2.4 LAGRANGE'S EQUATIONS

The equations governing the motion of a complicated mechanical system, such as a robot manipulator, can be expressed very efficiently through the use of a method developed by the eighteenth-century French mathematician Lagrange. The differential equations that result from use of this method are known as *Lagrange's equations* and are derived from Newton's laws of motion in most textbooks on advanced dynamics.[2, 3]

The fundamental principle of Lagrange's equations is the representation of the system by a set of generalized coordinates q_i ($i = 1, 2, \ldots, r$), one for each independent degree of freedom of the system, which completely incorporate the constraints unique to that system, i.e., the interconnections between the parts of the system. After having defined the generalized coordinates, the kinetic energy T is expressed in terms of these coordinates and their derivatives, and the potential energy V is expressed in terms of the generalized coordinates. (The potential energy is a function of only the generalized coordinates and *not* their derivatives.) Next, the *lagrangian* function

$$L = T(q_1, \ldots, q_r, \dot{q}_1, \ldots, \dot{q}_r) - V(q_1, \ldots, q_r)$$

is formed. And finally the desired equations of motion are derived using Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad i = 1, 2, \ldots, r \tag{2.11}$$

where Q_i denotes generalized forces (i.e., forces and torques) that are external to the system or not derivable from a scalar potential function.

6.3 EQUATIONS OF MOTION

In this section, we specialize the Euler–Lagrange equations derived in Section 6.1 to the special case when two conditions hold: First, the kinetic energy is a quadratic function of the vector $\dot{\mathbf{q}}$ of the form

$$K = \frac{1}{2}\sum_{i,j}^{n} d_{ij}(\mathbf{q})\dot{q}_i\dot{q}_j := \frac{1}{2}\dot{\mathbf{q}}^T D(\mathbf{q})\dot{\mathbf{q}} \tag{6.3.1}$$

where the $n \times n$ "inertia matrix" $D(\mathbf{q})$ is symmetric and positive definite for each $\mathbf{q} \in \mathbb{R}^n$. Second, the potential energy $V = V(\mathbf{q})$ is independent of $\dot{\mathbf{q}}$. We have already remarked that robotic manipulators satisfy this condition.

The Euler–Lagrange equations for such a system can be derived as follows. Since

$$L = K - V = \frac{1}{2}\sum_{i,j} d_{ij}(\mathbf{q})\dot{q}_i\dot{q}_j - V(\mathbf{q}) \tag{6.3.2}$$

we have that

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}(\mathbf{q})\dot{q}_j \tag{6.3.3}$$

and

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} &= \sum_j d_{kj}(\mathbf{q})\ddot{q}_j + \sum_j \frac{d}{dt} d_{kj}(\mathbf{q})\dot{q}_j \\ &= \sum_j d_{kj}(\mathbf{q})\ddot{q}_j + \sum_{i,j} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j\end{aligned} \tag{6.3.4}$$

Also

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial V}{\partial q_k} \tag{6.3.5}$$

Thus the Euler–Lagrange equations can be written

$$\sum_j d_{kj}(\mathbf{q})\ddot{q}_j + \sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j - \frac{\partial V}{\partial q_k} = \tau_k$$

$$k = 1, \cdots, n \tag{6.3.6}$$

By interchanging the order of summation and taking advantage of symmetry, we can show that

$$\sum_{i,j} \left\{ \frac{\partial d_{kj}}{\partial q_i} \right\} \dot{q}_i \dot{q}_j = \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} \right\} \dot{q}_i \dot{q}_j \tag{6.3.7}$$

Hence

$$\sum_{i,j} \left\{ \frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \dot{q}_j$$

$$= \sum_{i,j} \frac{1}{2} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \dot{q}_j \tag{6.3.8}$$

The terms

$$c_{ijk} := \frac{1}{2} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \tag{6.3.9}$$

are known as **Christoffel symbols** (of the first kind). Note that, for a fixed k, we have $c_{ijk} = c_{jik}$, which reduces the effort involved in computing these symbols by a factor of about one half. Finally, if we define

$$\phi_k = \frac{\partial V}{\partial q_k} \tag{6.3.10}$$

Hence

$$\sum_{i,j} \left\{ \frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \dot{q}_j$$

$$= \sum_{i,j} \frac{1}{2} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \dot{q}_j \tag{6.3.8}$$

