STATE-SPACE REPRESENTATION OF DYNAMIC SYSTEMS

A dynamic system consisting of a finite number of lumped elements may be described by ordinary differential equations in which time is the independent variable. By use of vector-matrix notation, an nth-order differential equation may be expressed by a first-order vector-matrix differential equation. If n elements of the vector are a set of state variables, then the vector-matrix differential equation is a *state* equation. In this section we shall present methods for obtaining state-space representations of continuous-time systems.

State-space representation of nth-order systems of linear differential equations in which the forcing function does not involve derivative terms. Consider the following nth-order system:

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u \tag{3–34}$$

Noting that the knowledge of $y(0), \dot{y}(0), \ldots \overset{(n-1)}{y}(0)$, together with the input $u(t)$ for $t \geq 0$, determines completely the future behavior of the system, we may take $y(t)$, $\dot{y}(t), \ldots, \overset{(n-1)}{y}(t)$ as a set of n state variables. (Mathematically, such a choice of state variables is quite convenient. Practically, however, because higher-order derivative terms are inaccurate, due to the noise effects inherent in any practical situations, such a choice of the state variables may not be desirable.)

Let us define

$$
\begin{aligned}
x_1 &= y \\
x_2 &= \dot{y} \\
&\cdot \\
&\cdot \\
&\cdot \\
x_n &= \overset{(n-1)}{y}
\end{aligned}
$$

Then Equation (3–34) can be written as

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\cdot \\
&\cdot \\
&\cdot \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -a_n x_1 - \cdots - a_1 x_n + u
\end{aligned}
$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{3–35}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

The output can be given by

$$y = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

or

$$y = \mathbf{C}\mathbf{x} \tag{3–36}$$

where

$$\mathbf{C} = [1 \quad 0 \quad \cdots \quad 0]$$

[Note that D in Equation (3–27) is zero.] The first-order differential equation, Equation (3–35), is the state equation, and the algebraic equation, Equation (3–36), is the output equation. A block diagram realization of the state equation and output equation given by Equations (3–35) and (3–36), respectively, is shown in Figure 3–13.

Note that the state-space representation for the transfer function system

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

is given also by Equations (3–35) and (3–36).

State-space representation of nth-order systems of linear differential equations in which the forcing function involves derivative terms. If the differential equation of the system involves derivatives of the forcing function, such as

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \ldots + b_{n-1}\dot{u} + b_n u \qquad (3\text{–}37)$$

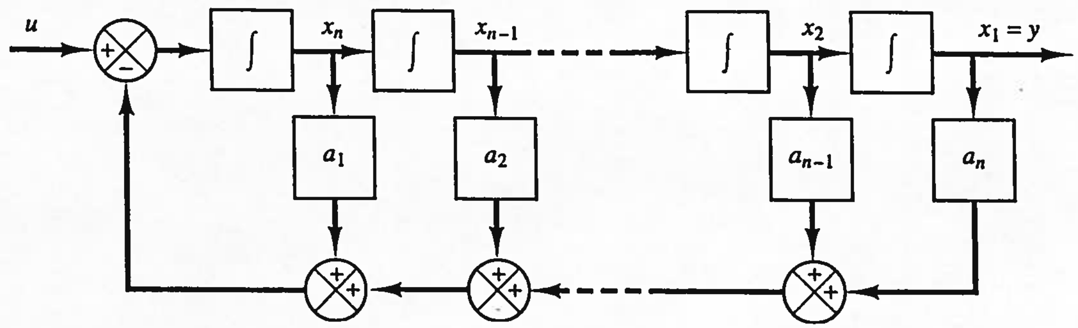
$$
\begin{aligned}
x_1 &= y - \beta_0 u \\
x_2 &= \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \\
x_3 &= \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u \\
&\vdots \\
x_n &= \overset{(n-1)}{y} - \beta_0 \overset{(n-1)}{u} - \beta_1 \overset{(n-2)}{u} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u
\end{aligned}
\tag{3–38}
$$

where $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ are determined from

$$
\begin{aligned}
\beta_0 &= b_0 \\
\beta_1 &= b_1 - a_1\beta_0 \\
\beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\
\beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 \\
&\vdots \\
\beta_n &= b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_n\beta_0
\end{aligned}
\tag{3–39}
$$

With this choice of state variables the existence and uniqueness of the solution of the state equation is guaranteed. (Note that this is not the only choice of a set of state variables.) With the present choice of state variables, we obtain

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \beta_1 u \\
\dot{x}_2 &= x_3 + \beta_2 u \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \beta_{n-1} u \\
\dot{x}_n &= -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u
\end{aligned}
\tag{3–40}
$$



$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \tag{3–43}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{3–41}$$

