Consider a problem of an axially loaded elastic bar as shown in Figure 5.10. Dimensions are in meters. Solve for the unknown displacement and stresses with a finite element (nel=3, nel=1) mesh consisting of a single three-node element ($n_{en}=3$, $n_{el}=1$) as shown in Figure 5.11.

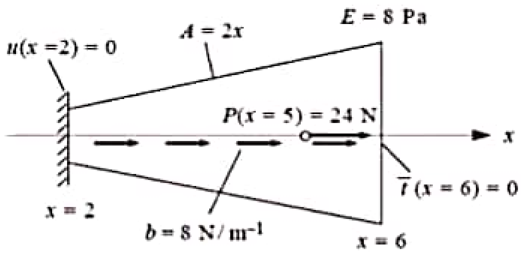


Figure 5.10 Geometry, loads and boundary conditions of Example 5.2.

- Element shape functions for three-node 1D elements

$$N_1^{(1)} = \frac{(x - x_2^{(1)})(x - x_3^{(1)})}{(x_1^{(1)} - x_2^{(1)})(x_1^{(1)} - x_3^{(1)})} = \frac{(x-4)(x-6)}{(-2)(-4)} = \frac{1}{8}(x-4)(x-6),$$

$$N_2^{(1)} = \frac{(x - x_1^{(1)})(x - x_3^{(1)})}{(x_2^{(1)} - x_1^{(1)})(x_2^{(1)} - x_3^{(1)})} = \frac{(x-2)(x-6)}{(2)(-2)} = -\frac{1}{4}(x-2)(x-6),$$

$$N_3^{(1)} = \frac{(x - x_1^{(1)})(x - x_2^{(1)})}{(x_3^{(1)} - x_1^{(1)})(x_3^{(1)} - x_2^{(1)})} = \frac{(x-2)(x-4)}{(4)(2)} = \frac{1}{8}(x-2)(x-4),$$

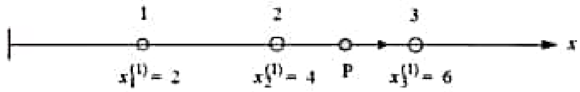
$$B_1^{(1)} = \frac{dN_1^{(1)}}{dx} = \frac{1}{4}(x-5), \qquad B_2^{(1)} = \frac{dN_2^{(1)}}{dx} = \frac{1}{2}(4-x), \qquad B_3^{(1)} = \frac{dN_3^{(1)}}{dx} = \frac{1}{4}(x-3),$$

$$\mathbf{B}^{(1)} = \frac{1}{4}[(x-5)\ (8-2x)\ (x-3)].$$

- Element stiffness

$$\mathbf{K}^{(1)} = \mathbf{K} = \int_{x_1}^{x_3} \mathbf{B}^{(1)T} A^{(1)} E^{(1)} \mathbf{B}^{(1)}\, dx = \int_2^6 \frac{1}{4}\begin{bmatrix}(x-5)\\(8-2x)\\(x-3)\end{bmatrix}(2x)(8)\frac{1}{4}[(x-5)\ (8-2x)\ (x-3)]\, dx$$

$$= \int_2^6 \begin{bmatrix} x(x-5)^2 & x(x-5)(8-2x) & x(x-5)(x-3) \\ x(8-2x)(x-5) & x(8-2x)^2 & x(8-2x)(x-3) \\ x(x-3)(x-5) & x(x-3)(8-2x) & x(x-3)^2 \end{bmatrix} dx.$$



It can be seen that the integrand is cubic ($p = 3$). So the number of quadrature points required for exact integration is $2n_{gp} - 1 \geq 3$, i.e. $n_{gp} \geq 2$, that is, two-point Gauss quadrature is adequate for exact integration of the integrand. The Jacobian is

$$J = \frac{b-a}{2} = 2.$$

Writing x in terms of ξ and transforming to the parent domain, we have

$$\int_2^6 f(x)\, dx = 2\int_{-1}^{1} f(x(\xi))\, d\xi = J\left[\underbrace{W_1}_{1} f(x(\xi_1)) + \underbrace{W_2}_{1} f(x(\xi_2))\right] = 2[f(x_1) + f(x_2)], \qquad (5.35)$$

where

$$x_1 = x(\xi_1) = 4 + 2\xi_1 = 4 + 2\left(-\frac{1}{\sqrt{3}}\right) = 2.8453,$$

$$x_2 = x(\xi_2) = 4 + 2\xi_2 = 4 + 2\left(\frac{1}{\sqrt{3}}\right) = 5.1547.$$

$$K_{11} = \int_2^6 x(x-5)^2\, dx = 2(2.8453(2.8453-5)^2 + 5.1547(5.1547-5)^2) = 26.667.$$

The stiffness matrix is given by

$$\mathbf{K} = \begin{bmatrix} 26.67 & -32 & 5.33 \\ & 85.33 & -53.33 \\ \text{sym} & & 48 \end{bmatrix} = \begin{bmatrix} 26.67 & -32 & 5.33 \\ -32 & 85.33 & -53.33 \\ 5.33 & -53.33 & 48 \end{bmatrix}.$$

