3.5.2. Sets and Indices

The model sets and indices are as follows:

I, J, K, V : The alias sets of all investable stocks.

i, j, k, v : The alias indices of the set of investable stocks.

T : The set of planning periods.

t : The index of the set of planning periods.

3.5.3. Decision Variables

The decision variables of the model are presented as follows:

$x_{i,t}$: The ratio of investment in the i^{th} stock to the total budget in the t^{th} period.

$x'_{i,t}$: The ratio of the purchased i^{th} stock to the total budget in the t^{th} period.

$x''_{i,t}$: The ratio of the i^{th} stock sold to the total budget in the t^{th} period.

$z_{i,t}$: A binary variable that equals one if the purchase is made and zero otherwise.

$y_{i,t}$: A binary variable that is defined for each share and indicates whether the i^{th} stock in the t^{th} period is present in the portfolio or not.

3.5.4. Parameters

The parameters of the model are presented as follows:

n: The number of risky assets that can be invested and selected in the portfolio.

$Budget_t$: The available budget for the investor in period t.

W_0: Available wealth at the beginning of period t.

$\widehat{r}_{i,t}$: The value of the predicted returns of the i^{th} stock in the t^{th} period.

$p_{i,t}\left(q_{i,t}\right)$: purchase (sales) price of each unit of the i^{th} stock in the t^{th} period.

$c_{i,t}$: transactional cost of i^{th} stock in the t^{th} period.

$\sigma^2_{i,t}$: The variance of the return of the i^{th} stock in the t^{th} period.

$\sigma_{i,j,t}$: Covariance between i^{th} and j^{th} stocks in the t^{th} period.

$S^3_{i,t}$: The skewness of the return of the i^{th} stock in the t^{th} period.

$S_{i,j,k,t}$: Co-skewness between i^{th},j^{th} , and k^{th} assets in the t^{th} period.

$K^4_{i,t}$: The kurtosis of the return of the i^{th} stock in the t^{th} period.

$K_{i,j,k,v,t}$: Co-kurtosis between i^{th}, j^{th}, k^{th}, and v^{th} assets in the t^{th} period.

$L_{i,t}$: The lower limit of the investment amount of the i^{th} stock in the t^{th} period.

$U_{i,t}$: The upper limit of the investment amount of the i^{th} stock in the t^{th} period.

K_{min}: The minimum number of assets that can be in the portfolio. (minimum number of assets in the stock portfolio).

K_{max}: The maximum number of shares that can be in the portfolio.

e_t: The degree of expected diversification of the portfolio in the t^{th} period.

3.5.5. Functions

$\widehat{r}_{pt}$: the predicted returns of the stock portfolio in t^{th} period.

C_t: Transaction cost of the portfolio in t^{th} period.

W_t: Expected wealth at the end of the t^{th} period.

$\sigma^2\left(\widehat{r}_{pt}\right)$: The variance of the predicted return of the portfolio in the t^{th} period.

$S^3\left(\widehat{r}_{pt}\right)$: The skewness of the predicted return of the portfolio in the t^{th} period.

$K^4\left(\widehat{r}_{pt}\right)$: The kurtosis of the predicted return of the portfolio in the t^{th} period.

$H_t(X)$: Entropy of investment of the t^{th} period.

In the following, the Equations related to the functions of the model are presented. The predicted returns of the portfolio in the t^{th} period without considering the transaction cost can be calculated through Eq. (16).

$$\widehat{r}_{pt} = \sum_{i=1}^{n} \widehat{r}_{it} x_{it}, t = 1, 2, \cdots, T \tag{16}$$

The transaction cost of the stock portfolio of the period t can be calculated through Eq. (17).

$$C_t = \sum_{i=1}^{n} c_{it} \times \left[\left|\frac{Budget_t \times x_{it}}{P_{it}} - \frac{Budget_{t-1} \times x_{it-1}}{P_{it-1}}\right| + \left|\frac{Budget_t \times x_{it}}{q_{it}} - \frac{Budget_{t-1} \times x_{it-1}}{q_{it-1}}\right|\right] t = 2, \cdots, T \tag{17}$$

The predicted returns of the portfolio in the t^{th} period by considering the transaction cost can be calculated through Eq. (18).

$$\widehat{r}_{pt} = \sum_{i=1}^{n} \widehat{r}_{it} x_{it} - \sum_{i=1}^{n} c_{it} \times \left[\left|\frac{Budget_t \times x_{it}}{P_{it}} - \frac{Budget_{t-1} \times x_{it-1}}{P_{it-1}}\right| + \left|\frac{Budget_t \times x_{it}}{q_{it}} - \frac{Budget_{t-1} \times x_{it-1}}{q_{it-1}}\right|\right] t = 2, \cdots, T \tag{18}$$

