Alexandria Engineering Journal (2020) 59, 3643–3652

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Alexandria Engineering Journal

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A new hybrid orthonormal Bernstein and improved block-pulse functions method for solving mathematical physics and engineering problems

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Received 30 October 2019; revised 5 June 2020; accepted 7 June 2020
Available online 27 June 2020

KEYWORDS

Fredholm integral equation;
Bernstein polynomial;
Improved Block-Pulse function;
Absolute error;
Accuracy

Abstract In this paper, numerical solution of the linear second kind Fredholm integral equations is studied. These integral equations are widely used for solving many problems in mathematical physics and engineering. A new hybrid Bernstein and Improved Block-Pulse Functions (*HBIBP*) method is introduced and utilized to convert linear (nonlinear) second kind Fredholm integral equations into an algebraic equation. The new methodology is a combination of Bernstein polynomials (BPs) and improved block-pulse functions (IBPFs) on the interval [0, 1). Convergence analysis and numerical examples that illustrate the pertinent features of the method are presented.

1. Introduction

Integral Equation is an equation in which the unknown function appears under one or more than integral sign. In recent years, many different basic functions have been used to estimate the solution of integral equations, such as Block-Pulse functions [1-6], Haar functions [7], Triangular functions [8–10], Hybrid Chebyshev and Block-Pulse functions [11,12], Hybrid Legendre and Block-Pulse functions [13–15], Hybrid Taylor, Block-Pulse functions [16], Hybrid Fourier and Block-Pulse functions [17]. Hybrid Bernstein polynomials and Block-Pulse functions [18-20]. In the first time, Block-Pulse functions were introduced to electrical engineering by Harmuth and several researchers discussed the Block-Pulse [21–24]. Bernstein polynomials play a prominent role in various areas of mathematics. These polynomials have been frequently used in the solution of integral equations, differentials and approximation theory [25–29].

The improved block pulse function is introduced by Farshid Mirzaee [30]. Modified block pulse functions for numerical solution of stochastic Volterra integral equations is introduced [31]. Authors of [34-38] introduced a numerical method for solving nonlinear Fredholm integral equations of the second kind using different methods.

In this paper, HBIBP functions are introduced to solve linear and nonlinear Fredholm integral equation (1.1), (1.2) . We use the definition of improved block-Pulse function with a slight change in the definition of function from [30] to match with new hybrid function.

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Peer review under responsibility of Faculty of Engineering, Alexandria University.

https://doi.org/10.1016/j.aej.2020.06.014
1110-0168

$$u(x) = f(x) + \int_{t=0}^{1} k(x,t)u(t)dt, \tag{1.1}$$

$$u(x) = f(x) + \int_{t=0}^{1} k(x,t)[u(t)]^m dt, \qquad \text{m} > 1 \tag{1.2}$$

This paper is organized as follows. In Section 2, review of improved block-pulse function and Bernstein polynomials and its properties is briefly presented. In Section 3, we introduce a new combination of Bernstein and Improved Block-Pulse functions. In Section 4, the proposed technique for numerical approximation of linear and nonlinear Fredholm integral equations by using new basis is introduced. In Section 5, the error estimate and convergence analysis is presented for the proposed method. Section 6 includes some numerical examples to illustrate the reliability and accuracy of our proposed scheme to handle linear and nonlinear Fredholm integral equations of the second kind. Finally, we give our concluding remarks.

2. Preliminaries

2.1. Improved block-pulse function

Definition:. An $(n+1)$-set of IBPFs consists of $(n+1)$ functions which are defined over district Das follows [30] with a slight change in the definition of function

$$\varphi_1(x) = \begin{cases} 1, & x \in \left[0, \frac{h}{2}\right), \\ 0, & otherwise, \end{cases}$$

$$\varphi_i(x) = \begin{cases} 1, & x \in \left[(i-2)h + \frac{h}{2}, (i-1)h + \frac{h}{2}\right), \\ 0, & otherwise, \end{cases} i = 2, 3, \cdots, n$$

$$\varphi_{n+1}(x) = \begin{cases} 1, & x \in \left[1 - \frac{h}{2}, 1\right), \\ 0, & otherwise, \end{cases}$$

where n is an arbitrary positive integer and $h = \frac{1}{n}$,

There are some properties for these functions; the most important properties are disjointness, orthogonality, and completeness.

Lemma. Let a set of improved block-pulse functions (IBPFs) $\varphi_i(x)$, $i = 1, 2, \cdots, N+1$ be defined on the interval [0, 1) such that:

$$\varphi_1(x) = \begin{cases} 1, x \in \left[0, \frac{h}{2}\right), \\ 0, otherwise, \end{cases}$$

$$\varphi_i(x) = \begin{cases} 1, x \in \left[(i-2)h + \frac{h}{2}, (i-1)h + \frac{h}{2}\right), \\ \quad 0, otherwise, \end{cases} i = 2, 3, \cdots, n$$

$$\varphi_{n+1}(x) = \begin{cases} 1, x \in \left[1 - \frac{h}{2}, 1\right), \\ \quad 0, otherwise. \end{cases}$$

Then, the following properties for these functions satisfy the following:

(i) disjointness,
(ii) orthogonality,
(iii) completeness.

