Many-Particle Physics

Homework 1

- **Instructors**: Kargarian - Vaezi
- **Semester**: Fall 1401 (2022)
- **Handed out**: Monday 25-Mehr-1401
- **Due date**: Monday 09-Aban-1401

Notice: the solutions to most problems in this and other assignments in future may be found in standard textbooks or in webs. Feel free to use them but write the solutions in your own words and cite the references used throughout.

1. Solve the classical vibrational modes of a one-dimensional chain of atoms of type A and B. They alternate on the chain with masses m_A and m_B. The harmonic spring between atoms has spring constant K_s.
2. Write down the Hamiltonian you obtained in the above problem. Solve it, and show that it may be reduced to the form
$$H = \sum_{k\lambda} \omega_{k\lambda} \left(a^\dagger_{k\lambda} a_{k\lambda} + \frac{1}{2} \right), \tag{1}$$
where the $\omega_{k\lambda}$ are the classical normal modes.
3. Find the exact solution to
$$H = E_0 a^\dagger a + E_1 (a^\dagger a^\dagger + aa) \tag{2}$$
where E_0 and E_1 are constants and a and $a^\dagger$ are boson operators.
4. Consider a tight-binding one-dimensional solid which has alternate atoms of type A and B. Then electron Hamiltonian in the nearest -neighbor model has the form
$$H = \sum_i \left(A a^\dagger_i a_i + B b^\dagger_i b_i \right) + t \sum_i \left(a^\dagger_i b_{i+\delta} + b^\dagger_{i+\delta} a_i \right) \tag{3}$$
where a_i and b_i are electron operators. Find the exact eigenvalues of this Hamiltonian.
5. In class we derived the Hubbard model as
$$H = \sum_{ij} t_{ij} a^\dagger_i a_j + U \sum_i n_{i\uparrow} n_{i\downarrow}. \tag{4}$$
Consider the atomic limit of this Hamiltonian by setting $t_{ij} = 0$. Calculate the partition function
$$Z = \mathrm{Tr} e^{-\beta(H-\mu N)}. \tag{5}$$
Then evaluate the average number of electrons
$$\bar{N} = -\frac{\partial \Omega}{\partial \mu}, \tag{6}$$
where $\beta\Omega = -\ln Z$.
6. Explicitly verify the equation of continuity
$$\partial_t \rho(\mathbf{r}, t) + \boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{r}, t) = 0. \tag{7}$$
for a gas of free fermions particles with
$$H = \sum_{\mathbf{k}\alpha} \varepsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha}, \quad \rho(\mathbf{r}, t) = \frac{1}{V} \sum_{\mathbf{kq}\alpha} e^{-i\mathbf{k}\cdot\mathbf{r}} a^\dagger_{\mathbf{k}+\mathbf{q},\alpha} a_{\mathbf{k},\alpha}. \tag{8}$$
Find the expression for $\mathbf{j}(\mathbf{r}, t)$.

7. Consider a model consisting of two sites each having one orbital. There is a hopping integral for overlap between two orbitals. Suppose you want to describe this system by the Hubbard model.
   (a) Determine the basis you need to construct the Fock space.
   (b) What's is the size of the Hamiltonian matrix in this space.
   (c) What are the symmetries of the system which you can use to block diagonalize the Hamiltonian. Classify the basis based on the symmetries and write the Hamiltonian which each class.