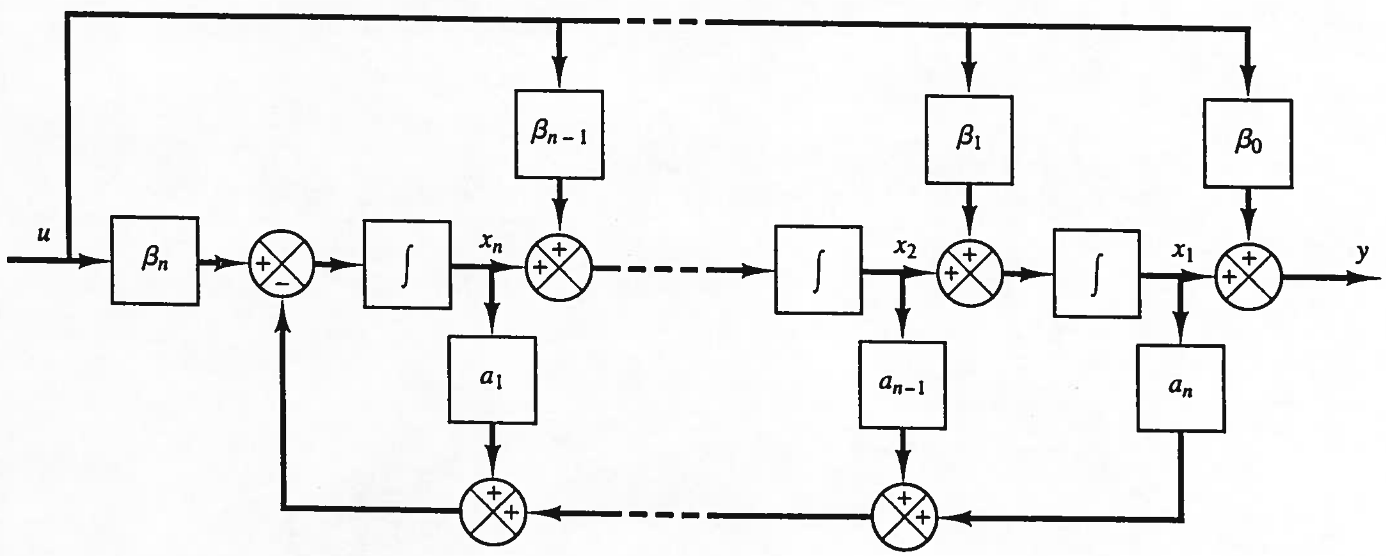
$$y = \mathbf{C}\mathbf{x} + Du \tag{3–42}$$

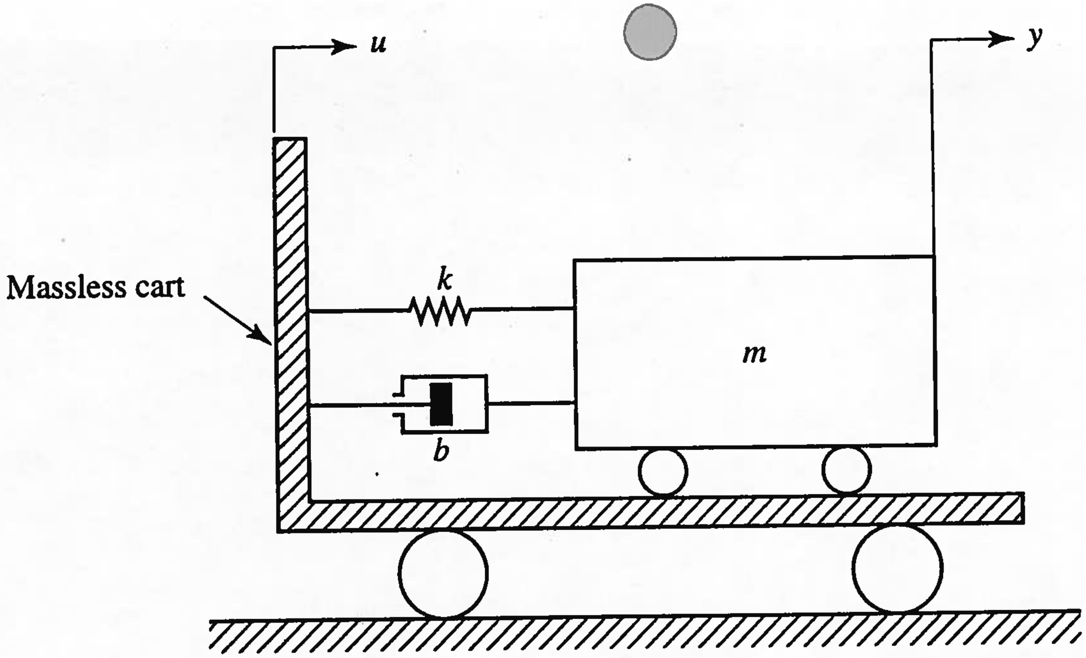
where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0 \quad \cdots \quad 0], \qquad D = \beta_0 = b_0$$

$$
\begin{aligned}
\beta_0 &= b_0 \\
\beta_1 &= b_1 - a_1\beta_0 \\
\beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\
\beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 \\
&\ \ \vdots \\
\beta_n &= b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_n\beta_0
\end{aligned}
\tag{3–39}
$$





