Decomposition Via QP

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***Abstract*—This brief presents an algorithm for decoupling multivariable systems based on quadratic programming (QP). A single framework is presented which can be used to design centralized, decentralized, and sparse structures of arbitrary dynamical order. A worked example and a case study are presented to demonstrate the usage and performance. It is shown that only previous methods based on Evolutionary Algorithms are able to achieve slightly higher performance than the proposed algorithm. However, these minor improvements are outweighed by the huge increase in time and costs associated with evolutionary optimizations.**

***Index Terms*—Decentralized control, pre-compensation, quadratic programming, simply structured control, sparse control structure.**

NOTATION

$\mathcal{R}(s)^{m\times m}$	Set of $m \times m$ rational transfer functions.
$\bar{\Omega}$	A vector of frequency points.
$\Omega(j\omega)$	The vector Ω in raised powers—see (16).
$G(s)$	Stable LTI transfer function matrix.
$K(s)$	Polynomial pre-compensator matrix—see (2).
$\tilde{G}$	Frequency response array of $G(s)$—see (18) and (14).
$\tilde{\mathbf{k}}$	Vector comprising coefficients of $K(s)$—see (11) and (12).
O	Matrix of integers, $o_{lj}-1$ specifies the order of $k_{lj}(s)$—see (2).
Γ	Dominance ratio function—see (5).
P	Permutation matrix.
$\Re(A)$	The real part of A.
$\otimes$	The Kronecker product.

I. INTRODUCTION

ROSENBROCK's contribution to the design of control systems for linear multivariable plants inspired much activity in the development of techniques for achieving diagonal dominance [1]. The primary objective of all such techniques is to reduce plant interactions by the introduction of a multivariable pre-compensator. The control system design can then be completed by using classical techniques to synthesize a set of single-loop controllers for the compensated plant [2], [3]. In addition to their low order and simplicity, dominance-based controllers guarantee that for open-loop stable systems, the closed-loop will not become unstable by independent adjustment (including down to zero) of the single-loop controller gains, nor by any individual or collective sensor failure. For open-loop unstable systems the method provides loop-wise upper and lower gain bounds. Indeed, diagonal dominance has been shown to be a sufficient condition for decentralized integral controllability (DIC) [4].

The study of diagonal dominance and generalized Nyquist Stability are almost exclusively performed in a linear framework. Nonlinear effects such as input or output saturation limits are outside the scope of the framework, and likewise this brief. Traditional techniques developed for the achievement of diagonal dominance by the use of static pre-compensators are the pseudo-diagonalization [5], [6], the function-minimization method using conjugate-direction optimization [7], and the ALIGN algorithm developed initially in conjunction with characteristic-locus methods [8]. More recently improved techniques based on Evolution Strategies [9], $\mathcal{H}_2$-norm [10], and the $\mathcal{H}_\infty$-norm [11] have been proposed. Nevertheless there is an upper bound on the performance of static pre-compensation which is highly dependant on the plant's frequency response characteristics [12].

Dynamic pre-compensation promises greater performance but with added complications. For example while Chughtai and Munro [11] extend their static formulation to dynamic designs [13], the pre-compensator order will be very high. The same is true for the method proposed in [14] which allows design of decoupling controllers through Hadamard weighted $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimization. One of the more versatile recent approaches has been with the use of Evolutionary Algorithms [9]. An evolutionary optimization offers greater design flexibility. These include consideration of multiple plant models and setting each element of the pre-compensator to have a specified order. Alas, these user benefits are countered by two important obstacles; a huge computational effort, and the "curse of dimensionality".

This brief aims to draw upon the main benefits of the previous techniques to propose a practical and usable method for the design of "general" dynamic pre-compensators. General refers to the ability to specify any arbitrary order for any element of the precompensator, or to force any element to be completely zero (giving rise to a sparse structure). Since the problem is formulated as a QP it executes many orders of magnitude faster than an Evolutionary Algorithm. The combined power of the fast execution time, with the ability to choose the structure of the pre-compensator wholly arbitrarily, and the possibility to consider several plant models simultaneously, makes this approach

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stand-out among the family of tools developed for the achievement of dominance.

The remainder of this brief is organized as follows; Section II-A states the main problem which is then formed as a QP optimization problem in Section II-B. Section II-C demonstrates how the algorithm may be easily augmented when the same pre-compensator is required to achieve dominance for a set of multiple models. In Section III-A a worked example is presented followed by a case study on the Spey Rolls-Royce gas turbine engine in Section III-B. This brief is concluded in Section IV.

II. Quadratic Program Approach for Dynamic Decomposition

A. Problem Statement

Consider a stable LTI system $G(s) = [g_{ij}(s)] \in \mathcal{R}(s)^{m\times m}$. The design problem is to find a dynamic precompensator $K(s)$ such that $Q(j\omega) = G(j\omega)K(j\omega)$ is dominant over a set of frequencies $\bar{\Omega} = \{\omega_k : k = 1, \ldots, N\}$ [5], where

$$q_{ij}(j\omega) = \sum_{l=1}^{m} g_{il}(j\omega)k_{lj}(j\omega). \qquad (1)$$

The pre-compensator $K(s)$ is defined as

$$\begin{aligned} k_{lj}(s) &= \sum_{r=1}^{o_{lj}} \alpha_{lj(r)} s^{r-1} \\ &= \alpha_{lj(o_{lj})} s^{o_{lj}-1} + \alpha_{lj(o_{lj}-1)} s^{o_{lj}-2} + \ldots + \alpha_{lj(1)} \end{aligned} \qquad (2)$$

