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Fuzzy inferior ratio method for multiple attribute decision making problems

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ABSTRACT

Multiple attribute decision making forms an important part of the decision process for both small (individual) and large (organization) problems. When available information is precise, many methods exist to solve this problem. But the uncertainty and fuzziness inherent in the structure of information make rigorous mathematical models inappropriate for solving this type of problems. This paper incorporates the fuzzy set theory and the basic nature of subjectivity due to the ambiguity to achieve a flexible decision approach suitable for uncertain and fuzzy environment. The proposed method can take both real and fuzzy inputs. An outranking intensity is introduced to determine the degree of overall outranking between competing alternatives, which are represented by fuzzy numbers. Numerical examples finally illustrate the approach.

1. Introduction

Multiple attribute decision-making (MADM) methods are widely used to rank real world alternatives or select the best alternative with respect to several competing criteria. In classical MADM methods, assessments of alternatives are precisely known [3,30,32]. Due to fuzziness and uncertainty of decision-making problems, and the inherent vagueness of human preferences, however, the best expression of decision makers comes in natural language. As a result, using linguistic (fuzzy) assessments are much more realistic than numerical values. In other words, linguistic variables, in which the values are words or sentences from natural or artificial languages, are used in the assessment of alternatives with respect to criteria [18,29,33,38,39].

An MADM problem can be concisely expressed in matrix format as

$$\boldsymbol{A} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & \cdots & c_m \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \end{array} \tag{1.1}$$

where $A_1, \ldots, A_n$ are feasible alternatives among which decision should be made, $c_1, \ldots, c_m$ are attributes with which the performance of alternatives are measured, and the entry a_{ij} of decision matrix $\boldsymbol{A}$ is the rating of alternative A_i with respect to the

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attribute c_j. Furthermore, the attributes normalized weight vector $\boldsymbol{w} = (w_1, \ldots, w_m)^t$ should be either appraised by pairwise comparisons or determined by decision maker's (DM) preferences [19,23,40].

Assume that the alternatives set is denoted by Λ and the attributes set is denoted by Υ; that is, $\Lambda = \{A_1, \ldots, A_n\}$ and $\Upsilon = \{c_1, \ldots, c_m\}$. In general terms, attributes are divided into benefit attributes and cost attributes. That is to say, Υ is partitioned into two distinct sets Υ^+ and Υ^-. Then, an MADM problem can be portrayed as follows [16,28,35]:

$$\begin{array}{rl} \text{Maximize} & \{a_{ij} : j \in \Upsilon^+\} \\ \text{Minimize} & \{a_{ij} : j \in \Upsilon^-\} \\ \text{Subject to} & A_i \in \Lambda \end{array} \tag{1.2}$$

In classical MADM methods, the attribute values (ratings) and weights are determined precisely. A survey of the methods has been presented in [17]. In real life management decision situations, MADM models and methods have been proposed. So far, a variety of applicable methods such as Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) and Analytic Hierarchy Process (AHP) have been developed for an MADM problem.

TOPSIS was developed for solving MADM problems based on the concept that the obtained solution should be nearest to the positive ideal solution (PIS) and be remotest from the negative ideal solution (NIS). TOPSIS is a popular method and has been widely used in the literature (Abo-Sinna and Amer [1]; Agrawal et al. [2]; Cheng et al. [9]; Mokhtarian and Hadi-Vencheh [31]; Deng et al. [13]; Feng and Wang [14,15]; Chen and Liu [7]; Hwang and Yoon [19]; Jee and Kang [20]; Kim et al. [24]; Li [27]). The method has also been extended to deal with fuzzy MADM problems. For example, Tsaur et al. [36] first convert a fuzzy MADM problem into a real one via centroid defuzzification and then solve the non-fuzzy MADM problem using the TOPSIS method. Chen and Tzeng [8] transform a fuzzy MADM problem into a non-fuzzy MADM using fuzzy integral. Instead of using distance, they employ gray relation grade to define the relative closeness of each alternative. Chu [10,11] and Chu and Lin [12] also change a fuzzy MADM problem into a real one and solve the real MADM problem using the TOPSIS method. Differing from the others, they first derive the membership functions of all the weighted ratings in a weighted normalization decision matrix using interval arithmetics of fuzzy numbers and then defuzzify them into real values using the ranking method of mean of removals [22]. Chen [6] extends the TOPSIS method to fuzzy group decision making situations by defining a real Euclidean distance between any two fuzzy numbers. Triantaphyllou and Lin [35] develop a fuzzy version of the TOPSIS method based on fuzzy arithmetic operations, which leads to a fuzzy relative closeness for each alternative. It is argued that fuzzy weights and fuzzy ratings should result in fuzzy relative closeness. Real relative closeness provides only one possible solution to a fuzzy MADM problem, but cannot reflect the whole picture of its all possible solutions. Wang and Elhang [37] propose a fuzzy TOPSIS method based on alpha level sets. They deal with the relative closeness as an optimal solution to a fractional programming.

In all aforementioned works, the authors develop a hybrid fuzzy TOPSIS. TOPSIS, as will be seen in next section, seeks for a compromise solution (alternative) that is closest to ideal solution and remotest from negative ideal solution. However, the compromise solution of TOPSIS is not essentially the remotest from negative ideal solution. Therefore, the ranking coefficients concern only to closeness to the ideal solution (see Example 1 in sub-Section 2.1). Motivated by such a fact, this paper proposes a method for solving fuzzy MADM.