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$$\phi_k = \frac{\partial V}{\partial q_k} \tag{6.3.10}$$

then we can write the Euler-Lagrange equations as

$$\sum_j d_{kj}(\mathbf{q}) \ddot{q}_j + \sum_{i,j} c_{ijk}(\mathbf{q}) \dot{q}_i \dot{q}_j + \phi_k(\mathbf{q}) = \tau_k , \quad k = 1, \ldots, n \tag{6.3.11}$$

In the above equation, there are three types of terms. The first involve the second derivative of the generalized coordinates. The second are quadratic terms in the first derivatives of $\mathbf{q}$, where the coefficients may depend on $\mathbf{q}$. These are further classified into two types. Terms involving a product of the type $\dot{q}_i^2$ are called centrifugal, while those involving a product of the type $\dot{q}_i \dot{q}_j$ where $i \neq j$ are called **Coriolis** terms. The third type of terms are those involving only $\mathbf{q}$ but not its derivatives. Clearly the latter arise from differentiating the potential energy. It is common to write (6.3.11) in matrix form as

$$D(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau \tag{6.3.12}$$

where the k, j-th element of the matrix $C(\mathbf{q}, \dot{\mathbf{q}})$ is defined as

$$\begin{aligned} c_{kj} &= \sum_{i=1}^{n} c_{ijk}(\mathbf{q})\dot{q}_i \qquad (6.3.13) \\ &= \sum_{i=1}^{n} \frac{1}{2} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \end{aligned}$$

We next derive an important relationship between the inertia matrix $D(\mathbf{q})$ and the matrix $C(\mathbf{q}, \dot{\mathbf{q}})$ appearing in (6.3.12) that will be of fundamental importance for the problem of manipulator control considered in later chapters.

In summary, the development in this section is very general and applies to *any* mechanical system whose kinetic energy is of the form (6.3.1) and whose potential energy is independent of $\dot{\mathbf{q}}$. In the next section we apply this discussion to study specific robot configurations.

(ii) *Theorem 6.3.1*

Define the matrix $N(\mathbf{q},\dot{\mathbf{q}}) = \dot{D}(\mathbf{q}) - 2C(\mathbf{q},\dot{\mathbf{q}})$. Then $N(\mathbf{q},\dot{\mathbf{q}})$ is skew symmetric, that is, the components n_{jk} of N satisfy $n_{jk} = -n_{kj}$.

Proof: Given the inertia matrix $D(\mathbf{q})$, the kj-th component of $\dot{D}(\mathbf{q})$ is given by the chain rule as

$$\dot{d}_{kj} = \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \tag{6.3.14}$$

Therefore, the kj-th component of $N = \dot{D} - 2C$ is given by

$$\begin{aligned} n_{kj} &= \dot{d}_{kj} - 2c_{kj} \\ &= \sum_{i=1}^{n} \left[\frac{\partial d_{kj}}{\partial q_i} - \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \right] \dot{q}_i \\ &= \sum_{i=1}^{n} \left[\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j} \right] \dot{q}_i \end{aligned} \tag{6.3.15}$$

Since the inertia matrix $D(\mathbf{q})$ is symmetric, that is, $d_{ij} = d_{ji}$, it follows from (6.3.15) by interchanging the indices k and j that

$$n_{jk} = -n_{kj} \tag{6.3.16}$$

which completes the proof.

(iii) *Example 6.4.1 Two-Link Cartesian Manipulator*

Consider the manipulator shown in Figure 6-2, consisting of two links and two prismatic joints. Denote the masses of the two links by m_1 and m_2, respectively, and denote the displacement of the two prismatic joints by q_1 and q_2, respectively. Then it is easy to see, as mentioned in Section 6.1, that these two quantities serve as generalized coordinates for the manipulator. Since the generalized coordinates have dimensions of distance, the corresponding generalized forces have units of force. In fact, they are just the forces applied at each joint. Let us denote these by f_i, $i = 1, 2$.