where $O = [o_{lj}] \in \mathbb{N}^{m\times m}$ is a matrix of integers. If $o_{lj} = 0$, then (2) becomes an empty sum and thus $k_{lj}(s) = 0$. Otherwise $o_{lj} - 1$ determines the order of the (l, j) element of the pre-compensator $K(s)$. We therefore seek $K(s)$ such that the off-diagonal terms of $Q(j\omega)$, i.e.,

$$\sum_{k=1}^{N} \sum_{\substack{i,j=1 \\ j\neq i}}^{m} |q_{ij}(j\omega_k)|^2 \qquad (3)$$

are minimized. If $G(s)$ is not functional controllable [15] the problem is not well defined. When $G(s)$ is functional controllable, one only has to ensure that the null solution is made infeasible. This is usually achieved by imposing a constraint on $K(s)$. In the original pseudo-diagonalization algorithm (for static pre-compensators [6]) the condition was

$$\sum_{i=1}^{m} k_{ij}^2 = 1, \quad \forall j. \qquad (4)$$

Consider the dominance ratio of the j^{th} column of $Q(s)$ defined as

$$\Gamma(GK, \Omega) = \sum_{k=1}^{N} \frac{\sum_{\substack{i=1 \\ i\neq j}}^{m} |q_{ij}(j\omega_k)|}{|q_{jj}(j\omega_k)|}. \qquad (5)$$

It is easy to verify that $\Gamma(GK, \Omega) = \Gamma(GKD, \Omega)$, where $D = \mathrm{diag}(d_1, \ldots, d_m)$. In a practical design study, D would assume the role of the diagonal loop shaping controller. This property ensures that tuning of the loop controllers will not alter (and possibly destroy) the dominance achieved by K. Therefore one may freely scale the columns of any K which satisfies (4) so that

$$k_{jj} = 1. \qquad (6)$$

Conversely (3) may be minimized subject to (6), still yielding the same optimum non-trivial value of (5). This latter constraint was used in [10] to formulate the $\mathcal{H}_2$-norm minimization pre-compensator design approach. When considering dynamic pre-compensators, the situation remains the same except that column scaling cannot be used to drive all polynomial coefficient to zero at the same time as ensuring the static term is 1. This requires that the off-diagonal terms be rational instead of polynomial functions of complex frequency s. Nevertheless, scaling the j^{th} column of $K(s)$ by a scalar factor of $k_{jj}(0)^{-1}$ will always bring it to the form in which $\alpha_{jj(1)} = 1$ where $\alpha_{ij(p)}$ is the coefficient of $(p-1)^{th}$ term (s^{p-1}) of the polynomial $k_{ij}(s)$ (see (2)). In resume, the problem considered in this brief is

$$\min_{K(s)} \sum_{k=1}^{N} \sum_{\substack{i,j=1 \\ j\neq i}}^{m} |q_{ij}(j\omega_k)|^2 \qquad (7)$$

subject to

$$\alpha_{jj(1)} = 1 \qquad (8)$$

where the pre-compensator is dynamic and defined according to (2).

B. QP Optimization Problem

In [10] it was shown that solving (7) for a static K can be represented as a $\mathcal{H}_2$-norm minimization. The minimization itself was then solved using LMIs. However, using LMIs for the design of dynamic pre-compensators relieves the designer from the ability to chose arbitrary order for each element, or to set them to zero. As a design framework, QP is not as powerful as LMIs. Nevertheless a key advantage of using QP in this case is that the optimization problem involving a $K(s)$ of arbitrary order can be represented as a QP with a globally optimal solution.

Theorem 1: Let $G(s)$ be a stable LTI system $G(s) = [g_{ij}(s)] \in \mathcal{R}(s)^{m\times m}$. Let $K(s) = [k_{ij}(s)] \in \mathcal{P}(s)^{m\times m}$ be polynomial pre-compensator matrix

$$k_{lj}(s) = \sum_{r=1}^{o_{lj}} \alpha_{lj(r)} s^{r-1} \qquad (9)$$

where $O = [o_{lj}] \in \mathbb{N}^{m\times m}$ is as defined previously. Then

$$\sum_{i,j=1, j\neq i}^{m} |q_{ij}(j\omega)|^2 = \left\| \tilde{\mathbf{k}} \bar{G}(j\omega) \right\|_2^2 \qquad (10)$$

where $Q(j\omega)$ is as defined in (1). In (10), $\tilde{\mathbf{k}}$ is defined as follows:

$$\tilde{\mathbf{k}} = [\tilde{k}_1, \ldots, \tilde{k}_m] \in \mathbb{R}^{1\times[O_m m^2]} \qquad (11)$$

where

$$\tilde{k}_j = \left\{[0^{1\times O_m - o_{1j}}, \alpha_{1j}], \ldots, [0^{1\times O_m - o_{mj}}, \alpha_{mj}]\right\} \in \mathbb{R}^{1\times mO_m}, \quad O_m = \max_{i,j} o_{ij} \tag{12}$$

$$\alpha_{lj} = \left(\alpha_{lj(o_{lj})}, \alpha_{lj(o_{lj}-1)}, \ldots, \alpha_{lj(1)}\right) \in \mathbb{R}^{1\times o_{lj}}. \tag{13}$$