In this paper, we develop a new approach for solving multiple attribute decision making problems based on the same concept as TOPSIS considering both distances to the PIS and from the NIS. Furthermore, constructing the model under the assumption of vagueness and imprecise environment enables the method to conform to real life decision situations.

This paper is organized as follows. In the following section the essential definitions and concepts, as well as, notations of the fuzzy set theory are exhibited. Section 3 is dedicated to the proposed method and its hints. Section 4 illustrates the proposed method with two numerical examples. Ultimately, discussion and conclusion are given in Section 5.

2. Preliminaries

Fuzzy sets are coherent extension of real sets and were first developed by Zadeh [41] as an aid for dealing with uncertainty/imprecision and vagueness in the real world. A fuzzy set is a collection of elements in a universe of information where the boundary of the set contained in the universe is ambiguous, vague and fuzzy. Each fuzzy set is specified by a membership function, which assigns to each element in the universe of discourse a value within the unit interval $[0, 1]$. The assigned value is called degree (or grade) of membership, which specifies the extent to which a given element belongs to the fuzzy set or is related to a concept. If the assigned value is 0, then the given element does not belong to the set. If the assigned value is 1, then the element totally belongs to the set. If the value lies within the interval $(0, 1)$, then the element only partially belongs to the set. Therefore, any fuzzy set can be uniquely determined by its membership function.

Let X be the universe of discourse. A fuzzy set $\widetilde{A}$ of the universe of discourse X is said to be convex if and only if for all x_1 and x_2 in X there always exists $\lambda \in [0, 1]$ such that:

$$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geqslant \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)) \tag{2.1}$$

where $\mu_{\tilde{A}}$ is the membership function of the fuzzy set $\widetilde{A}$. A fuzzy set $\widetilde{A}$ of the universe of discourse X is said to be normal if there exists a $x_i \in X$ satisfying $\mu_{\tilde{A}}(x_i) = 1$. Fuzzy numbers are special cases of fuzzy sets that are both convex and normal. The membership function of a fuzzy number is piecewise continuous and satisfies the following properties:

1. $\mu_{\tilde{A}}(x) = 0$ outside some interval $[a, d]$;
2. $\mu_{\tilde{A}}(x)$ is non-decreasing (monotonically increasing) on $[a, b]$ and non-increasing (monotonically decreasing) on $[c, d]$;
3. $\mu_{\tilde{A}}(x) = 1$ for each $x \in [b, c]$,

where $a \leqslant b \leqslant c \leqslant d$ belong to the real line $\mathbb{R}$.

Definition 1. A fuzzy number $\widetilde{A}$ is said to be positive if and only if $\mu_{\tilde{A}}(x) = 0$ for all $x < 0$.
We denote the set of all real fuzzy numbers by $\widetilde{\mathbb{R}}$. The binary operation on fuzzy numbers is shown in Definition 2.

Definition 2. If $\widetilde{A}$ and $\widetilde{B}$ are any two fuzzy numbers, then $\widetilde{C} = \widetilde{A} \odot \widetilde{B}$ is a fuzzy number whose membership function is defined as [26,34]

$$\mu_{\tilde{C}}(z) := \sup_{z=x*y} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\} \tag{2.2}$$

where $*$ stands for the corresponding non-fuzzy binary operation.

The most commonly used fuzzy numbers are triangular and trapezoidal fuzzy numbers, whose membership functions are, respectively, defined as

$$\mu_{\tilde{A}_1}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leqslant x \leqslant b; \\ \frac{d-x}{d-b}, & b \leqslant x \leqslant d; \\ 0, & \text{Otherwise.} \end{cases} \tag{2.3}$$

and

$$\mu_{\tilde{A}_2}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leqslant x \leqslant b; \\ 1, & b \leqslant x \leqslant c; \\ \frac{d-x}{d-c}, & c \leqslant x \leqslant d; \\ 0, & \text{Otherwise.} \end{cases} \tag{2.4}$$

For brevity, triangular and trapezoidal fuzzy numbers are often denoted as (a, b, d) and (a, b, c, d). It is obvious that triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers with $b = c$.

Definition 3. Let $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$ be any two positive triangular fuzzy numbers. The basic fuzzy arithmetic operations on these fuzzy numbers are defined as following [21,25].

Addition: $\tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$;
Subtraction: $\tilde{a} - \tilde{b} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$;
Multiplication: $\tilde{a} \times \tilde{b} \approx (a_1b_1, a_2b_2, a_3b_3)$;
Division: $\tilde{a} \div \tilde{b} \approx \left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}\right)$.

