# Runge-Kutta (RK) Method

- In the forward Euler method, we used the slope or the derivative of $y$ at the given time step to extrapolate the solution to the next time-step.
- RK method uses the information on the slope at more than one point to extrapolate the solution to the future time step.
- Only first order ordinary differential equations can be solved by using the RK method

# RK Method

1st Order formula is same as the Euler's method

- $y_1 = y_0 + hf(x_0, y_0)$

2nd Order RK method

- $y_1 = y_0 + (½)(k_1 + k_2)$

where,

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + h, y_0 + k_1)$

# Runge-Kutta Fourth Order (RK4) Method

- It is the most commonly used RK method
- The RK4 method provides the approximate value of y for a given point x
- It computes next value $y_{n+1}$ using current $y_n$ plus weighted average of four increments
- $y_1 = y_0 + (1/6)(k_1 + 2k_2 + 2k_3 + k_4)$

Where,

$k_1 = hf(x_0, y_0)$

$k_2 = hf[x_0 + (1/2)h, y_0 + (1/2)k_1]$

$k_3 = hf[x_0 + (1/2)h, y_0 + (1/2)k_2]$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$k_1$ is the increment based on the slope at the beginning of the interval using y
$k_2$ ------------------------------------------------------ at the midpoint of the interval using $y + hk_1/2$
$k_3$ ------------------------------------------------------ at the midpoint using $y + hk_2/2$
$k_4$ ------------------------------------------------------ at the end of the interval using $y + hk_3$

# Example

Consider an ordinary differential equation $dy/dx = x^2 + y^2$, $y(1) = 1.2$. Find $y(1.05)$ using the fourth order RK4 method.

$f(x, y) = x^2 + y^2$

$x_0 = 1$ and $y_0 = 1.2$

$h = 0.05$

$k_1 = hf(x_0, y_0)$

$= (0.05)\ [x_0^{\ 2} + y_0^{\ 2}]$

$= (0.05)\ [(1)2 + (1.2)2]$

$= (0.05)\ (1 + 1.44)$

$= (0.05)(2.44)$

$= 0.122$

$k_2 = hf[x_0 + (½)h, y_0 + (½)k_1]$

$= (0.05)\ [f(1 + 0.025, 1.2 + 0.061)]$

$\{h/2 = 0.05/2 = 0.025$ and $k_1/2 = 0.122/2 = 0.061\}$

$= (0.05)\ [f(1.025, 1.261)]$

$= (0.05)\ [(1.025)2 + (1.261)2]$

$= (0.05)\ (1.051 + 1.590)$

$= (0.05)(2.641)$

$= 0.1320$

# The Lorenz Equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

where σ, *ρ* and *β* > 0 are parameters they depend on conditions like the fluid, the heat input, the size of the pot, etc.

As per Lorenz, there are three quantities that characterize the state of the fluid:

x: the rate of convective motion i.e. how fast the rolls are rotating,

y: the temperature difference between the ascending and descending currents, and

z: the distortion of the vertical temperature profile.

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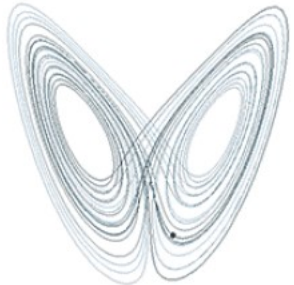

A sample solution in the Lorenz attractor when $\rho = 28$, $\sigma = 10$, and $\beta = 8/3$

- The Lorenz equations were published in 1963 by Edward N. Lorenz (a meteorologist and mathematician).
- They were derived to model some of the unpredictable behavior of weather.
- They represents a simple model of the unpredictability of weather based on a system of non-linear ordinary differential equations.
- The Lorenz model is deterministic i.e. if you know the exact starting values of the variables then in theory you can determine their future values as they change with time.