Proof:. The disjointness property can be clearly obtained from the definition of improved block-pulse functions as follows:

$$\varphi_i(x)\varphi_j(x) = \begin{cases} \varphi_i(x), & i = j \\ 0, & otherwise. \end{cases}$$

The other property of IBPFs is orthogonal to each other where $x \in D$

$$\int_0^1 \varphi_i(x)\varphi_j(x)dx = \begin{cases} \frac{h}{2}, & i = j \in \{1, n+1\}, \\ h, & i = j \in \{2, 3, \cdots, n\}, \\ 0, & otherwise, \end{cases}$$

The third property is completeness. For every $f \in L^2$ [0,1] when n approaches to infinity, Parsevals identity hold:

$$\int_0^1 f^2(x)dx = \sum_{i=0}^{\infty} f_i^2 \|\varphi_i(x)\|^2,$$

where $f_i = \frac{1}{h}\int_0^1 \varphi_i(x)dx$.

2.1.1. Vector forms of IBPFS

Consider the first $(n+1)$ terms of IBPFs and write them concisely *as* $(n+1)$-vector

$$\Phi_n(x) = \left[\varphi_1(x), \varphi_2(x), \cdots, \varphi_{n+1}(x)\right]^T, x \in D \tag{2.1}$$

from disjointness

$$\Phi_n(x)\Phi_n^T(x) = \begin{pmatrix} \varphi_1(x) & 0 & \cdots & 0 \\ 0 & \varphi_2(x) & \cdots & 0 \\ \vdots \ \vdots & & \ddots & \vdots \\ 0 \ 0 & & \cdots & \varphi_{n+1}(x) \end{pmatrix} = diag(\Phi_n(x)),$$

2.1.2. IBPFs expansions

A continues function $f(x) \in L^2(D)$ may be expand by the IBPFs as

$$f(x) \simeq f_n(x) = \sum_{i=1}^{n+1} f_i\varphi_i(x) = F_n^T\Phi_n(x) = \Phi_n^T(x)F_n, \tag{2.2}$$

where F_n is a $(n+1) \times 1$ vector given by

$$F_n = \left[f_1, f_2, \cdots, f_{n+1}\right]^T,$$

and $\Phi_n(x)$ is defined in (1) and f_i is obtained as

$$f_i = \begin{cases} 2n\int_0^{\frac{h}{2}} f(x)dx, i = 1 \\ n\int_{(i-2)h+\frac{h}{2}}^{(i-1)h+\frac{h}{2}} f(x)dx, i = 2, \cdots, n \\ 2n\int_{1-\frac{h}{2}}^{1} f(x)dx, i = n+1 \end{cases} \tag{2.3}$$

Similarly a function of two variables $K(x,y) \in L^2(D \times D)$ can be approximated by IBPFs as follows

$$K(x,y) \simeq \Phi_{n+1}^T(x)K_n\Phi_{n+1}(y)$$

where $\Phi_n(x)$ and $\Phi_n(y)$ are IBPFs vector of dimension *(n+1)* and $K_n = \left[k_{i,j}\right]$ is the $(n+1) \times (n+1)$ IBPFs coefficient matrix of$k(x, y)$.

2.2. Bernstein polynomials

Definition [29]: The Bernstein polynomials of the M-th degree are defined on the interval [0,1] as

$$B_{i,M}(x) = \binom{M}{i} x^i (1-x)^{M-i},\ i = 0, 1, \cdots, M,$$

where $\binom{M}{i} = \frac{M!}{i!(M-i)!}$.

By using binomial expansion of $(1-x)^{M-i}$, we have

$$\binom{M}{i} x^i (1-x)^{M-i} = \sum_{k=0}^{M-i} (-1)^k \binom{M}{i} \binom{M-i}{k} x^{i+k},$$

For mathematical convenience, we usually set $B_{i,M} = 0$ if $i < 0 \text{or} i > M$. A recursive definition can also be used to generate the Bernstein polynomials over [0,1] so that the i-th Bernstein polynomial of M-th degree can be written

$$B_{i,M}(x) = (1-x)B_{i,M-1}(x) + xB_{i-1,M-1}(x)$$

It can readily be shown that each of the Bernstein polynomials is positive and the sum of all the Bernstein polynomials is unity for all real$x \in [0,1]$, i.e.,$\sum_{i=0}^{M} B_{i,M}(x) = 1$ (unity partition property). It is easy to show that any given polynomial of M-th-degree can be expanded in terms of these basis functions.