Definition 4. Assume that $\tilde{x} = (x_1, x_2, x_3)$ and $\tilde{y} = (y_1, y_2, y_3)$ are any two triangular fuzzy numbers. Then the fuzzy p-distance between these two numbers is defined as

$$\rho_p(\tilde{x}, \tilde{y}) = \sqrt[p]{\frac{1}{3}[|y_1 - x_1|^p + |y_2 - x_2|^p + |y_3 - x_3|^p]}, \quad p \geqslant 1. \tag{2.5}$$

2.1. TOPSIS: an overview

The procedure of the TOPSIS is summarized as follows:

Step 1. Calculate normalized criterion values. The normalized criterion values v_{ij} $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ of a_{ij} can be calculated as follows:

$$v_{ij} := \frac{a_{ij}}{\sqrt{\sum_{k=1}^{n} a_{kj}^2}} \tag{2.6}$$

Step 2. Calculate weighted normalized criterion values. The weighted normalized criterion values υ_{ij} of v_{ij} can be calculated as follows:

$$\upsilon_{ij} := v_{ij} w_j \tag{2.7}$$

where $\boldsymbol{w} = (w_1, \ldots, w_m)$ is the criteria weight vector.

Step 3. Determine the ideal solution and the negative ideal solution. Denote the ideal solution and the negative ideal solution as A^+ and A^- whose weighted normalized criterion value vectors are, respectively, denoted by $\boldsymbol{v}^{*+} = (v_1^{*+}, \ldots, v_m^{*+})$ where

$$v_j^{*+} := \begin{cases} \max\limits_{1 \leqslant i \leqslant n} v_{ij}, & \text{if } c_j \text{ is a benefit criterion;} \\ \min\limits_{1 \leqslant i \leqslant n} v_{ij}, & \text{if } c_j \text{ is a cost criterion.} \end{cases} \tag{2.8}$$

and $\boldsymbol{v}^{*-} = (v_1^{*-}, \ldots, v_m^{*-})$ where

$$v_j^{*-} := \begin{cases} \min\limits_{1 \leqslant i \leqslant n} v_{ij}, & \text{if } c_j \text{ is a benefit criterion;} \\ \max\limits_{1 \leqslant i \leqslant n} v_{ij}, & \text{if } c_j \text{ is a cost criterion.} \end{cases} \tag{2.9}$$

Step 4. Calculate the separation measures. The separation measures of alternatives A_i from the ideal solution A^+ as well as the negative ideal solution A^- are, respectively, defined as follows:

$$d_i^+ := \sqrt{\sum_{j=1}^m \left(v_{ij} - v_j^{*+}\right)^2}, \quad i = 1, \ldots, n \tag{2.10}$$

and

$$d_i^- := \sqrt{\sum_{j=1}^m \left(v_{ij} - v_j^{*-}\right)^2}, \quad i = 1, \ldots, n \tag{2.11}$$

Step 5. Calculate the relative closeness coefficients to the ideal solution. The relative closeness coefficient of alternative A_i with respect to the ideal solution A^+ are defined as follows:

$$\xi_i := \frac{d_i^-}{d_i^- + d_i^+}, \quad i = 1, \ldots, n \tag{2.12}$$

Step 6. Rank the alternatives. Ranking order of all alternative A_i is generated according to the decreasing order of all coefficients ξ_i.

TOPSIS seeks a compromise solution based on closeness to ideal solution and remoteness from negative ideal solution, simultaneously. However, there exists some drawback in its logic. Consider the following example:

Example 1. Suppose the decision matrix of an MADM problem with three alternatives A_1, A_2, and A_3 against the cost criterion c_1 and benefit criterion c_2 is given as follows:

$$\boldsymbol{D} = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \end{matrix} \begin{matrix} c_1 & c_2 \\ \left(\begin{matrix} 1 & 3000 \\ 1.1 & 3750 \\ 5 & 4500 \end{matrix}\right) \end{matrix}$$

If the relative importance weight vector of criteria is $\boldsymbol{w} = (0.5, 0.5)$, then the weighted normalized decision matrix and final results are presented in Table 1.

As indicated in Table 1 the compromise solution, the alternative A_2, has the least distance to ideal solution, $D_2^{*+} = 0.058$, but it is not the remotest from negative ideal solution, since $D_1^{*-} = 0.383 > 0.378 = D_2^{*-}$. Nonetheless, this drawback is the major motivation for FIR method.

Table 1
The results for Example 1 using TOPSIS.

Description variables	Weighted normalized criterion values		TOPSIS results			
	c_1	c_2	D_i^{*+}	D_i^{*-}	ξ_i	Rank
A_1	0.096	0.228	0.114	0.383	0.771	2
A_2	0.105	0.285	0.058	0.378	0.867	1
A_3	0.479	0.342	0.383	0.114	0.229	3
Ideal solution	0.096	0.342				
Negative Ideal solution	0.479	0.228				

3. Proposed method: the fuzzy inferior ratio method

Our method focuses on rating and selecting from a set of alternatives in the presence of vagueness and multiple conflicting attributes. It determines a compromise solution (or ranking list) based on the concept that the chosen alternative should be the closest to the PIS and the farthest away from NIS, simultaneously. Differences between each alternative and the both PIS and NIS are measured with an extension of weight Minkowski L_1 metric.

In this paper, it is assumed that the alternatives set $\Lambda = \{A_1, \ldots, A_n\}$ and the attributes set $\Upsilon = \{c_1, \ldots, c_m\}$ are finite and that the MADM problem is expressed concisely in matrix format as follows:

$$\tilde{\boldsymbol{A}} = \begin{matrix} & c_1 & c_2 & \cdots & c_m \\ A_1 & \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1m} \\ A_2 & \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_n & \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nm} \end{matrix} \tag{3.1}$$

where $\tilde{a}_{ij} = \left(a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}\right)$ denotes fuzzy triangular numbers.