$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

or

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

with the standard form

$$\ddot{y} + a_1\dot{y} + a_2y = b_0\ddot{u} + b_1\dot{u} + b_2u$$

and identify a_1, a_2, b_0, b_1, and b_2 as follows:

$$a_1 = \frac{b}{m}, \qquad a_2 = \frac{k}{m}, \qquad b_0 = 0, \qquad b_1 = \frac{b}{m}, \qquad b_2 = \frac{k}{m}$$

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = \frac{b}{m}$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2$$

Then, referring to Equation (3–38), define

$$x_1 = y - \beta_0 u = y$$

$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u$$

From Equation (3–40) we have

$$\dot{x}_1 = x_2 + \beta_1 u = x_2 + \frac{b}{m}u$$

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + \beta_2 u = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]u$$

and the output equation becomes

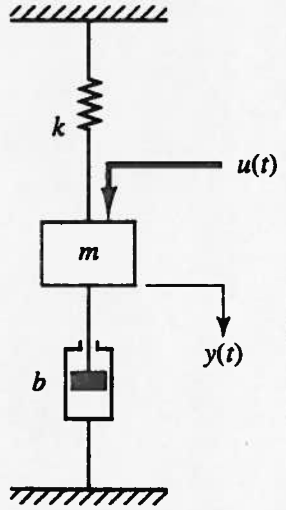
$$y = x_1$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{m} \\ \frac{k}{m} - \left(\frac{b}{m}\right)^2 \end{bmatrix} u \tag{3–45}$$

and

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3–46}$$



external force $u(t)$ is the input to the system, and the displacement $y(t)$ of the mass is the output. The displacement $y(t)$ is measured from the equilibrium position in the absence of the external force. This system is a single-input–single-output system.

From the diagram, the system equation is

$$m\ddot{y} + b\dot{y} + ky = u \tag{3–19}$$

This system is of second order. This means that the system involves two integrators. Let us define state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

Then we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

or

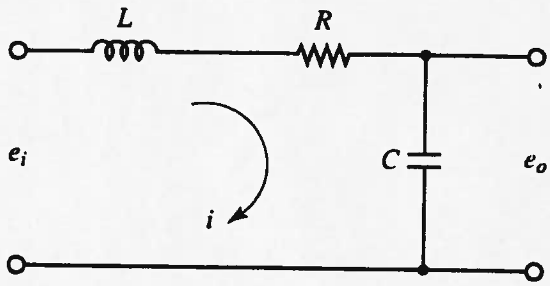
$$\dot{x}_1 = x_2 \tag{3–20}$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u \tag{3–21}$$

The output equation is

$$y = x_1 \tag{3–22}$$

In a vector-matrix form, Equations (3–20) and (3–21) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u \tag{3–23}$$



Electrical circuit.

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i$$

$$\frac{1}{C}\int i\,dt = e_o$$

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Ino.

Then by defining state variables by

$$x_1 = e_o$$
$$x_2 = \dot{e}_o$$

and the input and output variables by

$$u = e_i$$
$$y = e_o = x_1$$

we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{LC} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{LC} \end{bmatrix} u$$

and

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

These two equations give a mathematical model of the system in state space.

Example 3H Spring-coupled masses The equations of motion of a pair of masses M_1 and M_2 coupled by a spring, and sliding in one dimension in the absence of friction (see Fig. 3.7(a)) are

$$\begin{aligned} \ddot{x}_1 + \frac{K}{M_1}(x_1 - x_2) &= \frac{u_1}{M_1} \\ \ddot{x}_2 + \frac{K}{M_2}(x_2 - x_1) &= \frac{u_2}{M_2} \end{aligned} \tag{3H.1}$$

where u_1 and u_2 are the externally applied forces and K is the spring constant. Defining the state