Eq. (19) applies to the final wealth of the t^{th} period.

$$W_t \geq W_{t-1}\left[1 + \widehat{r}_{pt}\right] \tag{19}$$

The variance of predicted return of the portfolio in the t^{th} period can be calculated using Eq. (20).

$$\sigma^2\left(\widehat{r}_{pt}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij,t} x_{it} x_{jt} \tag{20}$$

where $\sigma_{ij} = E\left[(R_i - m_i)\left(R_j - m_j\right)\right]$, R_s denotes the return on asset S, and. $E(R_s) = m_s$

The third moment of predicted return of the portfolio in the t^{th} period can be calculated using Eq. (21).

$$S^3\left(\widehat{r}_{pt}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} x_{it} x_{jt} x_{kt} s_{ijk,t} \tag{21}$$

$$s_{ijk} = E\left[(R_i - m_i)\left(R_j - m_j\right)(R_k - m_k)\right]$$

The fourth moment of predicted return of the portfolio in the t^{th} period can be calculated using Eq. (22).

$$K^4\left(\widehat{r}_{pt}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{v=1}^{n} x_{it} x_{jt} x_{kt} x_{vt} K_{ijkv,t} \tag{22}$$

$$k_{ijkv} = E\left[(R_i - m_i)\left(R_j - m_j\right)(R_k - m_k)(R_v - m_v)\right]$$

Entropy of investment in the t^{th} period can be calculated using Eq. (23).

$$H_t(X) = -\sum_{i=1}^{n} x_{it} ln x_{it} \geq e_t t = 1, \cdots, T 0 \leq e_t \leq \ln n \tag{23}$$

3.5.6. Mathematical Model

Finally, the mathematical model of the proposed multi objective multi period portfolio optimization is presented using the Eq. (24) to Eq. (40).

$$\max W_T = W_0 + \sum_{t=1}^{T}\left[\left[1+\sum_{i=1}^{n}\widehat{r}_{it}x_{it} - \sum_{i=1}^{n} c_{it}\times\left[\left|\frac{Budget_t \times x_{it}}{P_{it}} - \frac{Budget_{t-1}\times x_{it-1}}{P_{it-1}}\right| + \left|\frac{Budget_t\times x_{it}}{q_{it}} - \frac{Budget_{t-1}\times x_{it-1}}{q_{it-1}}\right|\right]\right]\right] \quad (24)$$

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij.t}x_{it}x_{jt} \quad (25)$$

$$\max \sum_{i=1}^{n}\sum_{j=1}^{n}\sum\nolimits_{k=1}^{n} s_{ijk.t}x_{it}x_{jt}x_{kt} \quad (26)$$

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n}\sum\nolimits_{k=1}^{n}\sum_{v=1}^{n} k_{ijkv.t}x_{it}x_{jt}x_{kt}x_{vt} \quad (27)$$

Subject to:

$$W_t \geq W_{t-1}\left[1+\sum_{i=1}^{n}\widehat{r}_{it}x_{it} - \sum_{i=1}^{n} c_{it}\times\left[\left|\frac{Budget_t\times x_{it}}{P_{it}} - \frac{Budget_{t-1}\times x_{it-1}}{P_{it-1}}\right| + \left|\frac{Budget_t\times x_{it}}{q_{it}} - \frac{Budget_{t-1}\times x_{it-1}}{q_{it-1}}\right|\right]\right] \quad (28)$$

$t = 1, \cdots, T$

$$x'_{it} = argMax(x_{it} - x_{it-1}, 0), i = 1,2,\cdots,n t = 2,\cdots,T \quad (29)$$

$$x''_{it} = |argMin(x_{it} - x_{it-1}, 0)|, i = 1,2,\cdots,n t = 2,\cdots,T \quad (30)$$

$$x'_{it} \bullet x''_{it} = 0 i = 1,2,\cdots,n t = 2,\cdots,T \quad (31)$$

$$z_{it} = \begin{cases} 1 x_{it} - x_{it-1} > 0 \\ 0 Oterwise \end{cases} i = 1,2,\cdots,n t = 2,\cdots,T \quad (32)$$

$$\sum_{i=1}^{n} c_{it}\times\left[z_{it}\times\left|\frac{Budget_t\times x_{it}}{P_{it}} - \frac{Budget_{t-1}\times x_{it-1}}{P_{it-1}}\right| + (1-z_{it})\times\left|\frac{Budget_t\times x_{it}}{q_{it}} - \frac{Budget_{t-1}\times x_{it-1}}{q_{it-1}}\right|\right] + \left|\frac{Budget_t\times x'_{it}}{P_{it}} - \frac{Budget_t\times x''_{it}}{q_{it}}\right| \leq Budget_t t = 1,\cdots,T \quad (33)$$