In (10), $\bar{G}(j\omega)$ is defined as

$$\bar{G}(j\omega) = \mathrm{diag}\left\{M_1(j\omega), \ldots, M_m(j\omega)\right\} \in \mathbb{C}^{m^2 O_m \times m(m-1)} \tag{14}$$

where

$$M_i(j\omega) = \left(G(j\omega)\otimes\Omega(j\omega)\right)^T |_i \ \in \mathbb{C}^{mO_m\times(m-1)}, \quad i = 1, \ldots, m \tag{15}$$

$$\Omega(j\omega) = \left(j\omega^{O_m-1}, j\omega^{O_m-2}, \ldots, 1\right) \in \mathbb{C}^{1\times O_m}. \tag{16}$$

Construction of (15) corresponds to removing the diagonal entries of $Q(s)$ from the minimization as required by (7).

Proof:

$$\begin{aligned}
&\sum_{i,j=1,j\neq i}^{m} |q_{ij}(j\omega)|^2 \\
&= \sum_{i,j=1,j\neq i}^{m} \left|\sum_{l=1}^{m} g_{il}(j\omega)\sum_{r=1}^{o_{lj}}\alpha_{lj(r)}s^{r-1}\right|^2 \\
&= \sum_{i,j=1,j\neq i}^{m} \left|\sum_{l=1}^{m} g_{il}(j\omega)\left(\alpha_{lj(o_{lj})}j\omega^{o_{lj}-1} + \alpha_{lj(o_{lj}-1)}j\omega^{o_{lj}-2} + \ldots + \alpha_{lj(1)}\right)\right|^2 \\
&= \sum_{i,j=1,j\neq i}^{m} \left|\sum_{l=1}^{m} \left[0^{1\times(O_m-o_{lj})},\ \alpha_{lj}\right]\Omega(j\omega)^T g_{il}(j\omega)\right|^2 \\
&= \sum_{i,j=1,j\neq i}^{m} \left|\sum_{l=1}^{m} \tilde{k}_{lj}\Omega(j\omega)^T g_{il}(j\omega)\right|^2 \\
&= \sum_{i,j=1,j\neq i}^{m} \left|\sum_{l=1}^{m} \tilde{k}_{lj}\left(g_{il}(j\omega)\otimes\Omega(j\omega)^T\right)\right|^2 \\
&= \sum_{i,j=1,j\neq i}^{m} \left|\tilde{k}_j\left[(g_{i1}(j\omega) \quad \ldots \quad g_{im}(j\omega))\otimes\Omega(j\omega)\right]^T\right|_2^2 \\
&= \sum_{i,j=1,j\neq i}^{m} \left\|\tilde{k}_j\left(g_i(j\omega)\otimes\Omega(j\omega)\right)^T\right\|_2^2 \\
&= \sum_{j=1}^{m} \left\|\tilde{k}_j\left[\left(G(j\omega)\otimes\Omega(j\omega)\right)^T|_j\right]\right\|_2^2 \\
&= \left\|(\tilde{k}_1, \ldots, \tilde{k}_m)\mathrm{diag}\left\{M_1(j\omega), \ldots, M_m(j\omega)\right\}\right\|_2^2 \\
&= \left\|\tilde{\mathbf{k}}\bar{G}(j\omega)\right\|_2^2.
\end{aligned} \tag{17}$$

■

Lemma 1: Let

$$\tilde{G} = \left[\bar{G}(j\omega_1), \bar{G}(j\omega_2), \ldots, \bar{G}(j\omega_N)\right]. \tag{18}$$

Then

$$\min_{K(s)}\sum_{k=1}^{N}\sum_{\substack{i,j=1\\j\neq i}}^{m}|q_{ij}(j\omega_k)|^2 = \min_{\tilde{\mathbf{k}}}\|\tilde{\mathbf{k}}\tilde{G}\|_2. \tag{19}$$

Proof: From (10)

$$\begin{aligned}
&\sum_{k=1}^{N}\sum_{\substack{i,j=1\\j\neq i}}^{m}|q_{ij}(j\omega_k)|^2 \\
&= \sum_{k=1}^{N}\left\|\tilde{\mathbf{k}}\bar{G}(j\omega_k)\right\|_2^2 \\
&= \left\|\tilde{\mathbf{k}}\bar{G}(j\omega_1)\right\|_2^2 + \left\|\tilde{\mathbf{k}}\bar{G}(j\omega_2)\right\|_2^2 + \ldots + \left\|\tilde{\mathbf{k}}\bar{G}(j\omega_N)\right\|_2^2 \\
&= \left\|\tilde{\mathbf{k}}\left[\bar{G}(j\omega_1), \bar{G}(j\omega_2), \ldots, \bar{G}(j\omega_N)\right]\right\|_2^2 \\
&= \|\tilde{\mathbf{k}}\tilde{G}\|_2^2.
\end{aligned} \tag{20}$$

■

It is now straightforward to set up the QP. First, note that

$$\|\tilde{\mathbf{k}}\tilde{G}\|_2^2 = \tilde{\mathbf{k}}\tilde{G}\tilde{G}^H\tilde{\mathbf{k}}^T. \tag{21}$$