$$x = [x_1 \quad x_2 \quad \dot{x}_1 \quad \dot{x}_2]'$$

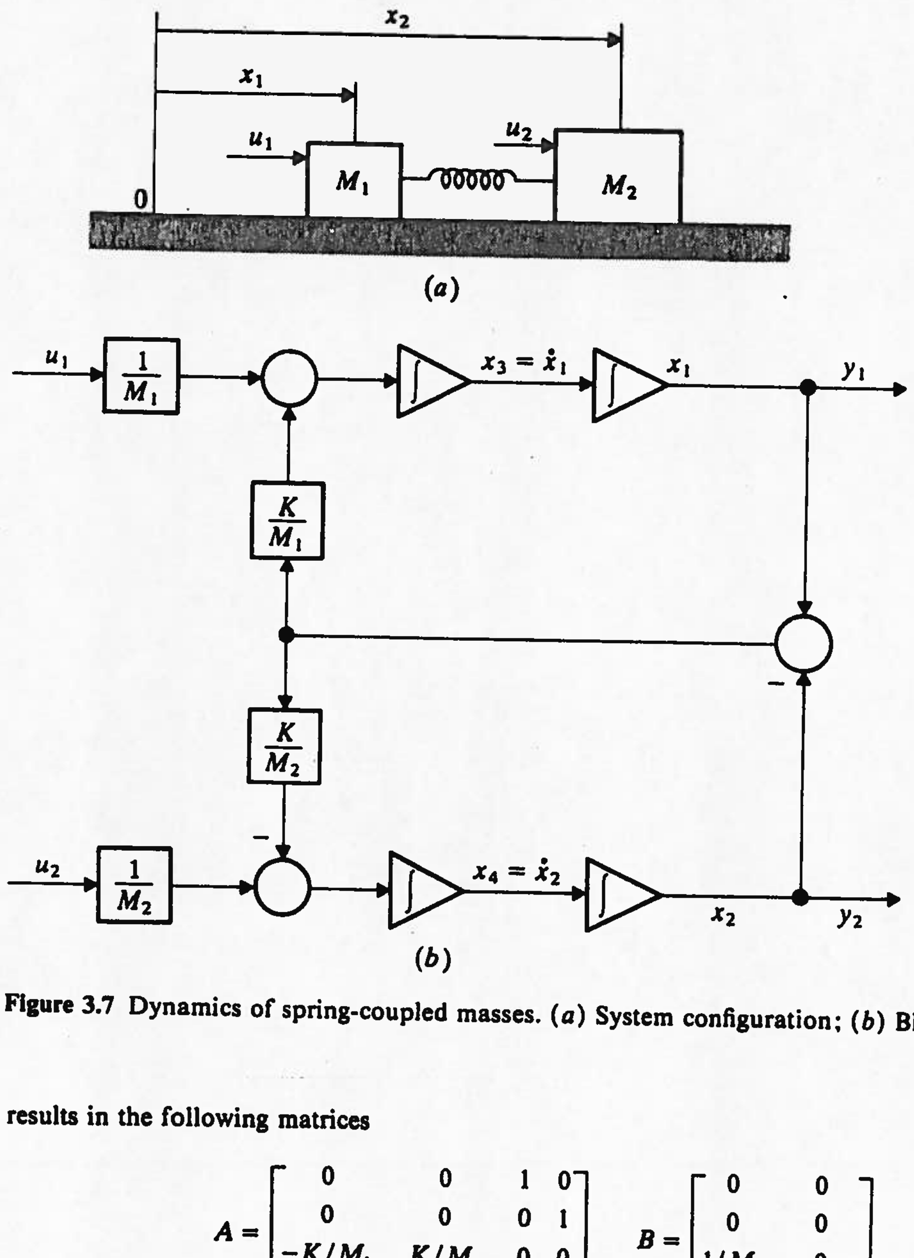


Figure 3.7 Dynamics of spring-coupled masses. (a) System configuration; (b) Block diagram.

results in the following matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K/M_1 & K/M_1 & 0 & 0 \\ K/M_2 & -K/M_2 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/M_1 & 0 \\ 0 & 1/M_2 \end{bmatrix} \tag{3H.2}$$

It is observed that the motion of the system is uniquely defined by the displacement of the cart from some reference point, and the angle that the pendulum rod makes with respect to the vertical.

The kinetic energy of the system is the sum of the kinetic energy of each mass. The cart is confined to move in the horizontal direction so its kinetic energy is

$$T_1 = \tfrac{1}{2}M\dot{y}^2$$

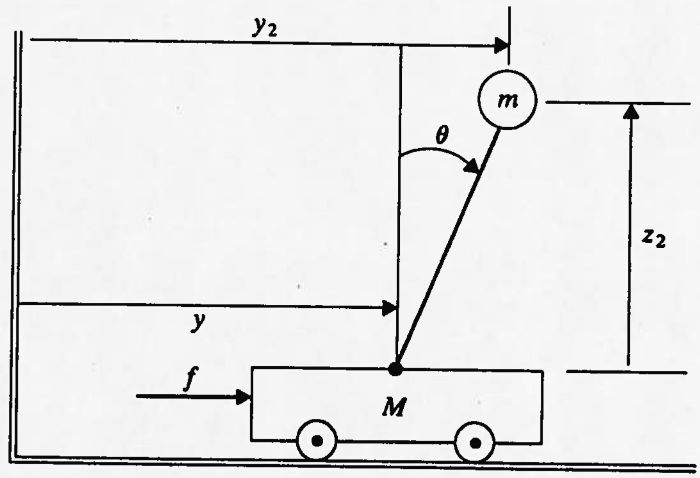


Figure 2.10 Inverted pendulum on moving cart.