Since we are using the joint variables as the generalized coordinates, we know that the kinetic energy is of the form (6.3.1) and that the potential energy is only a function of q_1 and q_2. Hence we can use the formulae in Section 6.3 to obtain the dynamical equations.[4] Also, since both joints are prismatic, the angular velocity Jacobian is zero and the kinetic energy of each link consists solely of the translational term.

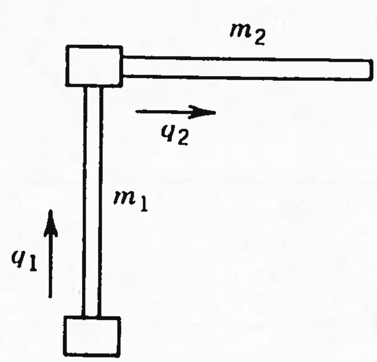


FIGURE 6-2
Two-link cartesian robot.

the velocity of the center of mass of link 1 is given by

$$v_{c1} = J_{v_{c1}}\dot{q} \tag{6.4.1}$$

where

$$J_{v_{c1}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{6.4.2}$$

Similarly,

$$v_{c2} = J_{v_{c2}}\dot{q} \tag{6.4.3}$$

where

$$J_{v_{c2}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{6.4.4}$$

Hence the kinetic energy is given by

$$K = \frac{1}{2}\dot{q}^T \left(m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}}\right)\dot{q} \tag{6.4.5}$$

Comparing with (6.3.1), we see that the inertia matrix D is given simply by

$$D = \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix}. \tag{6.4.6}$$

Next, the potential energy of link 1 is $m_1 g q_1$, while that of link 2 is $m_2 g q_1$, where g is the acceleration due to gravity. Hence the overall potential energy is

$$V = g(m_1 + m_2)q_1 \tag{6.4.7}$$

Now we are ready to write down the equations of motion. Since the inertia matrix is constant, all Christoffel symbols are zero. Further, the vectors ϕ_k are given by

$$\phi_1 = \frac{\partial V}{\partial q_1} = g(m_1 + m_2), \quad \phi_2 = \frac{\partial V}{\partial q_2} = 0 \tag{6.4.8}$$

Substituting into (6.3.11) gives the dynamical equations as

$$\begin{aligned}(m_1 + m_2)\ddot{q}_1 + g(m_1 + m_2) &= f_1 \\ m_2\ddot{q}_2 &= f_2\end{aligned} \tag{6.4.9}$$

(iv) *Example 6.4.2 Planar Elbow Manipulator*

Now consider the planar manipulator with two revolute joints shown in Figure 6-3. Let us fix notation as follows: For $i = 1,2$, q_i denotes the joint angle, which also serves as a generalized coordinate; m_i denotes the mass of link i, ℓ_i denotes the length of link i; ℓ_{ci} denotes the distance from the previous joint to the center of mass of link i; and I_i denotes the moment of inertia of link i about an axis coming out of the page, passing through the center of mass of link i.

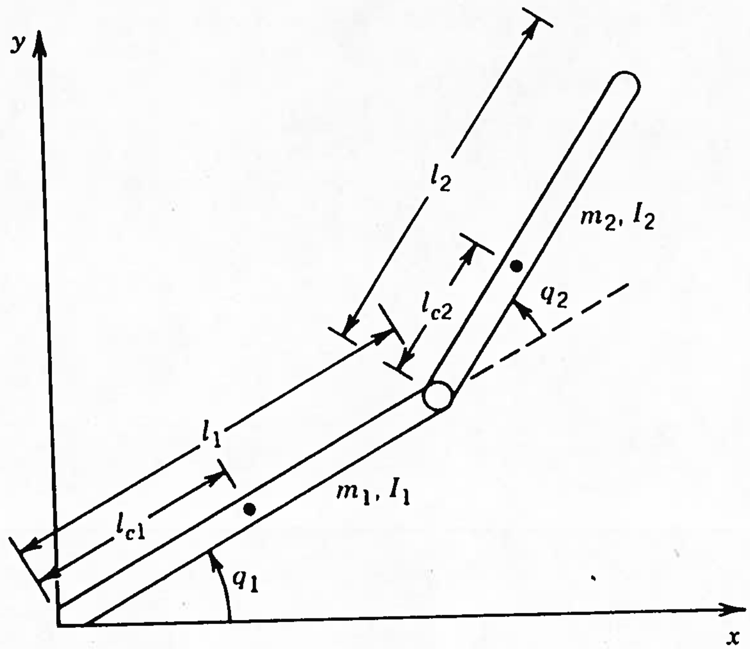


FIGURE 6-3
Two-link revolute joint arm.