Step I: Normalizing the decision matrix
Multiple attributes are usually incommensurable. Therefore, the decision matrix $\widetilde{\boldsymbol{A}}$ has to be normalized so that the units and dimensions of attribute values are eliminated. To avoid complicated normalization formula used in classical TOPSIS, obtain the normalized fuzzy decision matrix via the linear scale transformation as following:

$$\tilde{v}_{ij} := \left(\frac{a_{ij}^{(1)}}{\hat{a}_j^{(3)}}, \frac{a_{ij}^{(2)}}{\hat{a}_j^{(3)}}, \frac{a_{ij}^{(3)}}{\hat{a}_j^{(3)}}\right), \quad i = 1, \ldots, n \tag{3.2}$$

if $j \in \Upsilon^+$, where $\hat{a}_j^{(3)} = \max_{1 \leqslant i \leqslant n} a_{ij}^{(3)}$, and

$$\tilde{v}_{ij} := \left(\frac{\bar{a}_j^{(1)}}{a_{ij}^{(3)}}, \frac{\bar{a}_j^{(1)}}{a_{ij}^{(2)}}, \frac{\bar{a}_j^{(1)}}{a_{ij}^{(1)}}\right), \quad i = 1, \ldots, n \tag{3.3}$$

if $j \in \Upsilon^-$, where $\bar{a}_j^{(1)} = \min_{1 \leqslant i \leqslant n} a_{ij}^{(1)}$. So, the fuzzy normalized decision matrix is:

$$\tilde{\boldsymbol{N}} = \begin{matrix} & c_1 & c_2 & \cdots & c_m \\ A_1 & \tilde{\nu}_{11} & \tilde{\nu}_{12} & \cdots & \tilde{\nu}_{1m} \\ A_2 & \tilde{\nu}_{21} & \tilde{\nu}_{22} & \cdots & \tilde{\nu}_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_n & \tilde{\nu}_{n1} & \tilde{\nu}_{n2} & \cdots & \tilde{\nu}_{nm} \end{matrix} \tag{3.4}$$

In decision making process, different attributes have different importance. Suppose the attributes weight vector $\tilde{\boldsymbol{w}} = (\tilde{w}_1, \ldots, \tilde{w}_m)^t$, whose entries are positive triangular fuzzy numbers.

Step II: Determination of PIS and NIS, as well as, evaluation of the alternative distances to each of them
Denote A^+ as PIS whose normalized attribute value vector is defined as

$$\tilde{\boldsymbol{g}} = (\tilde{g}_1, \ldots, \tilde{g}_m)^t \tag{3.5}$$

where

$$\tilde{g}_j := \left(\max_{1 \leqslant i \leqslant n} v_{ij}^{(1)}, \max_{1 \leqslant i \leqslant n} v_{ij}^{(2)}, \max_{1 \leqslant i \leqslant n} v_{ij}^{(3)}\right), \quad j = 1, \ldots, m$$

In real situations, the MADM problem is often described by several conflicting (or competing) attributes and there may exist no solution (alternative) satisfying all attributes simultaneously. Thus, A^+ may not be a feasible alternative, i.e. $A^+ \notin \Lambda$. Otherwise, A^+ is an optimal solution of the MADM problem. In this paper, without loss of generality, we assume that $A^+ \notin \Lambda$. Difference between an alternative A_i $(i = 1, \ldots, n)$ and PIS A^+ is measured as follows:

$$d_p(A_i, A^+) := \sum_{j=1}^{m} \left(\frac{1}{3}\sum_{k=1}^{3}\left[w_j^{(k)}\left(g_j^{(k)} - v_{ij}^{(k)}\right)\right]^p\right)^{1/p} \tag{3.6}$$

where $p \geqslant 1$ is a distance parameter distinguished by DM according to the practical situations and decision making requirements.
The smaller $d_p(A_i, A^+)$ the better A_i. Let's denote

$$d_p(A^+) = \min_{1 \leqslant i \leqslant n} d_p(A_i, A^+), \tag{3.7}$$

hence the alternative A_i that satisfies $d_p(A^+) = d_p(A_i, A^+)$ is closed to PIS. However, such an alternative may not always be the farthest away from NIS.

Denote A^- as the NIS whose normalized attribute value vector is defined as

$$\tilde{\boldsymbol{b}} = (\tilde{b}_1, \ldots, \tilde{b}_m)^t \tag{3.8}$$

where

$$\tilde{b}_j := \left(\min_{1\leqslant i\leqslant n} v_{ij}^{(1)}, \min_{1\leqslant i\leqslant n} v_{ij}^{(2)}, \min_{1\leqslant i\leqslant n} v_{ij}^{(3)}\right), \quad j = 1, \ldots, m$$

A^- may not almost be a feasible alternative, i.e. $A^- \notin \Lambda$. Otherwise, A^- is an inferior solution of the MADM problem. Similarly, without loss of generality we assume that $A^- \notin \Lambda$.