The bob can move in the horizontal and in the vertical direction so its kinetic energy is

$$T_2 = \tfrac{1}{2}m(\dot{y}_2^2 + \dot{z}_2^2)$$

But the rigid rod constrains z_2 and y_2

$$y_2 = y + l \sin\theta \qquad \dot{y}_2 = \dot{y} + l\dot{\theta}\cos\theta$$
$$z_2 = l\cos\theta \qquad \dot{z}_2 = -l\dot{\theta}\sin\theta$$

Thus

$$\begin{aligned} T = T_1 + T_2 &= \tfrac{1}{2}M\dot{y}^2 + \tfrac{1}{2}m[(\dot{y} + l\dot{\theta}\cos\theta)^2 + l^2\dot{\theta}^2\sin^2\theta] \\ &= \tfrac{1}{2}M\dot{y}^2 + \tfrac{1}{2}m[\dot{y}^2 + 2\dot{y}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2] \end{aligned}$$

The only potential energy is stored in the bob

$$V = mgz_2 = mgl\cos\theta$$

Thus the lagrangian is

$$L = T - V = \tfrac{1}{2}(M+m)\dot{y}^2 + ml\cos\theta\,\dot{y}\dot{\theta} + \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \qquad \text{(2E.1)}$$

The generalized coordinates having been selected as (y, θ), Lagrange's equations for this system are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= f \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \end{aligned} \qquad \text{(2E.2)}$$

Now

$$\frac{\partial L}{\partial \dot{y}} = (M + m)\dot{y} + ml \cos\theta\dot{\theta}$$

$$\frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml \cos\theta\dot{y} + ml^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = mgl \sin\theta$$

Thus (2E.2) become

$$\begin{aligned} (M + m)\ddot{y} + ml \cos\theta\ddot{\theta} - ml\dot{\theta}^2 \sin\theta &= f \\ ml \cos\theta\ddot{y} - ml \sin\theta\dot{y}\dot{\theta} + ml^2\ddot{\theta} - mgl \sin\theta &= 0 \end{aligned} \tag{2E.3}$$

These are the exact equations of motion of the inverted pendulum on a cart shown in Fig. 2.10. They are nonlinear owing to the presence of the trigonometric terms $\sin\theta$ and $\cos\theta$ and the quadratic terms $\dot{\theta}^2$ and $\dot{y}\dot{\theta}$. If the pendulum is stabilized, however, then θ will be kept small. This justifies the approximations

$$\cos\theta \simeq 1 \qquad \sin\theta \simeq \theta$$

We may also assume that $\dot{\theta}$ and $\dot{y}$ will be kept small, so the quadratic terms are negligible. Using these approximations we obtain the linearized dynamic model

$$\begin{aligned} (M + m)\ddot{y} + ml\ddot{\theta} &= f \\ m\ddot{y} + ml\ddot{\theta} - mg\theta &= 0 \end{aligned} \tag{2E.4}$$

A state-variable representation corresponding to (2E.4) is obtained by defining the state vector

$$x = [y, \theta, \dot{y}, \dot{\theta}]'$$

Then

$$\begin{aligned} \frac{dy}{dt} &= \dot{y} \\ \frac{d\theta}{dt} &= \dot{\theta} \end{aligned} \tag{2E.5}$$

constitute the first two dynamic equations and on solving (2E.4) for $\ddot{y}$ and $\ddot{\theta}$, we obtain two more equations

$$\begin{aligned} \frac{d}{dt}(\dot{y}) &= \ddot{y} = \frac{f}{M} - \frac{mg}{M}\theta \\ \frac{d}{dt}(\dot{\theta}) &= \ddot{\theta} = -\frac{f}{Ml} + \left(\frac{M + m}{Ml}\right)g\theta \end{aligned} \tag{2E.6}$$

The four equations can be put into the standard matrix form

$$\dot{x} = Ax + Bu$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -mg/M & 0 & 0 \\ 0 & (M + m)g/Ml & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1/M \\ -1/Ml \end{bmatrix}$$

and

$$u = f = \text{external force}$$

A block-diagram representation of the dynamics (2E.5) and (2E.6) is shown in Fig. 2.11.