In addition to the optimization free variables, $\tilde{\mathbf{k}}$ will also contain a series of $1s \in \mathbb{R}^{1\times m}$ (imposed by the constraint (8)) and a series of zeros (arising from elements which have order less than the maximum). Let P be a permutation matrix constructed as follows:

$$P = [e_{P_i}], \quad i = \{1, 2, 3\} \tag{22}$$

where e_{P_i} denotes the set of P_i standard basis vectors. P_1 is defined as $P_1 = [(i(m+1)-m)O_m], i = 1, \ldots, m$. To define P_2, let $\tilde{O} = [\tilde{o}_k]$, $k = 1, \ldots, m^2$, where $\tilde{o}_k = [1^{1\times O_m - o_{lj}}, 0^{o_{lj}}]$. Then

$$P_2 = [i], \quad \forall i \text{ s.t. } \tilde{O}_i > 0. \tag{23}$$

Finally P_3 is defined as

$$P_3 = \overline{\left(P_1 \bigcup P_2\right)\bigcap\{1, 2, \ldots, O_m\times m^2\}}. \tag{24}$$

Multiplication of $\tilde{k}$ with P will bring it into this form

$$\tilde{\mathbf{k}}' = \tilde{\mathbf{k}}P = (\underbrace{1, \ldots, 1, 0, \ldots, 0}_{u}, \underbrace{\alpha_{11}, \ldots, \alpha_{m1}, \alpha_{12}, \ldots, \alpha_{m2}, \alpha_{1m}, \ldots, \alpha_{mm}}_{y}) \tag{25}$$

where

$$u \in \mathbb{R}^{1\times\left(m+\sum_{l,j=1}^{m}O_m - o_{lj}\right)}, \quad y \in \mathbb{R}^{1\times\sum_{l,j=1}^{m}o_{lj}-m} \tag{26}$$

TABLE I
TABLE OF LAPLACE TRANSFORMS

Order matrix	Precompensator
$O^1 = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}$	$K^1 = \begin{pmatrix} s+1 & 0.01s & 1.612s+0.9247 & 2.834 \\ -0.1s & 1 & 0.2613s+0.02114 & 1.468s+1.539 \\ 2s+5 & 0.12 & 1 & 1.793s+0.2167 \\ -0.02 & s+1 & 0.04874 & 1.634s+1 \end{pmatrix}$
$O^2 = \begin{pmatrix} 2 & 2 & 3 & 3 \\ 2 & 1 & 3 & 3 \\ 2 & 1 & 3 & 3 \\ 1 & 2 & 3 & 3 \end{pmatrix}$	$K^2(s) = \begin{pmatrix} s+1 & 0.01s & 5s^2+1.5s+1 & s+5 \\ -0.1s & 1 & -0.005 & 3s^2+s+2 \\ 2s+5 & 0.12 & 2s+1 & 0.1s \\ -0.02 & s+1 & 0.05s+0.025 & 1 \end{pmatrix}$
$O^{sp} = \begin{pmatrix} 2 & 0 & 3 & 2 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$	$K^{sp}(s) = \begin{pmatrix} 3.25s+1 & 0 & 4.8s^2+1.23s+1.13 & 0.96s+5 \\ 0 & 1 & 0 & 3s^2+s+2 \\ 8.69s+5.5 & 0 & 1.89s+1 & 0 \\ 0 & 1s+1 & 0 & 1 \end{pmatrix}$

since for any permutation matrix $PP^T = I$, then from (21)

$$\tilde{\mathbf{k}}\tilde{G}\tilde{G}^H\tilde{\mathbf{k}}^T = \tilde{\mathbf{k}}' \underbrace{P^T\tilde{G}\tilde{G}^H P}_{\tilde{Q}} \tilde{\mathbf{k}}'^T = (u \quad y)\Re(\tilde{Q})\begin{pmatrix} u^T \\ y^T \end{pmatrix}$$
$$= u\tilde{Q}_{11}u^T + y\tilde{Q}_{21}u^T + u\tilde{Q}_{12}y^T + y\tilde{Q}_{22}y^T. \quad (27)$$

The first term, u, is constant and does not effect the minimization. Moreover as the $\mathcal{L}_2$ norm [left-hand side of (27)] is always real, then $\tilde{\mathbf{k}}'\Im(\tilde{Q})\tilde{\mathbf{k}}'^T$ will always be necessarily zero. In summary, problem (7) is solved by the following QP problem:

$$\min_y \frac{1}{2}y\tilde{Q}_{22}y^T + u\tilde{Q}_{12}y^T \quad (28)$$

subject to

$$A.y \le b, \;\; A_{eq}.y = b_{eq}, \;\; lb \le y \le ub. \quad (29)$$

Matrices A, A_{eq} and vectors b, b_{eq}, lb, and ub may be specified to enforce additional constraints such as a limit on the gains of $K(s)$. They may also be used to ensure that elements of $K(s)$ remains minimum phase. For second-order (or less) polynomials, a necessary and sufficient condition is for all the coefficients of the polynomial to have the same sign. If higher order pre-compensation is required, dominance can be archived by successive application of second-order minimum phase pre-compensators. A suitable choice to enforce for the sign of an element is the steady-state sign of its corresponding element in the non-minimum phase unconstrained $K(s)$. Nevertheless, note that multivariable zeros are not a subset of element zeros. Therefore, even if elements of $K(s)$ are minimum phase, it may nonetheless have multivariable right-half plane zeros.