Difference between an alternative A_i $(i = 1, \ldots, n)$ and NIS A^- is also measured as follows:

$$d_p(A_i, A^-) := \sum_{j=1}^{m} \left(\frac{1}{3}\sum_{k=1}^{3}\left[w_j^{(k)}\left(v_{ij}^{(k)} - b_j^{(k)}\right)\right]^p\right)^{1/p} \tag{3.9}$$

The bigger $d_p(A_i, A^-)$ the better A_i. Therefore, the alternative satisfies

$$d_p(A^-) = \max_{1\leqslant i\leqslant n} d_p(A_i, A^-) \tag{3.10}$$

is farthest away from NIS.

Step III: The compromise solution

In this study, we seek an alternative satisfying both Eqs. (3.7) and (3.10). So, let

$$\zeta_p(A_i) := \frac{d_p(A_i, A^-)}{d_p(A^-)} - \frac{d_p(A_i, A^+)}{d_p(A^+)}, \quad i = 1, \ldots, n. \tag{3.11}$$

Evidently, $\zeta_p(A_i)$ measures the extent to which the alternative A_i closes to PIS and is far away from NIS, simultaneously.

Proposition 1. $\zeta_p(A_i) \leq 0$ for $i = 1, \ldots, n$.

Proof. According to Eqs. (3.7) and (3.10), for any $i = 1, \ldots, n$ we have

$$d_p(A^-) \geqslant d_p(A_i, A^-) \quad \& \quad d_p(A_i, A^+) \geqslant d_p(A^+)$$

thus,

$$\frac{d_p(A_i, A^-)}{d_p(A^-)} \leq 1 \quad \& \quad -\frac{d_p(A_i, A^+)}{d_p(A^+)} \leqslant -1$$

which yields to

$$\zeta_p(A_i) = \frac{d_p(A_i, A^-)}{d_p(A^-)} - \frac{d_p(A_i, A^+)}{d_p(A^+)} \leq 1 + (-1) = 0. \quad \square$$

We put the alternatives into order with respect to the decreasing values of ζ_i. The bigger $\zeta_p(A_i)$, the better A_i, viz.,

$$A^* := \left\{A_i : \left[i : \zeta_p(A_i) = \max_{1\leqslant k\leqslant n} \zeta_p(A_k)\right]\right\}. \tag{3.12}$$

If there exists an alternative, namely A_k, satisfying

$$d_p(A_k, A^-) = \max_{1\leqslant i\leqslant n} d_p(A_i, A^-) \quad \& \quad d_p(A_k, A^+) = \min_{1\leqslant i\leqslant n} d_p(A_i, A^+) \tag{3.13}$$

simultaneously, viz.,

$$d_p(A^-) = d_p(A_k, A^-) \quad \& \quad d_p(A^+) = d_p(A_k, A^+) \tag{3.14}$$

then $\zeta_p(A_k) = 0$ and A_k is a compromise solution (i.e. alternative) that is closest to PIS A^+ and farthest away from NIS A^- simultaneously. Hence, the ratio

$$IR_p(i) := \frac{\zeta_p(A_i)}{\min_{1\leqslant i\leqslant n} \zeta_p(A_i)} \tag{3.15}$$

suggests the "inferior ratio" for both attributes of the shortest distance from PIS and farthest away from NIS. It is perspicuous that $IR_p(i)$ is a value within the interval $[0,1]$.

Remark. Unfortunately, the provided ranking method does not result in a linear order, so ties may happen. To solve this problem, a possibility to overcome the drawback of possible ties would be the use of different approaches to measure proximity. In this sense, for instance, restricted equivalence functions based on distances could be of interest, since a whole family of different distances can be provided, see [4,5]. It would also be possible to consider the notion of penalty function [33].

4. Numerical examples

In this section, we examine two numerical examples using the proposed fuzzy method. These examples are taken from Chen [6] and Triantaphyllou and Lin [35] for the purpose of comparison. The values of $p(=2,3,7,29)$ have been chosen randomly.

Example 2. Reconsider the example investigated by Chen [6] in which a software company desires to hire a system analysis engineer among three candidates, A_1, A_2, and A_3 evaluated by a committee of three decision makers (DMs) against five benefit attributes, i.e. emotional steadiness (c_1), oral communication skills (c_2), personality (c_3), past experience (c_4) and self-confidence (c_5). The relative importance weights of the five attributes are described using linguistic variables such as Low, Medium, and High as defined in Table 2. The ratings (i.e. attribute values) are also characterized by linguistic variables such as Poor, Fair, and Good, as defined in Table 3. The three DMs express their opinions on the importance weights of the five attributes and the ratings of each candidate with respect to each attribute independently. Tables 4 and 5 show the original assessment information provided by the three DMs, where aggregated fuzzy numbers are obtained by averaging the fuzzy opinions of the three DMs (see Table 6). That is, $\tilde{w}_j = \left(\tilde{w}_j^{(1)} + \tilde{w}_j^{(2)} + \tilde{w}_j^{(3)}\right)/3$, where $\tilde{w}_j^{(k)}$ indicates the relative importance weight given by the kth DM. The results are presented in Table 7 which gives the ranking of $A_2 \succ A_3 \succ A_1$, where the symbol $\succ$ means "is superior" or "preferred to". It is easy to see that Chen's approach leads to the same ranking as ours.

Table 2
Linguistic variables for the relative importance weights for five attributes.