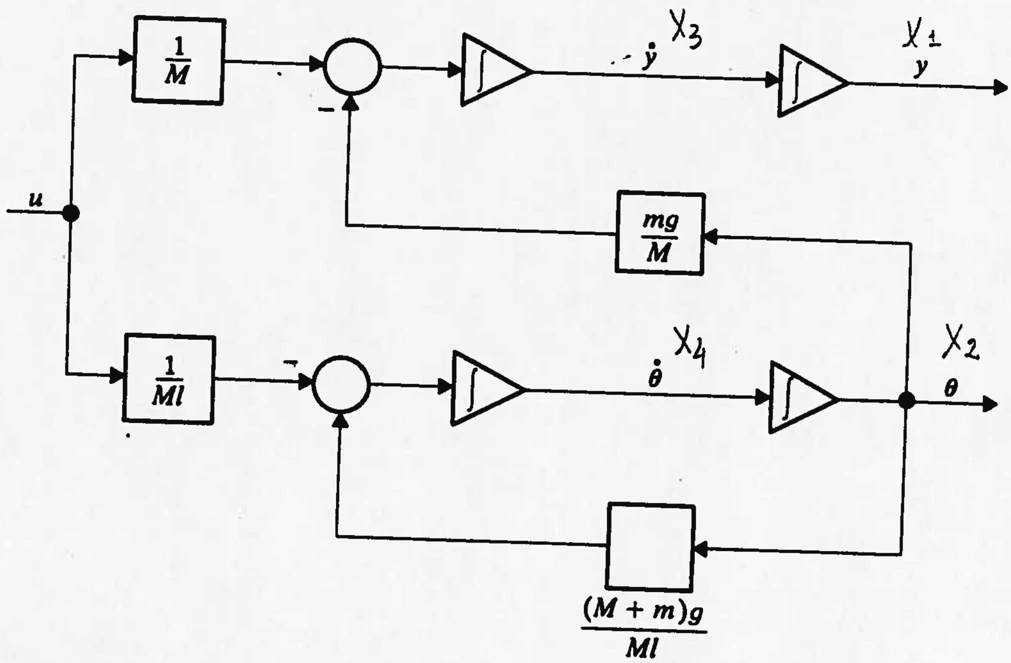


Figure 2.11 Block diagram of dynamics of inverted pendulum on moving cart.

A DC-motor basically works on the principle that a current carrying conductor in a magnetic field experiences a force $\mathbf{F} = \boldsymbol{\phi} \times \mathbf{i}$, where $\boldsymbol{\phi}$ is the magnetic flux and $\mathbf{i}$ is the current in the conductor. The motor itself consists of a fixed **stator** and a movable **rotor** that rotates inside the stator, as shown in Figure 7-2. If the stator produces a radial magnetic flux $\boldsymbol{\phi}$ and the current in the rotor (also called the **armature**) is $\mathbf{i}$ then there will be a torque on the rotor causing it to rotate. The magnitude of this torque is

$$\tau_m = K_1 \phi i_a \tag{7.2.2}$$

where τ_m is the motor torque (N-m), ϕ is the magnetic flux (webers), i_a is the armature current (amperes), and K_1 is a physical constant. In addition, whenever a conductor moves in a magnetic field, a voltage V_b is generated across its terminals that is proportional to the velocity of the conductor in the field. This voltage, called the **back emf**, will tend to oppose the current flow in the conductor.

Thus, in addition to the torque τ_m in (7.2.2), we have the back emf relation

$$V_b = K_2 \phi \omega_m \tag{7.2.3}$$

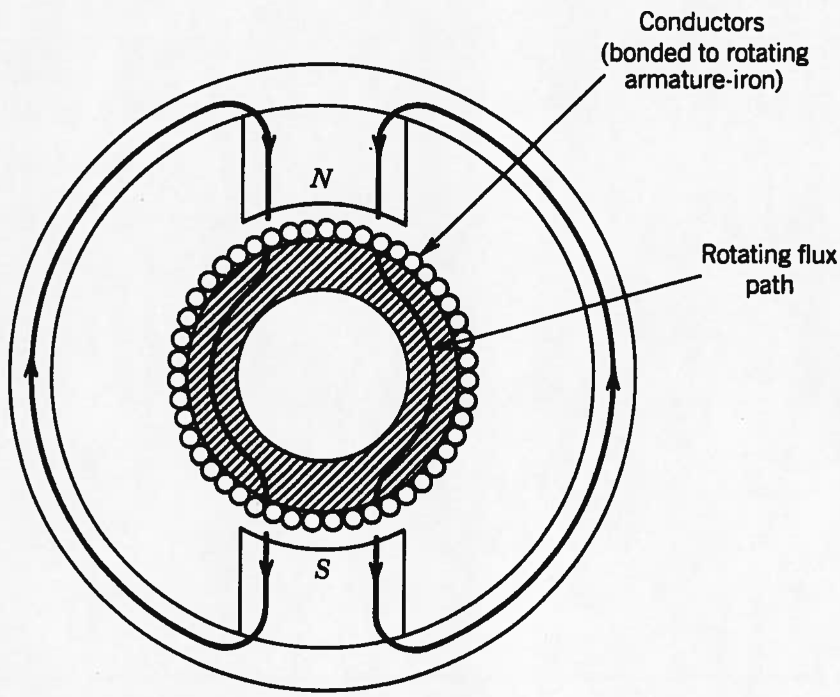


FIGURE 7-2
Cross-sectional view of a surface-wound permanent magnet DC motor.

where V_b denotes the back emf in Volts, ω_m is the angular velocity of the rotor (rad/sec), and K_2 is a proportionality constant.

DC-motors can be classified according to the way in which the magnetic field is produced and the armature is designed. Here we discuss only the so-called **permanent magnet** motors whose stator consists of a permanent magnet. In this case we can take the flux ϕ to be a constant. The torque on the rotor is then controlled by controlling the armature current i_a.