3. New Hybrid Bernstein improved block-pulse functions (HBIBPFs) method

Definition: $HBIBP_{i,j}(x)$ is the combination of Bernstein polynomials and Improved Block-Pulse functions where both are complete and orthogonal, then the set is a complete orthogonal complete system. Hybrid Orthonormal Bernstein and Improved Block-Pulse functions where $j = 0, 1, \cdots, M, i = 1, 2, \cdots, N+1$, $HBIBP_{i,j}(x)$have two arguments i and j are the order of IBPFs and BPs, respectively. $HBIBP(x)$ defined on the interval [0, 1) as follows:

$$HBIBP_{i,j}(x) = \begin{cases} B_{j,M}\left(\frac{2x}{h}\right), x \in \left[0, \frac{h}{2}\right), \\ 0, \qquad otherwise, \end{cases} for\ i = 1,\ j = 0, 1, \cdots, M \tag{3.1}$$

$$HBIBP_{i,j}(x) = \begin{cases} B_{j,M}\left(\frac{x}{h} + \frac{3}{2} - i\right), x \in \left[(i-2)h + \frac{h}{2}, (i-1)h + \frac{h}{2}\right) \\ 0, \qquad otherwise, \end{cases} for\ i = 2, 3, \cdots, N,\ j = 0, 1, \cdots, M \tag{3.2}$$

$$HBIBP_{i,j}(x) = \begin{cases} B_{j,M}\left(\frac{2x}{h} - \frac{2}{h} + 1\right), x \in \left[1 - \frac{h}{2}, 1\right), \\ 0, \qquad otherwise, \end{cases} for\ i = N+1,\ j = 0, 1, \cdots, M \tag{3.3}$$

Thus, our new basis is$\{HBIBP_{1,0}, HBIBP_{1,1}, \cdots, HBIBP_{N+1,M}\}$and we can approximate function with the base. In the next section, we deal with the problem of approximation of these functions.

3.1. Approximation of Functions by Using HBIBPFs

A function $u(x)$ may be expressed in terms of the HBIBP basis as follow:

$$u(x) = \sum_{i=1}^{N+1} \sum_{j=0}^{M} c_{i,j}.HBIBP_{i,j}(x) = C^T HBIBP(x), \tag{3.4}$$

where

$$HBIBP(x) = \left[HBIBP_{1,0}, HBIBP_{1,1}, \cdots, HBIBP_{N+1,M}\right]^T, \tag{3.5}$$

and

$$C = [c_{1,0}, c_{1,1}, \cdots, c_{N+1,M}]^T, \tag{3.6}$$

we have

$$C^T < HBIBP(x), HBIBP(x) >=< u(x), HBIBP(x), \tag{3.7}$$

then

$$C = L^{-1} < u(x), HBIBP >, \tag{3.8}$$

where $\langle .,. \rangle$ is the standard inner product and L is an $((N+1)(M+1) \times (N+1)(M+1))$ matrix that is said the dual matrix that is

$$L =< HBIBP(x), HBIBP(x) >$$

$$= \int_0^1 HBIBP(x).HBIBP^T(x)dx$$

$$= \begin{pmatrix} L_1 & 0 & 0 & \cdots & 0 \\ 0 & L_2 & 0 & \cdots & 0 \\ 0 & 0 & L_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L_{n+1} \end{pmatrix}, \tag{3.9}$$

$L_i (i = 1, 2, \cdots, n+1)$ is defined as follows

$$(L_1)_{i+1,j+1} = \int_0^{\frac{h}{2}} B_{i,M}\left(\frac{2x}{h}\right) B_{j,M}\left(\frac{2x}{h}\right) dx = \frac{h}{2} \int_0^1 B_{i,M}(x) B_{j,M}(x) dx$$

$$= \frac{h\binom{M}{i}\binom{M}{j}}{2(2M+1)\binom{2M}{i+j}}, for\ i,j = 0, \cdots, M, \tag{3.10}$$

$$(L_r)_{i+1,j+1} = \int_{(i-2)h+\frac{h}{2}}^{(i-1)h+\frac{h}{2}} B_{i,M}\left(\frac{x}{h} + \frac{3}{2} - i\right) B_{j,M}\left(\frac{x}{h} + \frac{3}{2} - i\right) dx,$$

$$for\ r = 2, \cdots, n$$

$$= h \int_0^1 B_{i,M}(x) B_{j,M}(x) dx = \frac{h\binom{M}{i}\binom{M}{j}}{(2M+1)\binom{2M}{i+j}}, \tag{3.11}$$

$$for\ i,j = 0, \cdots, M,$$

$$(L_{n+1})_{i+1,j+1} = \int_{1-\frac{h}{2}}^{1} B_{i,M}\left(\frac{2x}{h} - \frac{2}{h} + 1\right) B_{j,M}\left(\frac{2x}{h} - \frac{2}{h} + 1\right) dx$$

$$= \frac{h}{2}\int_{0}^{1} B_{i,M}(x)B_{j,M}(x)dx = \frac{h\binom{M}{i}\binom{M}{j}}{2(2M+1)\binom{2M}{i+j}}, \quad for\ i,j = 0,\cdots,M, \tag{3.12}$$