C. Multi-Model Optimization

Where the plant characteristics change over a range of operating conditions, it will be necessary to embrace sets of multivariable plants in order to ensure robustness. Consider a nonlinear system which has been linearized at several points. At each operating point g, one will obtain a linear $G^g(s) \in \mathbb{R}^{m\times m}$. When the overall optimization cost is taken as the sum of interactions of individual operating points, the design problem can be easily converted into a standard problem similar to one described in Section II-B. To see this consider

$$\sum_k^N \left(\beta_1 \sum_{i,j=1,i\neq j}^m \left|q_{ij}(j\omega_k)^1\right|^2 + \dots + \beta_n \sum_{i,j=1,i\neq j}^m \left|q_{ij}(j\omega_k)^n\right|^2 \right)$$
$$= \left\{ \beta_1 \|\tilde{\mathbf{k}}\tilde{G}^1\|_2^2 + \dots + \beta_n \|\tilde{\mathbf{k}}\tilde{G}^n\|_2^2 \right\}$$
$$= \left\{ \beta_1 \tilde{\mathbf{k}}\tilde{G}^1\tilde{G}^{1^H}\tilde{\mathbf{k}}^T + \dots + \beta_n \tilde{\mathbf{k}}\tilde{G}^n\tilde{G}^{n^H}\tilde{\mathbf{k}}^T \right\}$$
$$= \tilde{\mathbf{k}} \left(\beta_1\tilde{G}^1, \dots, \beta_n\tilde{G}^n\right)\left(\beta_1\tilde{G}^1, \dots, \beta_n\tilde{G}^n\right)^H \tilde{\mathbf{k}}^T$$
$$= \tilde{\mathbf{k}}\hat{G}\hat{G}^H\tilde{\mathbf{k}}^T \quad (30)$$
$$(31)$$

where

$$\hat{G} = \left(\beta_1\tilde{G}^1, \dots, \beta_n\tilde{G}^n\right) \quad (32)$$

and β_g is the frequency weight for model $G^g(s)$. Hence the multiple model problem is equivalent to solving the standard problem with a modified plant according to (32).

III. EXAMPLES

A. Worked Example

This section considers the model of a heavily interacting 4 × 4 system $G_{ex}(s)$. The A, B, C, D matrices of the model are presented in Appendix I. The open-loop Nyquist Array of $G_{ex}(s)$ is shown in Fig. 1. All figures are plotted over the range of frequencies from 10^{-3} to 10^2 rads/s. The same range is used for the subsequent QP optimization. Fig. 1 verifies the presence of large open-loop interactions.

We shall begin by designing a first order pre-compensator. This is achieved by setting $o^1_{lj} = 2$. The superscript k in O^k

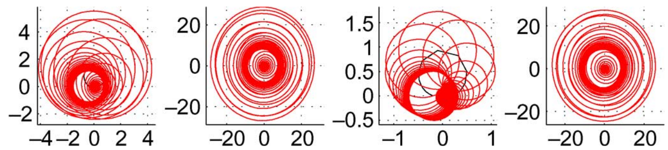

Fig. 1. Direct Nyquist Array of $G_{ex}(s)$ with column Gershgorin disks (only showing the diagonal entries).

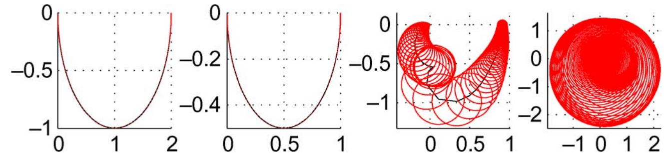

Fig. 2. Direct Nyquist Array of $G_{ex}(s)K^1(s)$ with column Gershgorin disks (only showing the diagonal entries).

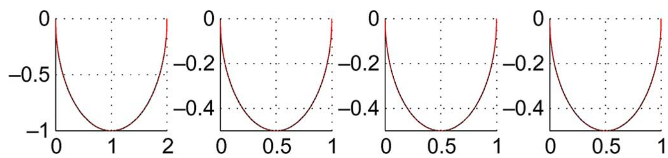

Fig. 3. Direct Nyquist Array of $G_{ex}(s)K^2(s)$ with column Gershgorin disks (only showing the diagonal entries).

denotes the order matrix at design iteration k. The resulting pre-compensator $K^1(s)$ is displayed in Table I. The Nyquist Array of $G_{ex}K^1(s)$ is shown in Fig. 2. The pre-compensator $K^1(s)$ has completely decoupled the first and second columns of $G_{ex}(s)$. Any further increase of dynamical order in the first two columns is futile. Moreover elements {(4,1), (2,2), (3,2)} are static despite the first order specification. Thus the order of these elements may be reduced to zero without effecting the decoupling performance. Accordingly the entries for elements {(4,1), (2,2), (3,2)} are reverted back to 1, and the elements of the last two columns are increased by one, as denotes in O^2. The resulting pre-compensator $K^2(s)$ completely decouples the system which is verified by Fig. 3. Once again we make the observation that not all elements of the last two columns are second order.

One may wish to explore the possibility of decoupling $G_{ex}(s)$ using a sparse structure. The optimal solution can point to a suitable choice of sparsity. Examination of elements of $K^2(s)$ reveals that the coefficients of elements $P = \{(2,1),(4,1),(3,2),(1,2),(4,3),(2,3),(3,4)\}$ are at least an order of magnitude smaller that the coefficients of the remaining elements. These elements are set to be zero (see O^{sp}) and the optimal sparse pre-compensator is recomputed. The Nyquist Array of $G_{ex}(s)K^{sp}(s)$ is shown in Fig. 4. As expected the interactions have increased compared to $K^2(s)$, but the increase pales into insignificance by considering that $K^{sp}(s)$ is highly sparse and contains less than 60% of the number of connections of $K^2(s)$.