We will make effective use of the Jacobian expressions in Chapter Five in computing the kinetic energy. Since we are using joint variables as the generalized coordinates, it follows that we can use the contents of Section 6.3. First,

$$\mathbf{v}_{c1} = J_{\mathbf{v}_{c1}}\dot{\mathbf{q}} \tag{6.4.10}$$

$$J_{\mathbf{v}_{c1}} = \begin{bmatrix} -\ell_{c1}\sin q_1 & 0 \\ \ell_{c1}\cos q_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{6.4.11}$$

Similarly,

$$\mathbf{v}_{c2} = J_{\mathbf{v}_{c2}}\dot{\mathbf{q}} \tag{6.4.12}$$

where

$$J_{\mathbf{v}_{c2}} = \begin{bmatrix} -\ell_1\sin q_1 - \ell_{c2}\sin(q_1 + q_2) & -\ell_{c2}\sin(q_1 + q_2) \\ \ell_1\cos q_1 + \ell_{c2}\cos(q_1 + q_2) & \ell_{c2}\cos(q_1 + q_2) \\ 0 & 0 \end{bmatrix} \tag{6.4.13}$$

Hence the translational part of the kinetic energy is

$$\frac{1}{2}m_1\mathbf{v}_{c1}^T\mathbf{v}_{c1} + \frac{1}{2}m_2\mathbf{v}_{c2}^T\mathbf{v}_{c2} = \frac{1}{2}\dot{\mathbf{q}}\left\{m_1 J_{\mathbf{v}_{c1}}^T J_{\mathbf{v}_{c1}} + m_2 J_{\mathbf{v}_{c2}}^T J_{\mathbf{v}_{c2}}\right\}\dot{\mathbf{q}} \tag{6.4.14}$$

Next we deal with the angular velocity terms. Because of the particularly simple nature of this manipulator, many of the potential difficulties do not arise. First, it is clear that

$$\boldsymbol{\omega}_1 = \dot{q}_1\mathbf{k}, \quad \boldsymbol{\omega}_2 = (\dot{q}_1 + \dot{q}_2)\mathbf{k} \tag{6.4.15}$$

when expressed in the base inertial frame. Now we pointed out in Section 6.2 that it is necessary to express these angular velocities in the link-bound coordinate frames. Fortunately, the z-axes of all of these frames are in the same direction, so the above expression is also valid in the link-bound frame. Moreover, since $\boldsymbol{\omega}_i$ is aligned with $\mathbf{k}$, the triple product $\boldsymbol{\omega}_i^T I_i \boldsymbol{\omega}_i$ reduces simply to $(I_{33})_i$ times the square of the magnitude of the angular velocity. This quantity $(I_{33})_i$ is indeed what we have labeled as I_i above. Hence the rotational kinetic energy of the overall system is

$$\frac{1}{2}\dot{\mathbf{q}}^T\left\{I_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I_2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}\dot{\mathbf{q}} \tag{6.4.16}$$

Now we are ready to form the inertia matrix $D(\mathbf{q})$. For this purpose, we merely have to add the two matrices in (6.4.14) and (6.4.16), respectively. Thus

$$D(\mathbf{q}) = m_1 J_{\mathbf{v}_{c1}}^T J_{\mathbf{v}_{c1}} + m_2 J_{\mathbf{v}_{c2}}^T J_{\mathbf{v}_{c2}} + \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix} \tag{6.4.17}$$