We can also approximate the function $k(x,t) \in L^2([0,1]\times[0,1])$ as follow:

$$k(x,t) = HBIBP^T(x).K.HBIBP(t)$$

where K is an $(M+1)(N+1)$ matrix that we can obtain as follows:

$$K = L^{-1}\langle HBIBP(x), \langle k(x,t), HBIBP(t)\rangle\rangle L^{-1}$$

3.2. Operational matrix of product

Suppose that $C^T = [C_1^T, C_2^T, \cdots, C_{N+1}^T]$is an arbitrary $1\times(N+1)(M+1)$ matrix which C_i^T is $1\times(M+1)$ matrix for $i = 1,2,\cdots,N+1$, then $\widehat{C}$ is $(N+1)(M+1)\times(N+1)(M+1)$ operational matrix of product whenever

$$C^T HBIBP(x)HBIBP(x)^T \simeq HBIBP(x)^T\widehat{C} \tag{3.13}$$

We know

$$C^T B(x)B(x)^T \simeq B(x)^T\widehat{C}_i, i = 1,2,\cdots,N+1$$

Which $\widehat{C}_i$ is operational matrix of product of Bernstein polynomials presented in [33], then

For $m = 0,\cdots,M$,which

$$\widehat{C} = \begin{bmatrix} \widehat{C}_1 & \bar{0} & \cdots & \bar{0} \\ \bar{0} & \widehat{C}_2 & \cdots & \bar{0} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \cdots & \widehat{C}_{N+1} \end{bmatrix}$$

Which $\bar{0}$ is $(m+1)\times(m+1)$ matrix.

4. Outline of solution

This section presents the derivation of the method for solving linear and nonlinear Fredholm equations (1.1) and (1.2).

Step 1: the function $u(x)$ is approximated by

$$u(x) = U^T HBIBP(x), \tag{4.1}$$

where U is unknown $(N+1)(M+1)$-vector and $HBIBP$ is given by Eqs. (3.1), (3.2) ,(3.3) and (3.4) .

Step 2: the function $f(x)$ and $k(x,t)$ are also expanded into the $HBIBP$ function

$$k(x,t) = HBIBP^T(x).K.HBIBP(t), \tag{4.2}$$

$$f(x) = F^T HBIBP(x) \tag{4.3}$$

where$K$ is an known function $(N+1)(M+1)\times(N+1)(M+1)$-matrix and F is known $(N+1)(M+1)$-vector.

Step3: In this step, a general formula is presented for approximating$[u(t)]^m$. By using Eqs. (3.4) and (12), we obtain

$$[u(x)]^2 = [U^T HBIBP(x)]^2 = U^T HBIBP(x)HBIBP^T(x)U = HBIBP^T(x)\widehat{U}U$$

$$C^T HBIBP(x)HBIBP(x)^T =$$

$$C^T\begin{bmatrix} HBIBP_{1,m}(x)HBIBP_{1,m}(x)^T & \bar{0} & \cdots & \bar{0} \\ \bar{0} & HBIBP_{2,m}(x)HBIBP_{2,m}(x)^T & \cdots & \bar{0} \\ \vdots\ \ \vdots & & \ddots & \vdots \\ \bar{0}\ \ \bar{0} & & \cdots & HBIBP_{N+1,m}(x)HBIBP_{N+1,m}(x)^T \end{bmatrix}$$

$$= \begin{bmatrix} C_1 HBIBP_{1,m}(x)HBIBP_{1,m}(x)^T & \bar{0} & \cdots & \bar{0} \\ \bar{0} & C_2 HBIBP_{2,m}(x)HBIBP_{2,m}(x)^T & \cdots & \bar{0} \\ \vdots\ \ \vdots & & \ddots & \vdots \\ \bar{0}\ \ \bar{0} & & \cdots & C_{N+1} HBIBP_{N+1,m}(x)HBIBP_{N+1,m}(x)^T \end{bmatrix}$$

$$= \begin{bmatrix} HBIBP_{1,m}(x)^T\widehat{C}_1 & \bar{0} & \cdots & \bar{0} \\ \bar{0} & HBIBP_{2,m}(x)^T\widehat{C}_2 & \cdots & \bar{0} \\ \vdots\ \ \vdots & & \ddots & \vdots \\ \bar{0}\ \ \bar{0} & & \cdots & HBIBP_{N+1,m}(x)^T\widehat{C}_{N+1} \end{bmatrix} = HBIBP(x)^T\widehat{C}$$

$$[u(x)]^3 = U^T HBIBP(x)\left[U^T HBIBP(x)\right]^2 = U^T HBIBP(x) HBIBP^T(x)\widehat{\mathrm{U}}\,U$$

$$= HBIBP^T(x)\widehat{\mathrm{U}}\,\widehat{\mathrm{U}}\,U = HBIBP^T(x)\left[\widehat{\mathrm{U}}\right]^2 U$$

And so

$$[u(x)]^q = HBIBP^T(x)\left[\widehat{\mathrm{U}}\right]^{q-1} U \tag{4.4}$$

where $\widehat{\mathrm{U}}$ is defined in (3.13). Now, using Eqs. (4.1) , (4.4) to substitute into linear and nonlinear Fredholm integral equation.