Linguistic variable	Fuzzy number
Very low (VL)	(0, 0, 0.1)
Low (L)	(0, 0.1, 0.3)
Medium low (ML)	(0.1, 0.3, 0.5)
Medium (M)	(0.3, 0.5, 0.7)
Medium high (MH)	(0.5, 0.7, 0.9)
High (H)	(0.7, 0.9, 1.0)
Very high (VH)	(0.9, 1.0, 1.0)

Table 3
Linguistic variables for attribute values.

Linguistic variable	Fuzzy number
Very poor (VP)	(0, 0, 1)
Poor (P)	(0, 1, 3)
Medium poor (MP)	(1, 3, 5)
Fair (F)	(3, 5, 7)
Medium good (MG)	(5, 7, 9)
Good (G)	(7, 9, 10)
Very good (VG)	(9, 10, 10)

Table 4
The relative importance weights of the five criteria by three DMs.

Attributes	DM1	DM2	DM3	$\tilde{w}_j$
c_1	H	VH	MH	(0.70, 0.87, 0.97)
c_2	VH	VH	VH	(0.90, 1.00, 1.00)
c_3	VH	H	H	(0.77, 0.93, 1.00)
c_4	VH	VH	VH	(0.90, 1.00, 1.00)
c_5	M	MH	MH	(0.43, 0.63, 0.83)

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Table 5
Ratings of three candidates with respect to the five criteria by the three DMs.

Attributes	Candidates	DMs		
		DM1	DM2	DM3
c_1	A_1	MG	G	MG
	A_2	G	G	MG
	A_3	VG	G	F
c_2	A_1	G	MG	F
	A_2	VG	VG	VG
	A_3	MG	G	VG
c_3	A_1	F	G	G
	A_2	VG	VG	G
	A_3	G	MG	VG
c_4	A_1	VG	G	VG
	A_2	VG	VG	VG
	A_3	G	VG	MG
c_5	A_1	F	F	F
	A_2	VG	MG	G
	A_3	G	G	MG

Table 6
Normalized data matrix for Example 2.

Attributes $\tilde{w}_j$	c_1 (0.70, 0.87, 0.97)	c_2 (0.90, 1.00, 1.00)	c_3 (0.77, 0.93, 1.00)	c_4 (0.90, 1.00, 1.00)	c_5 (0.43, 0.63, 0.83)
A_1	(0.00, 0.44, 0.89)	(0.00, 0.40, 0.73)	(0.00, 0.43, 0.77)	(0.40, 0.89, 1.00)	(0.00, 0.19, 0.50)
A_2	(0.12, 0.58, 0.97)	(0.72, 1.00, 1.00)	(0.47, 0.86, 1.00)	(0.60, 1.00, 1.00)	(0.26, 0.54, 0.83)
A_3	(0.12, 0.51, 0.81)	(0.36, 0.73, 0.93)	(0.36, 0.64, 0.92)	(0.00, 0.56, 0.89)	(0.22, 0.50, 0.83)

Table 7
Numerical results for Example 2.

p Values	A_i	$d_p(A_i, A^+)$	$d_p(A_i, A^-)$	$\zeta_p(A_i)$	$IR_p(i)$	Rank
$p = 2$	A_1	3.49	2.88	−1.80	1.00	3
	A_2	4.94	1.15	0.00	0.00	1
	A_3	4.00	2.31	−1.20	0.67	2
$p = 3$	A_1	3.45	2.88	−1.60	1.00	3
	A_2	4.69	1.23	0.00	0.00	1
	A_3	3.92	2.31	−1.04	0.65	2
$p = 7$	A_1	3.60	3.03	−1.42	1.00	3
	A_2	4.62	1.38	0.00	0.00	1
	A_3	4.04	2.44	−0.89	0.63	2
$p = 29$	A_1	3.80	0.15	−1.36	1.00	3
	A_2	4.74	1.01	0.00	0.00	1
	A_3	4.28	2.60	−0.84	0.62	2

Table 8
Fuzzy weights and fuzzy decision matrix for Example 3.

c_j $\tilde{w}_j$	c_1 (0.13, 0.20, 0.31)	c_2 (0.08, 0.15, 0.25)	c_3 (0.29, 0.40, 0.56)	c_4 (0.17, 0.25, 0.38)
A_1	(0.08, 0.25, 0.94)	(0.25, 0.93, 2.96)	(0.34, 0.70, 1.71)	(0.12, 0.24, 0.92)
A_2	(0.23, 1.00, 3.10)	(0.13, 0.60, 2.24)	(0.03, 0.05, 0.09)	(0.12, 0.40, 1.48)
A_3	(0.15, 0.40, 1.48)	(0.13, 0.20, 0.88)	(0.62, 1.48, 3.41)	(0.24, 1.00, 3.03)

Example 3. Reconsider the example investigated by Triantaphyllou and Lin [35], in which three alternatives $A_1 \sim A_3$ are evaluated against four benefit attributes $c_1 \sim c_4$. The fuzzy weights and the fuzzy decision matrix are duplicated in Table 8 (see Table 9). The results are recorded in Table 10. The $IR_p(i)$ values, presented in Table 10, lead to the ranking of $A_3 \succ A_1 \succ A_2$.

Table 9
Normalized data matrix for Example 3.