Carrying out the above multiplications and using the standard trigonometric identities $\cos^2\theta + \sin^2\theta = 1$, $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$ leads to

$$\begin{aligned} d_{11} &= m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos q_2) + I_1 + I_2 \\ d_{12} &= d_{21} = m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2 \\ d_{22} &= m_2\ell_{c2}^2 + I_2 \end{aligned} \tag{6.4.18}$$

Now we can compute the Christoffel symbols using the definition (6.3.9). This gives

$$\begin{aligned} c_{111} &= \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0 \\ c_{121} &= c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2\ell_1\ell_{c2}\sin q_2 =: h \\ c_{221} &= \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h \\ c_{112} &= \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h \\ c_{122} &= c_{212} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0 \\ c_{222} &= \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0 \end{aligned} \tag{6.4.19}$$

Next, the potential energy of the manipulator is just the sum of those of the two links. For each link, the potential energy is just its mass multiplied by the gravitational acceleration and the height of its center of mass. Thus

$$V_1 = m_1 g \ell_{c1} \sin q_1$$

$$V_2 = m_2 g(\ell_1 \sin q_1 + \ell_{c2}\sin(q_1 + q_2))$$

$$V = V_1 + V_2 = (m_1\ell_{c1} + m_2\ell_1)g\sin q_1 + m_2\ell_{c2}g\sin(q_1 + q_2) \quad (6.4.20)$$

Hence, the functions ϕ_k defined in (6.3.10) become

$$\phi_1 = \frac{\partial V}{\partial q_1} = (m_1\ell_{c1} + m_2\ell_1)g\cos q_1 + m_2\ell_{c2}g\cos(q_1 + q_2) \quad (6.4.21)$$

$$\phi_2 = \frac{\partial V}{\partial q_2} = m_2\ell_{c2}\cos(q_1 + q_2) \quad (6.4.22)$$

Finally we can write down the dynamical equations of the system as in (6.3.11). Substituting for the various quantities in this equation and omitting zero terms leads to

$$\begin{aligned} d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + \phi_1 &= \tau_1 \\ d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + \phi_2 &= \tau_2 \end{aligned} \quad (6.4.23)$$

In this case the matrix $C(\mathbf{q}, \dot{\mathbf{q}})$ is given as

$$C = \begin{bmatrix} h\dot{q}_2 & h\dot{q}_2 + h\dot{q}_1 \\ -h\dot{q}_1 & 0 \end{bmatrix} \quad (6.4.24)$$

(v) *Example 6.4.3 Planar Elbow Manipulator with Remotely Driven Link*

Now we illustrate the use of Lagrangian equations in a situation where the generalized coordinates are not the joint variables defined in earlier chapters. Consider again the planar elbow manipulator, but suppose now that both joints are driven by motors mounted at the base. The first joint is turned directly by one of the motors, while the other is turned via a gearing mechanism or a timing belt (see Figure 6-4). In this case one should choose the generalized coordinates as shown in Figure 6-5, because the angle p_2 is determined by driving motor number 2, and is not affected by the angle p_1. We will derive the dynamical equations for this configuration, and show that some simplifications will result.

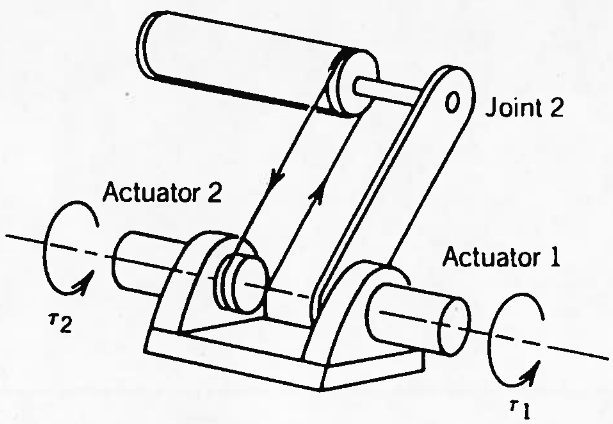


FIGURE 6-4
Two-link revolute joint arm with remotely driven link.