B. Rolls-Royce Spey Engine

In this section the proposed QP based design will be applied to the model of the twin spool Rolls-Royce RB.168 Spey Mk.

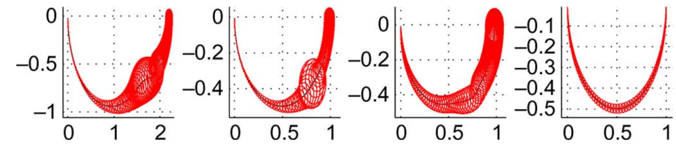

Fig. 4. Direct Nyquist Array of $G_{ex}(s)K^{sp}(s)$ with column Gershgorin disks (only showing the diagonal entries).

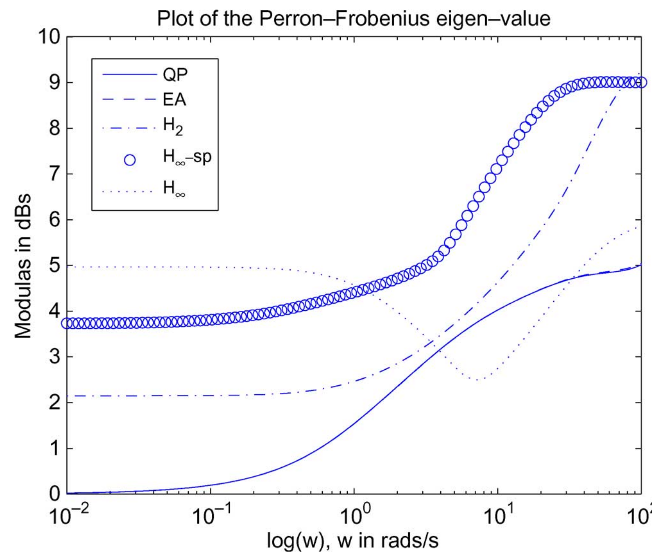

Fig. 5. Perron-Frobenius eigenvalue for static pre-compensator designs.

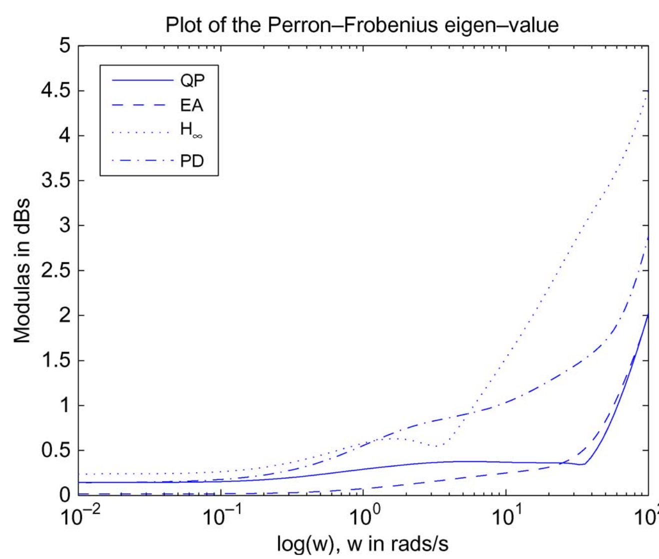

Fig. 6. Perron-Frobenius eigenvalue for dynamic pre-compensator designs.

202 Gas-Turbine engine. The Spey engine model has three inputs (*Fuel Flow, Inlet Guide Vanes, Nozzle Area*) and three outputs (*Low-Pressure Spool Speed, High-Pressure spool speed, Surge Margin*). The input and outputs are stated in the order by which they have been paired. The engine is highly non-linear and has been linearized at several operating points to result in a set of 21-state (engine plus actuators) LTI models. The model used for this example corresponds to the 85% High-pressure spool speed (NH). For more information in this model and previous case studies see [16]–[18].

There have been a number of previously reported results on the decoupling of this engine. They cover both static and dynamic designs. Since the QP method may be equally used to design static pre-compensators, two QP pre-compensators are designed; a static one and a dynamic one. The Perron Frobenius eigenvalue $\lambda_{pf}(j\omega)$ is used to measure the open-loop interactions of the system. For the significance of $\lambda_{pf}(j\omega)$ in system analysis see [19].

The static pre-compensator used for comparison purposes are designed using $\mathcal{H}_2$-norm LMI optimization [10] denoted by $K^{\mathcal{H}_2}$, $\mathcal{H}_\infty$-norm LMI optimization with and without the S-procedure [11] denoted respectively by $K^{\mathcal{H}_\infty}$ and $K^{\mathcal{H}_\infty-sp}$, Evolutionary Algorithms [9] denoted by K^{EA}, and the QP method proposed in this brief, denoted by K^{QP}. For the dynamic designs the methods used are the $\mathcal{H}_\infty$-norm approximate right inverse LMI optimization [20] denoted by $K_d^{\mathcal{H}_\infty}(s)$, the dynamic Pseudo-diagonalization [21] denoted by $K_d^{PD}(s)$, Evolutionary Algorithms [9] denoted by $K_d^{EA}(s)$ and the QP algorithm presented in this brief denoted by $K_d^{QP}(s)$.

