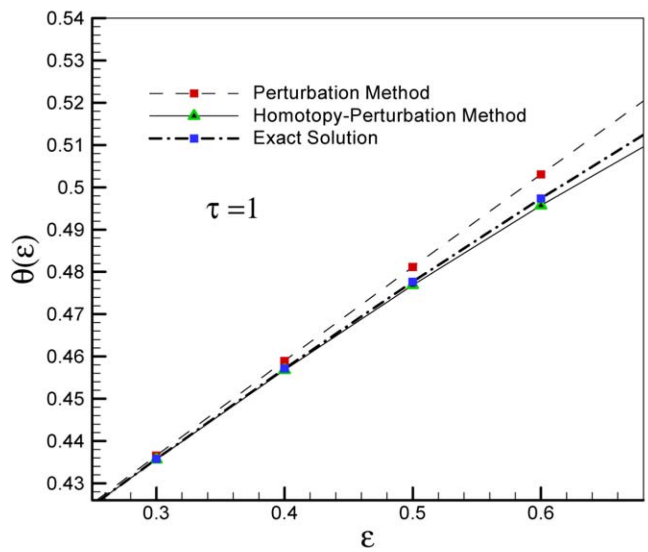

Fig. 3. Variation of $\theta(\varepsilon)$ with the ε for the first case.

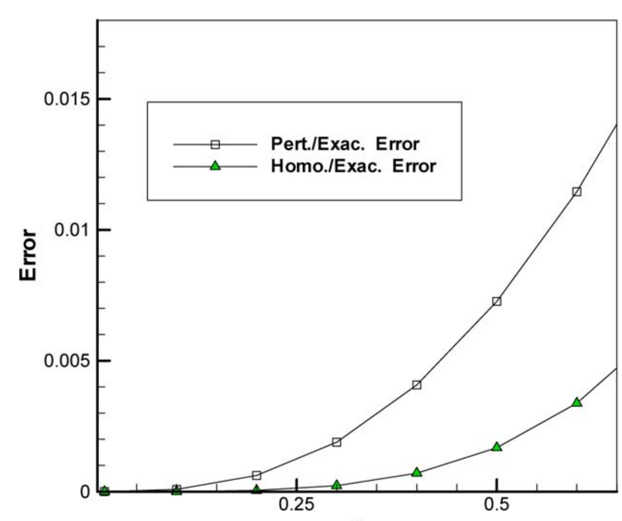

Fig. 4. Comparison of the error in answer resulted by three different methods for the first case.

tion to free space with nonlinearity of high order [12,14]:

$$\frac{d^2\theta}{dx^2} - \varepsilon\theta^4(x) = 0, \quad \theta(1) = 1, \quad \frac{d\theta}{dx}(0) = 0. \tag{27}$$

3.2.1. Homotopy method

We can create the following homotopy $v(r, p): \Omega \times [0, 1] \to \Re$ which satisfies:

$$\left(\frac{d^2}{dx^2}\theta(x)\right) - \left(\frac{d^2}{dx^2}y_0(x)\right) + p\left(\frac{d^2}{dx^2}y_0(x)\right) - p\varepsilon\theta^4(x) = 0. \tag{28}$$

With initial approximation $y_0 = 1$, suppose the solution of Eq. (27) has the form:

$$\theta = Y_0 + pY_1 + p^2Y_2 + \cdots. \tag{29}$$

Substituting Eq. (29) into Eq. (28), and equating the terms with identical powers of p,

$$p^0: \quad -\left(\frac{d^2}{dx^2}y_0(x)\right) + \left(\frac{d^2}{dx^2}Y_0(x)\right) = 0,$$
$$Y_0(1) = 1, \quad \frac{dY_0}{dx}(0) = 0, \tag{30}$$
$$p^1: \quad \left(\frac{d^2}{dx^2}Y_1(x)\right) + \left(\frac{d^2}{dx^2}y_0(x)\right) - \varepsilon Y_0^4(x) = 0,$$
$$Y_1(1) = 0, \quad \frac{dY_1}{dx}(0) = 0, \tag{31}$$
$$p^2: \quad \left(\frac{d^2}{dx^2}Y_2(x)\right) - 4\varepsilon\big(Y_1(x)Y_0^3(x)\big) = 0,$$
$$Y_2(1) = 0, \quad \frac{dY_2}{dx}(0) = 0, \tag{32}$$

and solving differential equations (30), (31) and (32), we have:

$$Y_0 = 1, \tag{33}$$
$$Y_1 = \frac{1}{2}\varepsilon\big(x^2 - 1\big), \tag{34}$$
$$Y_2 = \frac{1}{6}\varepsilon^2\big(x^4 - 1\big). \tag{35}$$