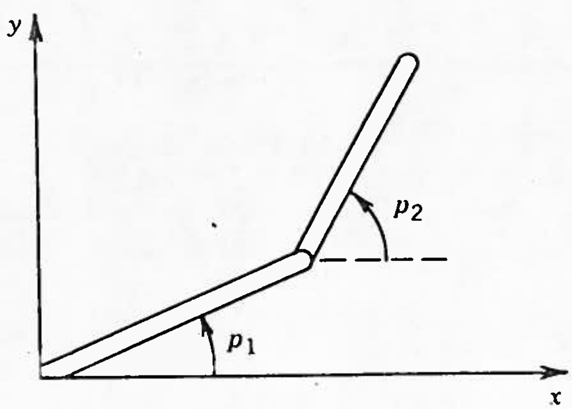


FIGURE 6-5
Generalized coordinates for robot of Figure 6-4.

Since p_1 and p_2 are not the joint angles used earlier, we cannot use the velocity Jacobians derived in Chapter Five in order to find the kinetic energy of each link. Instead, we have to carry out the analysis directly. It is easy to see that

$$\mathbf{v}_{c1} = \begin{bmatrix} -\ell_{c1}\sin p_1 \\ \ell_{c1}\cos p_1 \\ 0 \end{bmatrix} \dot{p}_1, \quad \mathbf{v}_{c2} = \begin{bmatrix} -\ell_1 \sin p_1 & -\ell_{c2}\sin p_2 \\ \ell_1 \cos p_1 & \ell_{c2}\cos p_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} \tag{6.4.25}$$

$$\omega_1 = \dot{p}_1 \mathbf{k}, \quad \omega_2 = \dot{p}_2 \mathbf{k} \tag{6.4.26}$$

Hence the kinetic energy of the manipulator equals

$$K = \dot{\mathbf{p}}^T D(\mathbf{p}) \dot{\mathbf{p}} \tag{6.4.27}$$

where

$$D(\mathbf{p}) = \begin{bmatrix} m_1\ell_{c1}^2 + m_2\ell_1^2 + I_1 & m_2\ell_1\ell_{c2}\cos(p_2-p_1) \\ m_2\ell_1\ell_{c2}\cos(p_2-p_1) & m_2\ell_{c2}^2 + I_2 \end{bmatrix} \tag{6.4.28}$$

Computing the Christoffel symbols as in (6.3.9) gives

$$\begin{aligned}
c_{111} &= \frac{1}{2}\frac{\partial d_{11}}{\partial p_1} = 0 \\
c_{121} = c_{211} &= \frac{1}{2}\frac{\partial d_{11}}{\partial p_2} = 0 \\
c_{221} &= \frac{\partial d_{12}}{\partial p_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial p_1} = -m_2\ell_1\ell_{c2}\sin(p_2-p_1) \\
c_{112} &= \frac{\partial d_{21}}{\partial p_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial p_2} = m_2\ell_1\ell_{c2}\sin(p_2-p_1) \\
c_{212} = c_{122} &= \frac{1}{2}\frac{\partial d_{22}}{\partial p_1} = 0 \\
c_{222} &= \frac{1}{2}\frac{\partial d_{22}}{\partial p_2} = 0
\end{aligned} \tag{6.4.29}$$

Next, the potential energy of the manipulator, in terms of p_1 and p_2, equals

$$V = m_1 g \ell_{c1}\sin p_1 + m_2 g(\ell_1 \sin p_1 + \ell_{c2}\sin p_2) \tag{6.4.30}$$

Hence

$$\phi_1 = (m_1\ell_{c1} + m_2\ell_1) g \cos p_1$$

$$\phi_2 = m_2\ell_{c2} g \cos p_2$$

Finally, the dynamical equations are

$$\begin{aligned}
d_{11}\ddot{p}_1 + d_{12}\ddot{p}_2 + c_{221}\dot{p}_2^2 + \phi_1 &= \tau_1 \\
d_{21}\ddot{p}_1 + d_{22}\ddot{p}_2 + c_{112}\dot{p}_1^2 + \phi_2 &= \tau_2
\end{aligned} \tag{6.4.31}$$