• Force vector

$$\mathbf{f}_\Omega = \mathbf{f}_\Omega^{(1)} = \int_{x_1}^{x_3} \mathbf{N}^{e\mathrm{T}} b \, \mathrm{d}x + \underbrace{(\mathbf{N}^{e\mathrm{T}} P)|_{x=5}}_{\text{contribution from the point force}} . \tag{5.36}$$

$$\mathbf{f}_\Omega = \int_2^6 \begin{bmatrix} 0.125(x-4)(x-6) \\ -0.25(x-2)(x-6) \\ 0.125(x-2)(x-4) \end{bmatrix} \times 8 \, \mathrm{d}x + \begin{bmatrix} 0.125(x-4)(x-6) \\ -0.25(x-2)(x-6) \\ 0.125(x-2)(x-4) \end{bmatrix}_{x=5} \times 24.$$

Two-point Gauss quadrature is needed because the function is quadratic, so

$$\int_2^6 f(x) \, \mathrm{d}x = 2[f(x_1) + f(x_2)].$$

Thus,

$$\mathbf{f}_\Omega = 8 \begin{bmatrix} 2\left[N_1^{(1)}(x_1) + N_1^{(1)}(x_2)\right] \\ 2[N_2^{(1)}(x_1) + N_2^{(1)}(x_2)] \\ 2[N_3^{(1)}(x_1) + N_3^{(1)}(x_2)] \end{bmatrix} + \begin{bmatrix} 3(5-4)(5-6) \\ -6(5-2)(5-6) \\ 3(5-2)(5-4) \end{bmatrix}$$

$$= \begin{bmatrix} 2((2.8453-4)(2.8453-6) + (5.1547-4)(5.1547-6)) \\ -4((2.8453-2)(2.8453-6) + (5.1547-2)(5.1547-6)) \\ 2((2.8453-2)(2.8453-4) + (5.1547-2)(5.1547-4)) \end{bmatrix} + \underbrace{\begin{bmatrix} -3 \\ 18 \\ 9 \end{bmatrix}}_{\text{sum}=24}$$

$$= \underbrace{\begin{bmatrix} 5.33 \\ 21.33 \\ 5.33 \end{bmatrix}}_{8\cdot 4} + \underbrace{\begin{bmatrix} -3 \\ 18 \\ 9 \end{bmatrix}}_{24} = \begin{bmatrix} 2.33 \\ 39.33 \\ 14.33 \end{bmatrix}.$$

Note that the boundary force matrix vanishes, except for the reaction at node 1. Thus the RHS of (5.32) is:

$$\mathbf{f} + \mathbf{r} = \begin{bmatrix} r_1 + 2.33 \\ 39.33 \\ 14.33 \end{bmatrix}.$$

The resulting global system of equations is

$$\left[\begin{array}{c|cc} 26.67 & -32 & 5.33 \\ \hline & 85.33 & -53.33 \\ \text{sym} & & 48 \end{array}\right] \left[\begin{array}{c} 0 \\ \hline u_2 \\ u_3 \end{array}\right] = \left[\begin{array}{c} r_1 + 2.33 \\ \hline 39.33 \\ 14.33 \end{array}\right],$$

where we have partitioned the equations after the first row and column. The reduced system of equations are:

$$\mathbf{K}_F \mathbf{d}_F = \mathbf{f}_F - \underbrace{\mathbf{K}_{EF}^{\mathrm{T}} \bar{\mathbf{d}}_E}_{0} .$$

Solving the above

$$\begin{bmatrix} 85.33 & -53.33 \\ -53.33 & 48 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 39.33 \\ 14.33 \end{bmatrix} \Rightarrow \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2.1193 \\ 2.6534 \end{bmatrix}.$$

$r_1 = -56.005$

- Postprocessing

$$u = N_1^{(1)} u_1 + N_2^{(1)} u_2 + N_3^{(1)} u_3, \qquad \mathbf{d} = \mathbf{d}^{(1)} = \begin{bmatrix} 0 \\ 2.1193 \\ 2.6534 \end{bmatrix}.$$

$$u(x) = \frac{1}{8}(x-4)(x-6)(0) + \frac{-1}{4}(x-2)(x-6)(2.1193) + \frac{1}{8}(x-2)(x-4)(2.6534)$$
$$= -0.198\,15x^2 + 2.248\,55x - 3.7045.$$

- Stress field

$$\sigma(x) = E\frac{du}{dx} = E\frac{d}{dx}(\mathbf{N}^{(1)}\mathbf{d}^{(1)}) = E\mathbf{B}^{(1)}\mathbf{d}^{(1)}$$
$$= 8\frac{1}{4}[(x-5)\ (8-2x)\ (x-3)]\begin{bmatrix} 0 \\ 2.1193 \\ 2.6534 \end{bmatrix} = -3.17x + 17.99.$$

- Comparison of the FEM (solid line) and exact (dashed line) solutions