4.1. *Solving linear second kind Fredholm integral equation using HBIBP method*

In this subsection we are dealing with the following linear Fredholm integral equation of the second kind:

$$u(x) = f(x) + \int_{t=0}^{1} k(x,t)u(t)dt,$$

Substituting Eq. (4.1)-(4.3) into Eq. (1.1) produces

$$HBIBP^T(x)U = HBIBP^T(x).F + \int_0^1 HBIBP^T(x)K.HBIBP(t).HBIBP^T(t)U dt. \tag{4.5}$$

Appling Eq. (3.9), Eq. (4.5) becomes

$$HBIBP^T(x)U = HBIBP^T(x)F + HBIBP^T(x)KLU. \tag{4.6}$$

Therefore

$$U = F + KLU, \tag{4.7}$$

where the dimensional subscripts have been dropped to simplify the notation. Rewriting Eq. (4.7), we have

$$U = (I - KL)^{-1}F, \tag{4.8}$$

where I is $(N+1)(M+1)$ identity matrix. The unknown vector U can be obtained by solving Eq. (4.8). Thus the solution $u(x)$ can be calculated in the HBIBP function expansion by using U and Eq. (4.1).

4.2. *Solving nonlinear second kind Fredholm integral equation using HBIBP method*

In this subsection we are dealing with the following nonlinear Fredholm integral equation of the second kind:

$$\boldsymbol{u(x) = f(x) + \int_{t=0}^{1} k(x,t)[u(t)]^m dt, m > 1}$$

Substituting Eq. (4.1)-(4.4) into Eq. (1.2) produces

$$HBIBP^T(x)U = HBIBP^T(x).F + \int_0^1 HBIBP^T(x)K.HBIBP(t).HBIBP^T(t)\left[\widehat{\mathrm{U}}\right]^{m-1} U dt \tag{4.9}$$

Appling Eq. (3.9), Eq. (4.9) becomes

$$HBIBP^T(x)U = HBIBP^T(x).F + HBIBP^T(x)KL\left[\widehat{\mathrm{U}}\right]^{m-1} U \tag{4.10}$$

Therefore

$$U = F + KL\left[\widehat{\mathrm{U}}\right]^{m-1} U \tag{4.11}$$

Rewriting Eq. (4.11), we have

$$\left(I - KL\left[\widehat{\mathrm{U}}\right]^{m-1}\right)U = F \tag{4.12}$$

where I is $(N+1)(M+1)$ identity matrix. The unknown vector U can be obtained by solving Eq. (4.12). Thus the solution $u(x)$ can be calculated in the HBIBP function expansion by using U and Eq. (4.1).

5. Convergence analysis of the proposed method

We assume the following conditions on $k(x,t)$ and $G(u(t)) = [U(t)]^m, m \geq 1$ where $k(x,t)$ is known function and $u(t)$ is an unknown function:

1. $S = \sup_{0 \leqslant x,t \leqslant 1} |k(x,t)| < \infty,$
2. $G(u(t))$ is continuous on [0,1] and satisfies Lipschitz condition

$$|G(u(t)) - G(z(t)) \leqslant| L\,|u(t) - z(t)|$$

We denote the error of term of the solution obtained by *HBIBP* method by $e_{HBIBP}(x) = \| u_{exact}(x) - \tilde{u}(x) \|$, where $u_{exact}(x)$ and $\tilde{u}(x)$are the exact and approximate solution of Eq. ((1.1) , (1.2)) , respectively. $\tilde{u}(x)$, given as

$$u(x) = \sum_{i=0}^{N+1}\sum_{j=0}^{M} c_{i,j} HBIBP_{i,j}(x)$$

Theorem.. The solution of Fredholm integral equation ((1.1), (1.2)) using *HBIBP* method converges if

$0 < SL < 1.$

Proof.. We have

$$e_{HBIBP}(x) = \| u_{exact}(x) - \tilde{u}(x) \| = \max_{x\in[0,1]} \left|u_{exact}(x) - \tilde{u}(x)\right| \leqslant \max_{x\in[0,1]} \int_0^1 |k(x,s)| \left|G(u_{exact}(s)) - G\left(\tilde{u}\right)\right| ds.$$

This implies that$(1 - SL)\, e_{HBIBP}(x) < 1$.
Hence, by choosing $0 < (1 - SL) < 1$ when $(N+1)M \to +\infty$, it implies that $e_{HBIBP} \to 0$.