$$K_{min} \leq \sum_{i=1}^{n} y_{it} \leq K_{max} t = 1,\cdots,T \quad (34)$$

$$L_{it}y_{it} \leq x_{it} \leq U_{it}y_{it} i = 1,2,\cdots,n t = 1,\cdots,T \quad (35)$$

$$H_t(X) = -\sum_{i=1}^{n} x_{it} ln x_{it} \geq e_t t = 1,\cdots,T 0 \leq e_t \leq \ln n \quad (36)$$

$$\sum_{i=1}^{n} x_{it} = 1 t = 1,\cdots,T \quad (37)$$

$$x_{it} \geq 0 i = 1,2,\cdots,n t = 1,\cdots,T \quad (38)$$

$$y_{it} \in [0,1] i = 1,2,\cdots,n t = 1,\cdots,T \quad (39)$$

$$z_{it} \in [0,1] i = 1,2,\cdots,n t = 1,\cdots,T \quad (40)$$

3.5.7. Description of the mathematical model

Eq. (24) is the maximization of the investor's input wealth. Eq. (25), Eq. (26), and Eq. (27) presents the minimization of the variance, maximization of the skewness, and minimization of the kurtosis of the predicted return of the portfolio, respectively. Eq. (28) which is written for each period assures that the total wealth of the investors at the end of each period must be greater than or equal to the previous period. Eq. (29) and Eq. (30) which are written for each stock and each period present the buying and selling variables, respectively. Eq. (31) which is written for each stock and each period assures that buy and sell of stock i cannot be accomplished concurrently in the same period. Eq. (32) which is written for each stock and each period shows that if the purchase is made, z_{it} is equal to one and otherwise is equal to zero. Eq. (33) which is written for each period assures that the total costs of the portfolio do not violate the investor's budget. Eq. (34) which is written for each period controls the lower and upper bounds of cardinality of assets in the portfolio. Eq. (35) which is written for each stock and each period controls the lower and upper bounds of investment amount in the i^{th} stock in the i^{th} period. Eq. (36) which is written for each period controls diversification of the portfolio through entropy index. Eq. (37) which is written for each period assures that all the capital is invested. The Eq. (38) which is written for each stock and each period represents that short selling is not allowed. Eq. (39) and Eq. (40) defines the type of decision variables.

3.6. Solution approach: Goal programming method

Goal programming is used to solve the proposed model in this study. For solving model using goal programming, firstly, systemic constraints should be identified. Systemic constraints must be satisfied. Goal constraints are satisfied whenever possible. On the other hand, the decision maker settles for the closest available solution relative to the predetermined objective value. Goal constraints are expressed as $f(x) + d_P^- - d_P^+ = G_P$ where G_P denotes the desired level of goal for objective function $f(x)$. Unlike systemic constraints, goal constraints allow solutions to deviate from objective values. Therefore, d_P^- refers to the underachievement deviational variable and d_P^+ refers to over-achievement deviational variable. Finally, the objective of goal programming is to minimize total deviations (Bakhtavar et al., 2020; Ghoseiri, and Ghannadpour, 2010).

First, we solve four single-objective models to achieve the desired levels of the objectives (i.e., G1, G2, G3, and G4). Afterwards, instead of solving the original four-objective model, we solve the model (41) -(46). The new Objective function, i.e., Eq. (34) is formulated to minimize the total deviations from four goals. Eq. (42) to Eq. (45) denotes the goal constraints.

MIN= $\sum_{P=1}^{4}(d_P^- + d_P^+)$ (41).

Subject to:

$$W_T = W_0 + \sum_{t=1}^{T}\left[\left[1+\sum_{i=1}^{n}\widehat{r}_{it}x_{it} - \sum_{i=1}^{n} c_{it}\times\left[\left|\frac{Budget_t\times x_{it}}{P_{it}} - \frac{Budget_{t-1}\times x_{it-1}}{P_{it-1}}\right| + \left|\frac{Budget_t\times x_{it}}{q_{it}} - \frac{Budget_{t-1}\times x_{it-1}}{q_{it-1}}\right|\right]\right]\right] + d_1^- - d_1^+ = G1 \quad (42)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij.t}x_{it}x_{jt} + d_2^- - d_2^+ = G2 \quad (43)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum\nolimits_{k=1}^{n} s_{ijk.t}x_{it}x_{jt}x_{kt} + d_3^- - d_3^+ = G3 \quad (44)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum\nolimits_{k=1}^{n}\sum_{v=1}^{n} k_{ijkv.t}x_{it}x_{jt}x_{kt}x_{vt} + d_4^- - d_4^+ = G4 \quad (45)$$

Constraint of Eq. (28) to Eq. (40) (46).

where $d_P^-, P = 1,2,3,4$ refer to the underachievement deviational