Attributes	c_1	c_2	c_3	c_4
$\tilde{w}_j$	(0.13, 0.20, 0.31)	(0.08, 0.15, 0.25)	(0.29, 0.40, 0.56)	(0.17, 0.25, 0.38)
A_1	(0.00, 0.01, 0.09)	(0.00, 0.04, 0.25)	(0.03, 0.08, 0.28)	(0.00, 0.01, 0.10)
A_2	(0.01, 0.06, 0.31)	(0.00, 0.02, 0.19)	(0.00, 0.00, 0.01)	(0.00, 0.02, 0.18)
A_3	(0.03, 0.02, 0.14)	(0.00, 0.00, 0.07)	(0.05, 0.17, 0.56)	(0.01, 0.08, 0.38)

Table 10
Numerical results for Example 3.

p Values	A_i	$d_2(A_i, A^+)$	$d_2(A_i, A^-)$	$\zeta_2(A_i)$	$IR_p(i)$	Rank
$p = 2$	A_1	0.47	0.95	−0.75	0.96	2
	A_2	0.48	0.98	−0.78	1.00	3
	A_3	0.85	0.73	0.00	0.00	1
$p = 3$	A_1	0.53	0.90	−0.66	0.92	2
	A_2	0.53	0.94	-0.72	1.00	3
	A_3	0.92	0.73	0.00	0.00	1
$p = 7$	A_1	0.63	0.88	−0.58	0.82	2
	A_2	0.61	0.95	−0.71	1.00	3
	A_3	1.04	0.73	0.00	0.00	1
$p = 29$	A_1	0.70	0.90	−0.57	0.77	2
	A_2	0.67	1.01	−0.74	1.00	3
	A_3	1.12	0.76	0.00	0.00	1

5. Discussion and conclusion

The present method is based on an aggregating function representing closeness to the ideal solution and being far away from the negative ideal solution simultaneously, whereas the TOPSIS is characterized for considering ideal and nadir solutions. The proposed method introduces the inferior ratio to reflect some balance between the shortest distance from the ideal solution and the farthest distance from the negative ideal solution. The relative importance of the distances from the positive ideal solution and the negative ideal solution, which has been considered in our method, is a major concern in real life decision making. In contrast, the TOPSIS is based on the concept that the chosen alternative should have the shortest distance to the positive ideal solution and the farthest distance from the negative ideal solution. However, such a chosen alternative may not always guarantee to be the remotest from the negative ideal solution. Moreover, the TOPSIS introduces two reference points, i.e. the positive ideal solution and the negative ideal solution, but it does not consider the relative importance of the distances from these points. The proposed method and the TOPSIS use different normalization methods to eliminate the units and dimensions of attribute functions. In contrast to the TOPSIS, the normalized attribute values in our method do not depend on evaluation units and dimensions of attribute functions. The above comparative analysis from two aspects of the theory and numerical computation shows that the proposed method has some advantages over the TOPSIS.

In this paper, we have developed a method for solving multiple criteria decision making problems incorporating the basic nature of subjectivity due to ambiguity with concepts of objectivity based on aggregating decision functions. An inferior ratio was proposed to indicate the outranking intensity, as well as, the optimum scale due to closeness to the ideal point and being remotest from the negative ideal point, simultaneously. Considering the fuzzy nature of concepts, the flexibility of our method can increase under vague conditions.

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References

[1] M.A. Abo-Sinna, A.H. Amer, Extensions of the TOPSIS for multi-objective large-scale nonlinear programming problems, Appl. Math. Comput. 162 (2005) 243–256.
[2] V.P. Agrawal, V. Kohli, S. Gupta, Computer aided robot selection: the multiple attribute decision making approach, Int. J. Prod. Res. 29 (1991) 1629–1644.
[3] S. Alonso, E. Herrera-Viedma, F. Chiclana, F. Herrera, A web based consensus support system for group decision making problems and incomplete preferences, Inform. Sci. 180 (23) (2010) 4477–4495.
[4] H. Bustince, E. Barrenechea, M. Pagola, Restricted equivalence functions, Fuzzy Sets Syst. 157 (2006) 2337–2346.