6. Numerical examples

In this section, we solve some test problems to test the accuracy of the proposed method. All the computations have been performed using MATLAB software (R2018b).

Table 1 The numerical results for example 1 with M = 1,N = 2.

x	Exact solution	Presented Method with M = 1,N = 2	Absolute Error	OBH with $M=1,N=2$	Absolute Error
0	0.000000	-0.007002	7.002000e-003	0.028005	2.800467e-02
0.1	0.157983	0.160999	3.015807e-003	0.156826	1.157056e-03
0.2	0.329412	0.329000	4.117647e-004	0.341657	1.224518e-02
0.3	0.514286	0.501340	1.294571e-002	0.526488	1.220203e-02
0.4	0.712605	0.719720	7.114958e-003	0.644107	6.849769e-02
0.5	0.924370	0.938100	1.373025e-002	0.896149	2.822039e-02
0.6	1.149580	1.156480	6.900168e-003	1.148191	1.388467e-03
0.7	1.388235	1.374860	1.337529e-002	1.400233	1.199808e-02
0.8	1.640336	1.639600	7.361345e-004	1.652275	1.193924e-02
0.9	1.905882	1.908800	2.917647e-003	1.904317	1.565000e-03

Table 2 The numerical results for example 1 with M = 2, N = 3.

x	Exact solution	Presented Method M = 2,N = 3	Absolute Error
0	0.000000	0.000000	0.000000e + 00
0.1	0.157983	0.157983	2.305072e-13
0.2	0.329412	0.329412	2.279087e-13
0.3	0.514286	0.514286	1.351249e-13
0.4	0.712605	0.712605	5.823823e-14
0.5	0.924370	0.924370	4.585688e-13
0.6	1.149580	1.149580	1.601934e-13
0.7	1.388235	1.388235	2.567985e-14
0.8	1.640336	1.640336	9.905085e-14
0.9	1.905882	1.905882	6.419963e-13

Example 1. Consider linear Fredholm integral equation

$$u(x) = x + \int_{t=0}^{1} (tx^2 + xt^2)u(t)dt,$$

with exact solution $u(x) = \frac{180}{119}x + \frac{80}{119}x^2$.

Table 1 shows the comparison of the absolute error between the approximate solution for presented method, Hybrid orthonormal Bernstein and Block-Pulse functions (OBH) and the exact solution for$M = 1, N = 2$.

From the above numerical results in Table 2, we can see that the accuracy gets improved as M and N increase.

Example 2. Consider linear Fredholm integral equation

$$u(x) = \sin(x) - x + (x+1)\cos(1) - \sin(1) + \int_{t=0}^{1} (t+x)u(t)dt,$$

with exact solution$u(x)\sin(x)$.

Table 3 shows the comparison of the approximate solution between presented method and result of Hybrid Orthonormal Bernstein and Block-Pulse Functions method (OBH) in [18] with $M = 3, N = 2$ (the degree of Bernstein polynomials BPs and order of block pulse function BPFs, respectively).

Table 4 shows the comparison of the absolute error between exact solution and approximate solution of the block pulse function (BPFs), Bernstein polynomials (BPs), Hybrid Bernstein and Block-Pulse functions method (HBBPFM) [20], Hybrid Orthonormal Bernstein and Block-Pulse Functions (OBH) [18] and presented method. With Bernstein polynomials BPs degree is $M = 3$ and block pulse function BPFs order is $N = 4$. Note that the results of absolute error of the block pulse function (BPFs) and Bernstein polynomials (BPs) in the above table are taken from Table 1, page 3, and reference [20].

Example 3. Consider linear Fredholm integral equation

$$u(x) = -2x^3 + 3x^2 - x - \int_{t=0}^{1} (-x^2 + x + t^2 - t)u(t)dt,$$

with exact solution $u(x) = -2x^3 + 3x^2 - x$.

Table 3 The numerical results for example 2 with M = 3, N = 2.

x	Exact solution	Presented Method with M = 3,N = 2	Absolute Error	
			Presented method	OBH [18] with M = 3,N = 2
0	0.000000	0.000000	0.000000	——
0.1	0.099833	0.099833	5.216013×10^{-8}	4.4×10^{-6}
0.2	0.198669	0.198670	1.695559×10^{-7}	8.4×10^{-7}
0.3	0.295520	0.295524	3.770738×10^{-6}	2.1×10^{-6}
0.4	0.389418	0.389421	2.250715×10^{-6}	9.7×10^{-8}
0.5	0.479426	0.479419	6.679718×10^{-6}	8.8×10^{-6}
0.6	0.564642	0.564644	1.764123×10^{-6}	9.0×10^{-6}
0.7	0.644218	0.644222	4.504215×10^{-6}	7.1×10^{-6}
0.8	0.717356	0.717356	3.187132×10^{-7}	4.5×10^{-6}
0.9	0.783327	0.783326	6.799744×10^{-7}	9.1×10^{-6}

Table 4 The numerical results for example 2 when M = 3,N = 4.