The QP and EA methods allow the user to specify a frequency range of interest which has been set to $\Omega_d \in (10^{-2}, 10^2)$ rads/s. This range covers the main spectrum of interest and extends to above the natural bandwidth of the engine. The data for all the pre-compensators are presented in Appendix II (all dynamic pre-compensators have been column scaled to bring them into the normalized rational form). While the dynamic pre-compensators have similar dynamical order, due to the nature of each algorithm was not possible to ensure that they have precisely the same number of poles and zeros in each elements. Only the QP and EA designs share this characteristic (made possible by the fact that in the proposed QP approach any arbitrary choice of dynamics can be imposed). Nevertheless, the dimensions of the state-space matrices for the four designs are almost identical, with seven states for $K_d^{\mathcal{H}_\infty}(s)$ $K_d^{QP}(s)$ $K_d^{EA}(s)$ and eight states for $K_d^{PD}(s)$.

Fig. 5 shows the comparison plot of the Perron-Frobenius eigenvalue of the engine compensated with the various static designs. The larger the Perron-Frobenius eigenvalue, the more interactions are present in the system. The two worst performing static designs are $K^{\mathcal{H}_\infty}$ and $K^{\mathcal{H}_\infty-sp}$. This is not a surprise. The $\mathcal{H}_\infty$-norm is a worst-case norm and in its standard form will attempt to distribute the interactions across all frequencies (since it will minimize the peak value). This is the reason the Perron-Frobenius eigenvalue of $G(s)K^{\mathcal{H}_\infty}$ is roughly the same at both high and low frequencies. This is slightly improved by adoption of the S-procedure, which has the effect of reducing

$$A=\begin{pmatrix} -0.4824 & -0.4909 & -1.201 & -0.7466 & 0.05548 & -0.5519 & 0.5274 & -0.3957 & 2.11 & 0.8493 \\ -0.8667 & -2.155 & -0.241 & -0.6438 & 0.9091 & 0.9481 & 1.265 & 2.892 & 2.159 & -0.4327 \\ 0.2067 & 0.3694 & 0.03283 & -0.3465 & -0.4688 & 0.1483 & -0.8458 & -1.473 & 1.376 & 0.3067 \\ 0.2962 & -0.13 & 0.8214 & -0.2695 & -1.472 & -0.003819 & -1.082 & 1.322 & 0.6081 & 0.448 \\ 0.5143 & -0.2307 & 0.7053 & -0.496 & -2.129 & -0.2124 & -1.838 & 1.706 & 0.4911 & 1.105 \\ 0.3074 & -0.1772 & -0.313 & -0.1566 & 0.654 & 0.0431 & 0.6305 & -0.8722 & 0.5128 & 1.761 \\ -0.3355 & 0.1721 & -0.05654 & 0.1713 & 0.3204 & -0.06062 & -0.2437 & -0.9761 & 0.5608 & -1.002 \\ 0.004687 & -0.0002641 & 0.04078 & -0.002432 & -0.07369 & 0.005009 & -0.1001 & -0.5984 & 0.2225 & -0.04543 \\ -0.1011 & 0.05367 & 0.01737 & 0.05158 & 0.03695 & -0.01468 & 0.1508 & 0.03978 & -0.6329 & -0.353 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad (33)$$

$$B=\begin{pmatrix} -0.0143 & 0.3087 & 0.2384 & -0.4292 \\ 0.05574 & -0.2094 & 0.7917 & -0.2903 \\ -0.1431 & 0.5053 & -0.0225 & -0.3259 \\ -0.02573 & 0.2862 & 0.2544 & 0.3722 \\ 0.001328 & 0.4586 & 0.1451 & 1.052 \\ 0.2253 & 0.4523 & -0.1162 & -0.2193 \\ -0.409 & 0.3793 & -0.101 & 0.2811 \\ -0.2052 & 0.4418 & -0.4836 & -0.3789 \\ 0.1967 & -0.687 & -0.4051 & 0.2317 \\ -0.5731 & -0.6542 & 0.02155 & -0.03909 \end{pmatrix}$$

$$C=\begin{pmatrix} -0.05004 & -0.07916 & 0.05022 & -0.2724 & 0.09555 & \dots \\ 0.01217 & 0.05651 & -0.02893 & -0.01561 & -0.08384 & \dots \\ 0.07043 & -0.05712 & -0.1289 & 0.6074 & -0.324 & \dots \\ 0.05205 & -0.01726 & -0.04797 & -0.09695 & 0.04658 & \dots \\ \dots & -0.05014 & -0.1054 & -0.08775 & -0.1773 & 0.01954 \\ \dots & 0.2392 & 0.2735 & -0.1452 & 0.1161 & 0.003643 \\ \dots & 0.1234 & 0.1976 & -0.04584 & 0.1713 & -0.02045 \\ \dots & -0.09428 & -0.0335 & 0.03216 & -0.04912 & -0.02807 \end{pmatrix} \quad (34)$$

$$K^{\mathcal{H}_2} = \begin{pmatrix} 1 & -0.0053 & 0.4336 \\ -129.4379 & 1 & -40.2740 \\ 0.5622 & -0.0158 & 1 \end{pmatrix}, \quad K^{\mathcal{H}_\infty} = \begin{pmatrix} 1 & 0.0073412 & 0.38368 \\ -49.166 & 1 & -48.286 \\ 0.2595 & -0.0018075 & 1 \end{pmatrix} \tag{35}$$