Dynamic Equation of Mechanical Systems

The Lagrange equations describing motion of a mechanical system are

$$D(\mathbf{q})\ddot{\mathbf{q}} + C(\dot{\mathbf{q}}, \mathbf{q})\dot{\mathbf{q}} + G(\mathbf{q}) = B(\mathbf{q})\mathbf{u} \tag{1}$$

where

$\mathbf{q} \in R^n$	Generalized coordinates
$\mathbf{u} \in R^m$	Generalized external forces
$D(\mathbf{q})$	Positive definite symmetric inertia matrix
$C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$	Nonlinear vector functions
$B(\mathbf{q}) \in R^{n\times r}$	input transformation matrix

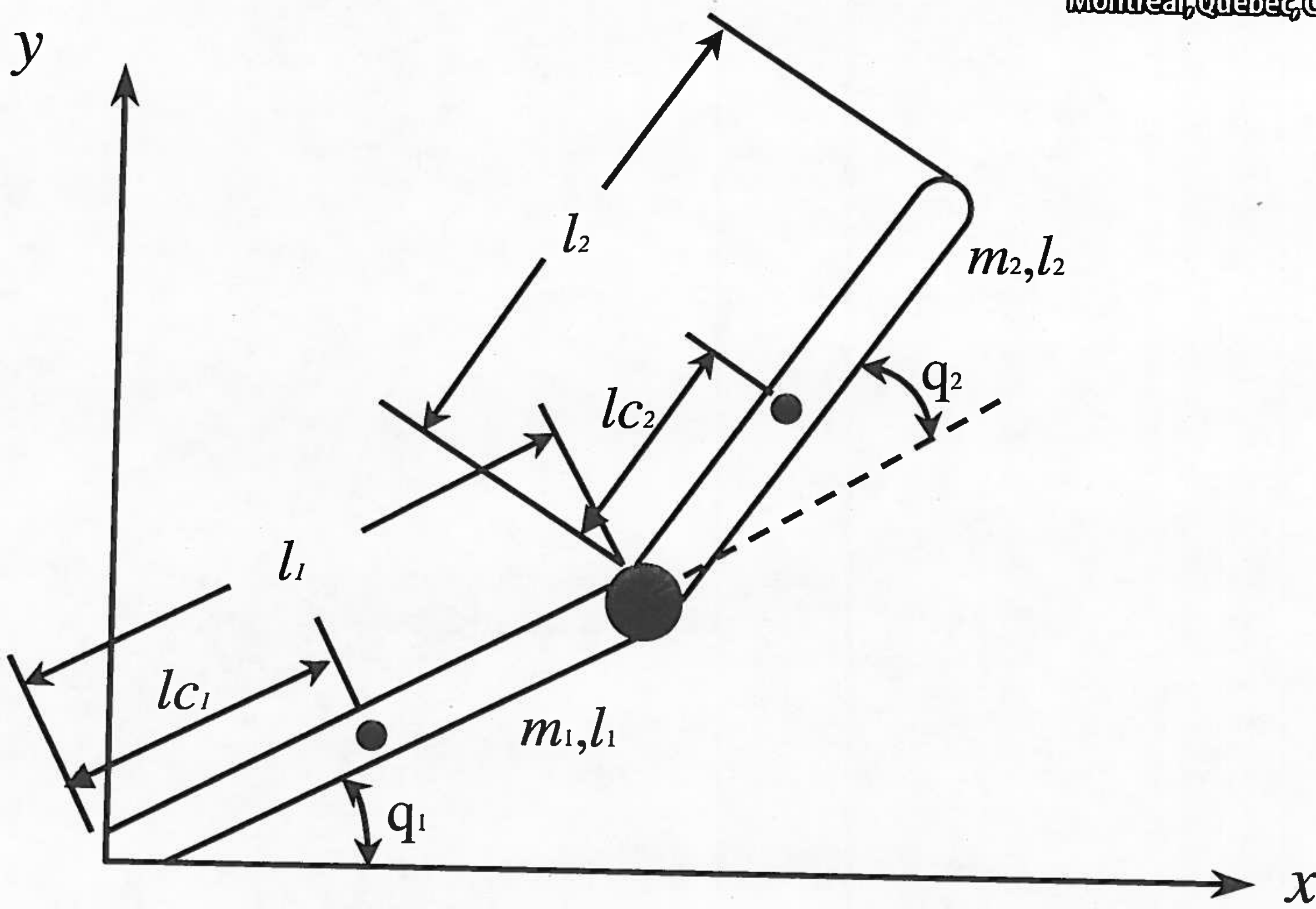
Properties

Property 1: A suitable definition of $C(\mathbf{q}, \dot{\mathbf{q}})$ makes the matrix $(\dot{D} - 2C)$ skew-symmetric.