Consider the schematic diagram of Figure 7-3 where

$$\begin{aligned} V(t) &= \text{armature voltage} \\ L &= \text{armature inductance} \\ R &= \text{armature resistance} \\ V_b &= \text{back emf} \\ i_a &= \text{armature current} \\ \theta_m &= \text{rotor position (radians)} \\ \tau_m &= \text{generated torque} \\ \\ \phi &= \text{magnetic flux due to stator.} \end{aligned}$$

The differential equation for the armature current is then

$$L\frac{di_a}{dt} + Ri_a = V - V_b \tag{7.2.4}$$

Since the flux ϕ is constant the torque developed by the motor is

$$\tau_m = K_1\phi i_a = K_m i_a \tag{7.2.5}$$

where K_m is the torque constant in N-m/amp. From (7.2.3) we have

$$V_b = K_2\phi\omega_m = K_b\,\omega_m = K_b\frac{d\theta_m}{dt} \tag{7.2.6}$$

where K_b is the back emf constant.

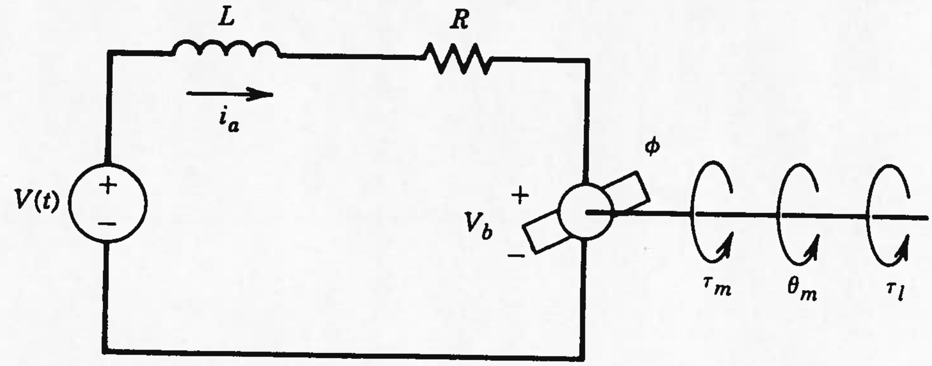


FIGURE 7-3
Circuit diagram for armature controlled DC motor.

Assume the system is stabilized, then θ_m will be kept small. This justifies the approximations

$$\sin(\theta_m/n) = \theta_m/n.$$

Then, one has

$$J\ddot{\theta}_m + B\dot{\theta}_m + C/n\,\theta_m = K_m i_a$$

$$L\frac{di_a}{dt} + Ki_a = V - K_b\frac{d\theta_m}{dt}$$

The state variable of the system can be defined as θ_m, w_m, and i_a. Then, the state equation can be written as

$$\begin{bmatrix} \frac{di_a}{dt} \\ \frac{dw_m}{dt} \\ \frac{d\theta_m}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{K_b}{L} & 0 \\ \frac{K_m}{J} & -\frac{B}{J} & -\frac{C}{Jn} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ w_m \\ \theta_m \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} V$$

Mathematic model

A commonly used model for a nonlinear system is

1 $$\dot{\mathbf{x}}(t) = \mathbf{f}[t, \mathbf{x}(t), \mathbf{u}(t)], \ \forall t \geq 0,$$

where t denotes time; $\mathbf{x}(t)$ denotes the value of the function $\mathbf{x}(\cdot)$ at time t and is an n-dimensional vector; $\mathbf{u}(t)$ is similarly defined and is an m-dimensional vector; and the function $\mathbf{f}$ associates, with each value of t, $\mathbf{x}(t)$, and $\mathbf{u}(t)$, a corresponding n-dimensional vector. Following common convention, this is denoted as: $t \in \mathbf{R}_+$, $\mathbf{x}(t) \in \mathbf{R}^n$, $\mathbf{u}(t) \in \mathbf{R}^m$, and $\mathbf{f}: \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^n$. Note that (1) is a first-order vector differential equation. The quantity $\mathbf{x}(t)$ is generally referred to as the **state** of the system at time t, while $\mathbf{u}(t)$ is called the **input** or the **control** function.