According to Eq. (29) and the assumption $p = 1$,

$$\theta = 1 + \frac{1}{2}\varepsilon\big(x^2 - 1\big) + \frac{1}{6}\varepsilon^2\big(x^4 - 1\big). \tag{36}$$

3.2.2. Perturbation method

For very small ε, let us assume a regular perturbation expansion and calculate the first three terms, thus we assume:

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \cdots. \tag{37}$$

Substituting Eq. (37) into Eq. (27) and separating terms with powers of ε,

$$\varepsilon^0: \quad \frac{d^2\theta_0}{dx^2} = 0, \quad \theta_0(x = 1) = 1, \quad \frac{d\theta_0}{dx}(x = 0) = 0, \tag{38}$$
$$\varepsilon^1: \quad \frac{d^2\theta_1}{dx^2} - \theta_0^4 = 0, \quad \theta_1(x = 1) = 1,$$
$$\frac{d\theta_1}{dx}(x = 0) = 0, \tag{39}$$
$$\varepsilon^2: \quad \frac{d^2\theta_2}{dx^2} - 4\theta_1\theta_0^3 = 0, \quad \theta_2(x = 1) = 1,$$
$$\frac{d\theta_2}{dx}(x = 0) = 0, \tag{40}$$

the solutions of Eqs. (38)–(40) are:

$$\theta_0 = 1, \qquad \theta_1 = \frac{x^2}{2} - \frac{1}{2}, \qquad \theta_2 = \frac{1}{6}\big(x^4 - 6x^2 + 5\big). \tag{41}$$

The three-term expansion in Eq. (37) now becomes:

$$\theta = 1 + \varepsilon\left(\frac{x^2}{2} - \frac{1}{2}\right) + \varepsilon^2\left(\frac{1}{6}\big(x^4 - 6x^2 + 5\big)\right). \tag{42}$$

3.2.3. Numerical method

Since, Eq. (27) cannot be easily solved by the analytical method; Eq. (27) is, therefore, solved by the numerical method using the software MAPLE whose results are given in Table 1, and also the consequent results of the three different methods of perturbation, homotopy and numerical are compared in Fig. 5. As you can see in $\varepsilon = 0.7$, the HPM has a high accuracy.

Considering Fig. 5, it can be specially achieved that the perturbation method is valid only for small parameter ε, but it should be pointed out that the HPM is valid for all the nonlinear equations with different parameters.

Table 1
The results of the numerical solution for the second case in $\varepsilon = 0.09$

x	$\theta(x)$	$\frac{d\theta(x)}{dx}$
0	0.9606243	0
0.1	0.9610076	0.0076680
0.2	0.9621587	0.0153607
0.3	0.9640814	0.0231026
0.4	0.9667818	0.0309192
0.5	0.9702686	0.0388363
0.6	0.9745533	0.0468813

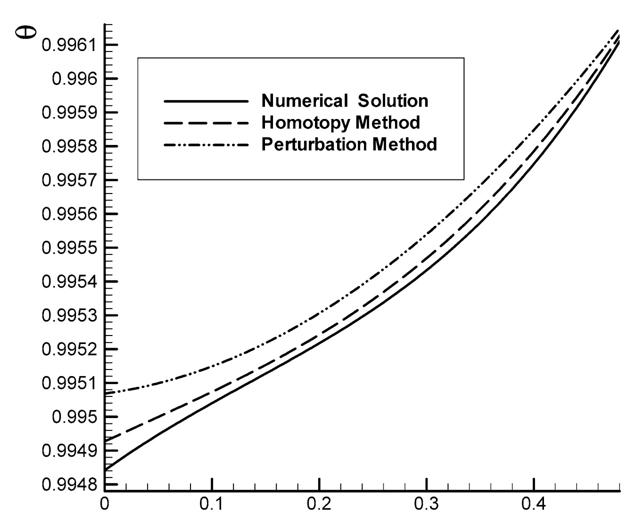

Fig. 5. Comparison of the three different methods for the second case in $\varepsilon = 0.7$.

4. Conclusion

Considering the results, it can be said that, as the small parameter of ε increases perturbation method is much different with the exact solution. This means, in high ε there will be more malfunctionality whereas homotopy method has got a much higher accuracy. In this research, it has been shown that the perturbation method is valid only for small parameter ε, but it should be pointed out that the HPM is valid for all the nonlinear equations with high order of nonlinearity containing different parameters.

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