X	Exact solution	Presented Method $M=3, N=4$	Absolute Error For $M=3, N=4$				
			BPFs $N=4$	BPs $M=3$	HBBPFM [20]	OBH Method [18]	Presented method
0	0.000000	0.000000	0.159448	0.000252739	2.57612×10^{-7}	————	0.000000e + 00
0.1	0.099833	0.099833	0.0596148	0.0000539886	5.73616×10^{-8}	4.7×10^{-6}	2.257289×10^{7}
0.2	0.198669	0.198670	0.0392211	0.000110834	1.23088×10^{-7}	5.1×10^{-7}	2.278492×10^{-7}
0.3	0.295520	0.295520	0.118936	0.0000398714	3.42659×10^{-7}	7.5×10^{-6}	1.773369×10^{-8}
0.4	0.389418	0.389418	0.0250381	0.0000566614	2.06685×10^{-7}	5.3×10^{-8}	1.049474×10^{-7}
0.5	0.479426	0.479425	0.167325	0.000106028	1.3331×10^{-6}	3.4×10^{-6}	4.957365×10^{-7}
0.6	0.564642	0.564643	0.0821085	0.0000743689	3.07359×10^{-7}	9.0×10^{-6}	3.848718×10^{-7}
0.7	0.644218	0.644218	0.00253325	0.0000243121	5.58694×10^{-7}	1.4×10^{-5}	8.459576×10^{-10}
0.8	0.717356	0.717356	0.125405	0.000119641	7.24512×10^{-7}	1.2×10^{-5}	1.460351×10^{-7}
0.9	0.783327	0.783325	0.0594347	0.000076931	4.20127×10^{-7}	9.8×10^{-7}	1.651769×10^{-6}

Table 5 The numerical results for example 3 with $M=3, N=2$.

x	Exact solution	Presented Method	Absolute Error	Approximate solution[19] with m = 32
0	0.000000	0.000000	0.000000e + 00	-0.014664
0.1	-0.072000	-0.072000	1.091394e-14	-0.075912
0.2	-0.096000	-0.096000	2.182787e-14	-0.095963
0.3	-0.084000	-0.084000	1.637090e-14	-0.084702
0.4	-0.048000	-0.048000	1.909939e-14	-0.052017
0.5	0.000000	0.000000	0.000000e + 00	0.007797
0.6	0.048000	0.048000	1.909939e-14	0.052017
0.7	0.084000	0.084000	1.637090e-14	0.084702
0.8	0.096000	0.096000	2.182787e-14	0.095963
0.9	0.072000	0.072000	1.091394e-14	0.075912

Table 6 The numerical results for example 4 when $M=3, N=4$.

x	Exact solution	Presented Method $M=3, N=4$	Absolute Error for $M=3, N=4$			
			BPFs $N=4$	BPs $M=3$	HBBPFM [20]	Presented method
0	1.000000	1.000000	0.134438	0.000939946	2.60043×10^{-6}	0.000000
0.1	1.105171	1.105139	0.0292675	0.000210236	6.00397×10^{-7}	1.169911×10^{-5}
0.2	1.221403	1.221215	0.0869644	0.000396173	1.08124×10^{-6}	2.411456×10^{-7}
0.3	1.349859	1.349856	0.103935	0.000126329	1.37399×10^{-6}	2.056056×10^{-7}
0.4	1.491825	1.491594	0.0380311	0.000213179	7.99831×10^{-7}	1.583865×10^{-6}
0.5	1.648721	1.648668	0.216077	0.00037144	4.28735×10^{-6}	1.528640×10^{-6}
0.6	1.822119	1.822330	0.0426798	0.000246979	9.89894×10^{-7}	1.507022×10^{-7}
0.7	2.013753	2.013844	0.148954	0.0000965353	1.78268×10^{-6}	1.842955×10^{-7}
0.8	2.225541	2.225408	0.167943	0.000412916	2.26538×10^{-6}	6.407956×10^{-7}
0.9	2.459603	2.459808	0.0661188	0.000254268	1.31873×10^{-6}	1.553279×10^{-5}

Table 5 shows the comparison of the absolute error between presented method and Numerical expansion-iterative block pulse method (with the number of iteration 32) [19] (see Tables 6 and 7).

Example 4:. Consider linear Fredholm integral equation

$$u(x) = e^x - x + \int_{t=0}^{1} (tx)u(t)dt,$$

with exact solution $u(x) = e^x$.

Example 5. Consider linear Fredholm integral equation

$$u(x) = x - \frac{2}{3} + \int_{t=0}^{1} (t+x)u(t)dt,$$

with exact solution $u(x) = 2x$.