There is no loss of generality in assuming that the system at hand is described by a first-order (differential or difference) equation. To see this, suppose the system is described by the n-th order scalar differential equation

3 $$\frac{d^n y(t)}{dt^n} = h\left[t, y(t), \dot{y}(t), \cdots, \frac{d^{n-1} y(t)}{dt^{n-1}}, u(t)\right], \ \forall t \geq 0.$$

This equation can be recast in the form (1) by defining the n-dimensional state vector $\mathbf{x}(t)$ in the familiar way, namely

4 $$x_1(t) = y(t), x_2(t) = \dot{y}(t), \cdots, x_n(t) = \frac{d^{n-1} y(t)}{dt^{n-1}}.$$

Then (3) is equivalent to

5 $$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t), \\ &\vdots \\ \dot{x}_{n-1}(t) &= x_n(t) \\ \dot{x}_n(t) &= h[t, x_1(t), x_2(t), \cdots, x_n(t), u(t)] \end{aligned}$$

Now (5) is of the form (1) with

6 $$\mathbf{x}(t) = [x_1(t) \cdots x_n(t)]',$$

7 $$\mathbf{f}(t, \mathbf{x}, u) = [x_1 \ x_2 \ \cdots x_n \ h(t, x_1, \cdots, x_n, u)]'.$$

In studying the system (1), one can make a distinction between two aspects,[1] generally referred to as analysis and synthesis, respectively. Suppose the input function $\mathbf{u}(\cdot)$ in (1) is specified (i.e., fixed), and one would like to study the behaviour of the corresponding function $\mathbf{x}(\cdot)$; this is usually referred to as **analysis**. Now suppose the problem is turned around: the system description (1) is given, as well as the desired behaviour of the function $\mathbf{x}(\cdot)$, and the problem is to find a suitable input function $\mathbf{u}(\cdot)$ that would cause $\mathbf{x}(\cdot)$ to behave in this desired fashion; this is usually referred to as **synthesis**.

Comments

Theory

Part II
State-space representation of dynamic systems

2.1 Notations

General state-space model of a dynamic system

$$\begin{aligned}
\dot{x}_1 &= \frac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r, t) \\
\dot{x}_2 &= \frac{dx_2}{dt} = f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r, t) \\
&\cdot \;\cdot \;\cdot \\
\dot{x}_n &= \frac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r, t)
\end{aligned} \tag{2.1}$$

Designations:

State vector $x = [x_1, x_2, \ldots, x_n]^T$

Control (input) vector $u = [u_1, u_2, \ldots, u_r]^T$ (2.2)

Mathematical model in vector form

$$\dot{x} = f(x,u,t) \tag{2.3}$$

where $x \in R^n$; $f \in R^n$; $u \in R^r$; $t \in R$

Note. If t does not appear explicitly in (2.3) the system is called *time-invariant*

State-space representation of a linear system is

$$\begin{aligned}
\dot{x}_1 &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + b_{11}(t)u_1 + b_{12}(t)u_2 + \ldots + b_{1r}(t)u_r \\
\dot{x}_2 &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + b_{21}(t)u_1 + b_{22}(t)u_2 + \ldots + b_{2r}(t)u_r \\
&\cdot \;\cdot \;\cdot \\
\dot{x}_n &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + b_{n1}(t)u_1 + b_{n2}(t)u_2 + \ldots + b_{nr}(t)u_r
\end{aligned}$$

or in vector notations

$$\dot{x} = A(t)x + B(t)u \tag{2.4}$$

where

$$A(t) = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n1} & \ldots & a_{nn} \end{bmatrix} ; \quad B(t) = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1r} \\ b_{21} & b_{22} & \ldots & b_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n1} & \ldots & b_{nr} \end{bmatrix} \tag{2.5}$$

or

$$A \in R^{n \times n} \qquad B \in R^{n \times r}$$

If the system is time-invariant, then matrices A and B are constants.

Comments

Theory

Output (or observation) vector

$$y = [y_1, y_2, \dots, y_m]^T \qquad (2.6)$$

relates to the state and the control variable as (for linear systems)

$$y = C(t)x + D(t)u \qquad (2.7)$$

where $C \in R^{m \times n}$; $D \in R^{m \times r}$.

Equation (2.7) is called output or observation equation.

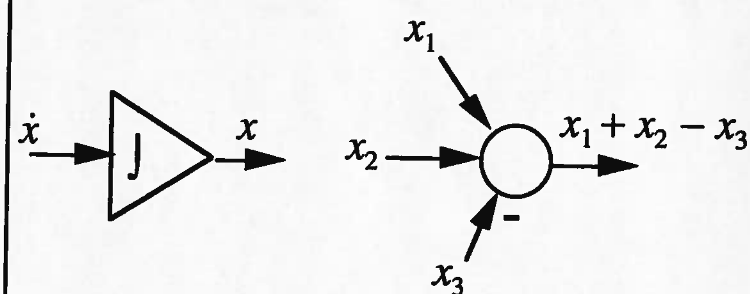
In the majority of applications $D = 0$.

The state equations may depend on external disturbances (or exogenous input). Then for a linear system we will write

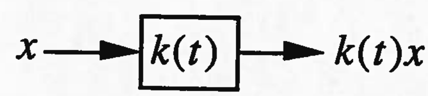
$$\dot{x} = A(t)x + B(t)u + E(t)d \qquad (2.8)$$

2.2 Block-diagram representation

Three kinds of elements:



Integrator *Summer*



Gain element (amplifier)

Example. Block-diagram of general 2nd-order linear system.