Property 2:

$$D(\mathbf{q})\dot{\mathbf{v}} + C(\mathbf{q}, \dot{\mathbf{q}})\mathbf{v} + G(\mathbf{q}) = \Phi(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}, \dot{\mathbf{v}})\alpha \qquad (2)$$

where $\Phi \in R^{n \times p}$ is a regressor matrix; and α is the p-vector of inertia parameters.


y
x
l_2
m_2, l_2
lc_2
q_2
l_1
lc_1
m_1, l_1
q_1

Defining parameters $\theta_1, \ldots, \theta_9$ *as* :

$$\theta_1 = m_1 l_{c_1}^2 \qquad \theta_2 = m_2 l_1^2 \qquad \theta_3 = m_2 l_{c_2}^2$$

$$\theta_4 = m_2 l_1 l_{c_2} \qquad \theta_5 = I_1 \qquad \theta_6 = I_2$$

$$\theta_7 = m_1 l_{c_1} g \qquad \theta_8 = m_2 l_1 g \qquad \theta_9 = m_2 l_{c_2} g$$

we can write as :

$$Y(q, \dot{q}, \ddot{q})\theta = \tau$$

where

$$\theta = [\theta_1, \ldots, \theta_9]^T$$

and Y *is given by* :

$$Y(q, \quad \dot{q}, \quad \ddot{q},) =$$

$$\begin{bmatrix} \ddot{q}_1 & \ddot{q}_2 & \ddot{q}_1 + \ddot{q}_2 & 2\cos q_2 \ddot{q}_1 + \cos q_2 \ddot{q}_2 - 2\sin q_2 \dot{q}_1 \dot{q}_2 - \sin q_2 q_2^2 & \ddot{q}_1 & \ddot{q}_1 + \ddot{q}_2 & \cos q_1 & \cos q_1 & \cos(q_1 + q_2) \\ 0 & 0 & \ddot{q}_1 + \ddot{q}_2 & \cos q_2 \ddot{q}_1 + \sin q_2 & \ddot{q}_2 & \ddot{q}_2 & 0 & 0 & \cos(q_1 + q_2) \end{bmatrix}$$

$$\theta_1 = m_1 l_{c_1}^2 + m_2 l_1^2 + I_1 \qquad \theta_2 = m_1 l_{c_2}^2 + \cancel{m_2 l_1^2} + I_2$$

$$\theta_3 = m_2 l_1 l_{c_2} \qquad \theta_4 = m_1 l_{c_1}$$

$$\theta_5 = m_2 l_1 \qquad \theta_6 = m_2 l_{c_2}$$

With this parametrization the dynamic equation can be written as:

$$Y(q, \quad \dot{q}, \quad \ddot{q})\theta = \tau \qquad ; \qquad \theta \in R^6$$

by Spong paper

Where the components y_{ij} of Y are given as:

$$y_{11} = \ddot{q}_1 \qquad y_{12} = \ddot{q}_1 + \ddot{q}_2 \qquad y_{14} = g\cos(q_1)$$

$$y_{13} = \cos(q_2)(2\ddot{q}_1 + \ddot{q}_2) - \sin(q_2)(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)$$

$$y_{15} = g\cos(q_1) \qquad y_{16} = g\cos(q_1 + q_2)$$

$$y_{21} = 0 \qquad y_{22} = \ddot{q}_1 + \ddot{q}_2 \qquad y_{23} = \cos(q_2)\ddot{q}_1 + \sin(q_2)\dot{q}_1^2$$

$$y_{24} = 0 \qquad y_{25} = 0 \qquad y_{26} = g\cos(q_1 + q_2)$$