Example 6. Consider linear Fredholm integral equation

$$u(x) = \sin(x) - x(\cos(1) + 2\sin(1) - 2) + \int_{t=0}^{1} (xt^2)u(t)dt,$$

Table 7 The numerical results for example 5 when $M = 3, N = 4$.

x	Exact solution	Presented Method	Absolute Error
0	0.000000	0.000000	0.000000e+00
0.1	0.200000	0.200000	6.548362e-14
0.2	0.400000	0.400000	2.482921e-13
0.3	0.600000	0.600000	3.246896e-13
0.4	0.800000	0.800000	1.800800e-13
0.5	1.000000	1.000000	6.821210e-13
0.6	1.200000	1.200000	3.110472e-13
0.7	1.400000	1.400000	3.055902e-13
0.8	1.600000	1.600000	3.055902e-13
0.9	1.800000	1.800000	3.492460e-13

with exact solution $u(x) = \sin(x)$.

Table 8 shows the comparison of the absolute error between exact solution and approximate solution for k = 2, m = 3, among Legendre wavelets (Leg for short), CAS wavelets, the second Chebyshev wavelets (Che for short) methods and presented method for M = 3, N = 4. Note that the results of absolute error of among Legendre wavelets (Leg for short), CAS wavelets, and the second Chebyshev wavelets (Che for short) methods in the above table are taken from Table 1, page 4, and reference [32].

Example 7. Consider nonlinear Fredholm integral equation

$$u(x) = \frac{-5}{12}x - \frac{4}{3} + \int_{t=0}^{1} (xt+1)(u(t))^2 dt,$$

with exact solution $u(x) = x + 1$.

Table 9 shows the comparison between the exact solution and the approximate solution for presented method

Table 9 The numerical results for example 7 when M = 3, N = 3.

x	Exact solution	Presented Method N = 3, M = 3	Absolute Error
0.0	1.00000	1.00157	1.57500e-03
0.1	1.10000	1.10111	1.11025e-03
0.2		1.20065	6.45771e-04
0.3	1.20000	1.30018	1.81784e-04
0.4	1.30000	1.39972	2.82527e-04
0.5	1.40000	1.49925	7.47000e-04
0.6	1.50000	1.59879	1.21128e-03
0.7		1.69832	1.67535e-03
0.8	1.60000	1.79786	2.13953e-03
0.9	1.70000	1.89740	2.60375e-03
	1.80000		
	1.90000		

Example 8. Consider nonlinear Fredholm integral equation [34].

$$u(x) = e^{x+1} - \int_{t=0}^{1} e^{x-2t}(u(t))^3 dt,$$

with exact solution $u(x) = e^x$.

Table 10 shows the exact solution and the comparison of the approximate solution for presented method and Haar wavelets method [34]. We can improve the accuracy by increasing M and N.

Table 10 The numerical results for example 8 when M = 1, N = 2.

x	Exact solution	Presented Method N = 2, M = 1	Absolute Error	Approximate solution [34] with k = 32
0.1	1.105170918	1.10181	3.36092e-03	1.107217811
0.2	1.221402758	1.22165	2.47242e-04	1.218102916
0.3	1.349858808	1.34293	6.92881e-03	1.341165462
0.4	1.491824698	1.49495	3.12530e-03	1.474918603
0.5	1.648721271	1.62698	2.17413e-02	1.667402633
0.6	1.822118800	1.82900	6.88120e-03	1.833861053
0.7	2.013752707	2.01102	2.73271e-03	2.016679830
0.8	2.225540928	2.22428	1.26093e-03	2.217456630
0.9	2.459603111	2.40752	5.20831e-02	2.437978177

7. Conclusion

The integral equations are important for studying and solving a large proportion of the problems in many topics. In the present work, we present a new hybrid function for solving linear Fredholm integral equation numerically. The new hybrid is combination Bernstein and improved block-pulse functions. Illustrative numerical examples are included in order to test

Table 8 The numerical results for example 6 when $M = 3, N = 4$.

x	Exact solution	Presented Method	Absolute Error			
			Che	Leg	CAS	Presented method
0.0	0.000000	0.000000	0.001269	0.001013	0.123699	0.000000e+00
0.2	0.198669	0.198669	0.000235	0.000280	0.008219	1.076863e-07
0.4	0.389418	0.389419	0.000199	0.000358	0.008096	2.949670e-07
0.6	0.564642	0.564643	0.000181	0.000284	0.008002	3.688978e-07
0.8	0.717356	0.717356	0.000160	0.000219	0.006031	1.083650e-07
1.0	0.841471	0.841471	0.000936	0.000734	0.080036	6.222436e-08

the accuracy of the proposed method. Moreover we see that accuracy of solutions in HBIBPFs is more satisfactory than the methods of BPFs and BPs. Form the obtained numerical results we can see that the method is very promising to handle more general nonlinear integral equations which is under investigation by the authors.

Declaration-of-competing-interests

The authors declare no conflict of interest.

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