[5] H. Bustince, E. Barrenechea, M. Pagola, Relationship between restricted dissimilarity functions, restricted equivalence functions and normal E_N-functions: Image thresholding invariant, Pattern Recogn. Lett. 29 (2008) 525–536.
[6] C.T. Chen, Extensions of the TOPSIS for group decision making under fuzzy environment, Fuzzy Sets Syst. 114 (2000) 1–9.
[7] J. Chen, Y. Liu, A model and its application for uncertainly group decision making, World J. Model. Simulat. 2 (1) (2006) 45–54.
[8] M.F. Chen, G.H. Tzeng, Combining grey relation and TOPSIS concepts for selecting an expatriate host country, Math. Comput. Model. 40 (2004) 1473–1490.
[9] S. Cheng, C.W. Chan, G.H. Huang, An integrated multi-criteria decision analysis and inexact mixed integer linear programming approach for solid waste management, Eng. Appl. Artif. Intell. 16 (2003) 543–554.
[10] T.C. Chu, Facility location selection using fuzzy TOPSIS under group decisions, Int. J. Uncertainty 10 (2002) 687–701.
[11] T.C. Chu, Selecting plant location via a fuzzy TOPSIS approach, Int. J. Adv. Manuf. Technol. 20 (2002) 859–864.
[12] T.C. Chu, Y.C. Lin, A fuzzy TOPSIS method for robot selection, Int. J. Adv. Manuf. Technol. 21 (2003) 284–290.
[13] H. Deng, C.H. Yeh, R.J. Willis, Inter-company comparison using modified TOPSIS with objective weights, Comput. Oper. Res. 27 (2000) 963–973.
[14] C.M. Feng, R.T. Wang, Performance evaluation for airlines including the consideration of financial ratios, J. Air Transp. Manage. 6 (2000) 133–142.
[15] C.M. Feng, R.T. Wang, Considering the financial ratios on the performance evaluation of highway bus industry, Transp. Rev. 21 (2001) 449–467.
[16] D. Feng-Li, An approach to fuzzy multiattribute decision making under uncertainty, Inform. Sci. 169 (1–2) (2005) 97–112.
[17] J. Figueira, S. Greco, M. Ehrgott, Multiple Criteria Decision Analysis: State of the Art Surveys, Springer Science + Business Media, Boston, 2005.
[18] A. Hadi-Vencheh, M.N. Mokhtarian, A new fuzzy MCDM approach based on centroid of fuzzy numbers, Expert Syst. Appl. 38 (2011) 5226–5230.
[19] C.L. Hwang, K. Yoon, Multiple Attributes Decision Making Methods and Applications, Springer, Berlin Heidelberg, 1981.
[20] D.H. Jee, K.J. Kang, A method for optimal material selection aided with decision making theory, Mater. Des. 21 (2000) 199–206.
[21] C. Kahraman, Fuzzy Multiple Criteria Decision Making, Springer, Istanbul, 2008.
[22] A. Kaufmann, M.M. Gupta, Introduction to Fuzzy Arithmetic: Theory and Applications, VanNostrand-Reinhold, New York, 1991.
[23] S. Khademolqorani, A. Zeinal Hamadani, An adjusted decision support system through data mining and multiple criteria decision making, Proc. – Soc. Behav. Sci. 73 (2013) 388–395.
[24] G. Kim, C.S. Park, K.P. Yoon, Identifying investment opportunities for advanced manufacturing systems with comparative-integrated performance measurement, Int. J. Prod. Econ. 50 (1997) 23–33.
[25] G.J. Klir, T.A. Folger, Fuzzy Sets, Uncertainty, and Information, Prentice-Hall of India, New Delhi, 2006.
[26] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logics – Theory and Applications, Prentice-Hall of India, New Delhi, 2007.
[27] D-F. Li, Relative ratio method for multiple attribute decision making problems, Int. J. Inform. Technol. Decis. Mak. 8 (2) (2009) 289–297.
[28] X. Liu, J.M. Mendel, D. Wu, Analytical solution methods for the fuzzy weighted average, Inform. Sci. 187 (2012) 151–170.
[29] D. Meng, P. Zheng, On weighted unbalanced linguistic aggregation operators in group decision making, Inform. Sci. 223 (2013) 31–41.
[30] J.M. Merigó, A.M. Gil-Lafuente, New decision-making techniques and their application in the selection of financial products, Inform. Sci. 180 (2010) 2085–2094.
[31] M.N. Mokhtarian, A. Hadi-Vencheh, A new fuzzy TOPSIS method based on left and right scores: an application for determining an industrial zone for dairy products factory, Appl. Soft Comput. 12 (2012) 2496–2505.
[32] R.O. Parreiras, P.Y. Ekel, J.S.C. Martini, R.M. Palhares, A flexible consensus scheme for multicriteria group decision making under linguistic assessments, Inform. Sci. 180 (7) (2010) 1075–1089.
[33] D. Paternain, A. Jurio, E. Barrenechea, H. Bustince, B. Bedregal, E. Szmidt, An alternative to fuzzy methods in decision-making problems, Fuzzy Sets Syst. 39 (2012) 7729–7735.
[34] W. Pedrycz, Granular Computing: Analysis and Design of Intelligent Systems, CRC Press/Taylor & Francis, Boca Raton, 2013.
[35] E. Triantaphyllou, C.T. Lin, Development and evaluation of five fuzzy multi-attribute decision-making methods, Int. J. Approx. Reason. 14 (1996) 281–310.
[36] S.H. Tsaur, T.Y. Chang, C.H. Yen, The evaluation of airline service quality by fuzzy MCDM, Tourism Manage. 23 (2002) 107–115.
[37] Y.M. Wang, T.M.S. Elhag, Fuzzy TOPSIS method based on alpha level sets with an application to bridge risk assessment, Expert Syst. Appl. 31 (2006) 309–319.
[38] Z. Xu, Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment, Inform. Sci. 168 (2004) 171–184.
[39] Z. Xu, A method based on the dynamic weighted geometric aggregation operator for dynamic hybrid multi-attribute group decision making, Int. J. Uncertainty, Fuzz. Knowl. – Based Syst. 17 (1) (2008) 15–33.
[40] H.B. Yan, V.N. Huynh, T. Ma, Y. Nakamori, Non-additive multi-attribute fuzzy target-oriented decision analysis, Inform. Sci. 240 (2013) 21–44.
[41] L.A. Zadeh, Fuzzy sets, Inform. Contr. 8 (1965) 338–353.