$$K^{\mathcal{H}_\infty - sp} = \begin{pmatrix} 1 & -0.014561 & 0.43388 \\ -97.458 & 1 & -39.35 \\ 2.0604 & -0.029747 & 1 \end{pmatrix}, \quad K^{QP} = \begin{pmatrix} 1 & 0.00065197 & 0.44136 \\ -188.75 & 1 & -40.374 \\ -3.1793 & -0.00043268 & 1 \end{pmatrix} \tag{36}$$

$$K^{EA} = \begin{pmatrix} 1 & 0.00072755 & 0.44136 \\ -189.07 & 1 & -40.374 \\ -3.1971 & -0.00027284 & 1 \end{pmatrix} \tag{37}$$

$$K_d^{PD}(s) = \begin{pmatrix} 0.11111 & 0.0072709\frac{(s+0.35)}{(s+3)(s+19)} & 0.0063889\frac{(s+70)}{(s+36)} \\ -38.5802\frac{(s+5)}{(s+9)} & 0.052632 & -1.4583\frac{(s+28)}{(s+36)} \\ -0.080247\frac{(s+40)}{(s+9)} & -0.004627\frac{(s+0.5)}{(s+8)(s+19)} & 0.027778 \end{pmatrix} \tag{38}$$

$$K_d^{QP}(s) = \begin{pmatrix} 0.26208 & 0.002442\frac{(s+0.6638)}{(s+2.539)(s+22.89)} & 0.0063344\frac{(s+69.67)}{(s+35.02)} \\ -66.6101\frac{(s+2.696)}{(s+3.816)} & 0.017206 & -1.5587\frac{(s+25.9)}{(s+35.02)} \\ -0.043511\frac{(s+61.04)}{(s+3.816)} & -4.0696e-005\frac{(s+8.802)(s+14.63)}{(s+2.539)(s+22.89)} & 0.028558 \end{pmatrix} \tag{39}$$

$$K_d^{EA}(s) = \begin{pmatrix} 0.24006 & 0.003397\frac{(s+0.213)}{(s+26.03)(s+1.867)} & 0.0067025\frac{(s+65.85)}{(s+33.97)} \\ -62.184\frac{(s+2.997)}{(s+4.166)} & 0.020581 & -1.6038\frac{(s+25.16)}{(s+33.97)} \\ -0.00062174\frac{(s+4988)}{(s+4.166)} & -3.8985e-005\frac{(s+23.11)(s+0.3258)}{(s+26.03)(s+1.867)} & 0.029442 \end{pmatrix} \tag{40}$$

$$K_d^{\mathcal{H}_\infty}(s) = \begin{pmatrix} 1428.5714\frac{(s+0.7)}{(s+1000)} & 1.6149\frac{(s+100)(s+0.3)}{(s+1000)(s+40)} & 8.805\frac{(s+50)}{(s+1000)} \\ -247694.0382\frac{(s+1)}{(s+1000)} & 666.6667\frac{(s+1.5)}{(s+1000)} & -1010.0629\frac{(s+40)}{(s+1000)} \\ -978.6277\frac{(s+7)}{(s+1000)} & -1.0093\frac{(s-0.6)}{(s+1000)} & 16.6667\frac{(s+60)}{(s+1000)} \end{pmatrix} \tag{41}$$

the interactions at the low frequency regions at the expense of the higher frequency regions. The next best design is $K^{\mathcal{H}_2}$. Although this pre-compensator outperforms the $\mathcal{H}_\infty$ designs, it essentially still suffers from the problem associated with the $\mathcal{H}_\infty$-norm. Both of these norms will take into account all frequencies from 0 to $+\infty$ which extends substantially beyond the range of interest represented by Ω_d. Only the QP and the EA methods allow the user to specify a frequency range of interest. Not surprisingly, these two designs are significantly better than the norm minimization based designs. Fig. 5, and an examination of the pre-compensator data presented in Appendix II reveal that the QP and EA designs are almost identical, suggesting that for the static case, the QP solution is extremely close to the globally optimum design.

The results of the dynamic designs are presented in Fig. 6. Quite clearly, the QP and EA designs are the superior choices. The EA design performs exceptionally well at low frequencies. The QP design is only slightly behind in most frequencies. At the same time, the minor advantage of the EA design needs to be counterweighted against is massive computational cost. For this example, the CPU clock times for the EA computation were three orders of magnitude more than the QP design. This difference will only increase with system dimensions since it is well known that Evolutionary Algorithm optimization problems do not scale well whereas a large-scale QP is solved much more easily.

IV. Conclusion

This brief demonstrated that the problem of designing dynamic pre-compensators may be posed as a QP optimization problem. The more prominent features of the proposed methodology are the ability to:

- choose dynamical order of elements arbitrarily;
- specify a frequency range for the decomposition effort;
- set any cross-coupling channel of the pre-compensator to zero;
- incorporate data from multiple plant models in the design optimization.

Application of the algorithm on a real-life case study demonstrated that the QP-based pre-compensators perform exceptionally well, exceeded only by the EA based design. A benefit of the QP formulation in terms of future development is the possibility to incorporate the problem into a Vapnik Support Vector Machine [22] (to robustify the design against parameter changes).

Appendix I

The state-space data for $G_{ex}(s)$ are as shown in (33) and (34) at the bottom of the previous page.

Appendix II

See (35)–(41) at the top of the page.

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