Aleksandr Mikhailovich Lyapunov

- Father astronomer, 7 children, 3 survived
- Professor of mechanics at Kharkov and St. Petersburg
  - Research on orbital mechanics and probability theory
- A. M. Lyapunov died tragically at age 61
- Completed his doctoral dissertation in 1892 under Chebyshev
  - Stability of rotating fluids applied to celestial bodies
  - Formulated his first and second methods (L1M and L2M)
- French translation appeared in 1907 = 1892 + 15
  - English translation didn't appear until 1992 = 1892 + 100

Comments

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Theory

Part 3

Stability in the sense of Lyapunov

There are different kinds of stability problems that arise in the study of dynamic systems: stability of equilibrium point, input-output stability, stability of periodic orbits, etc. We will study the stability of equilibrium or stability in the sense of Lyapunov, a Russian mathematician and engineer.

<u>Plant:</u>

$$\dot{x}(t) = f(t, x(t)), \quad t \geq 0, \tag{3.1}$$

where $x \in R^n$ and $f: R_+ \times R^n \rightarrow R^n$,
f satisfies the Lipschitz condition.

Let

$$f(t,0) = 0 \quad \forall t \geq t_0 \tag{3.2}$$

that is 0 is the equilibrium point.

<u>Definitions of stability</u>

1. The equation (1) is <u>stable</u> at time t_0 if
   for each $\varepsilon > 0$ there exist a $\delta(t_0, \varepsilon) > 0$ such that

$$\|x(t_0)\| < \delta(t_0, \varepsilon) \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq t_0 \tag{3.3}$$

2. The point "0" is <u>uniformly stable</u> over $[t_0, \infty)$ if
   for each $\varepsilon > 0$ there exist a $\delta(\varepsilon) > 0$ such that

$$\|x(t_1)\| < \delta(\varepsilon), \quad t_1 \geq t_0 \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq t_1 \tag{3.4}$$

Comments

Theory

3. The point "0" is <u>asymptotically stable</u> at time t_0 if
   (a) it is stable at t_0 ;
   (b) there exist a number $\delta_1(t_0) > 0$ such that

$$\|x(t_0)\| < \delta_1(t_0) \Rightarrow \|x(t)\| \to 0 \text{ as } t \to \infty \qquad (3.5)$$

The set

$$B_{\delta_1(t_0)} = \{x \in R^n : \|x\| < \delta_1(t_0)\}$$

is called a <u>region of attraction.</u>

4. The point "0" is <u>uniformly asymptotically stable</u> over $[t_0, \infty)$ if

   (a) it is stable over $[t_0, \infty)$;
   (b) there exist $\delta_1 > 0$ such that

$$\|x(t_1)\| < \delta_1, \quad t_1 \ge t_0 \Rightarrow \|x(t)\| \to 0 \quad \text{as } t \to \infty \quad (3.6)$$

The Lyapunov method

Definitions

1. $V(x): R^n \to R^+$ is positive definite (PD) if

$$V(0) = 0 \text{ and } V > 0 \quad \forall x \neq 0$$

2. Any PD function $L(x)$ where $x \in R^n$ is the system state vector is called a Lyapunov Function Candidate (LFC).

3. The Euler derivative, or the derivative along the trajectories (3.1) is

$$\dot{L} = \langle dL, f \rangle = \frac{\partial L}{\partial x_1} f_1 + \frac{\partial L}{\partial x_2} f_2 +,\dots,+ \frac{\partial L}{\partial x_n} f_n \quad (3.7)$$

Comments

Theory

Consider an autonomous (time-invariant) system given by the equation

$$\dot{x} = f(x); \quad f(0) = 0 \tag{3.8}$$

Theorem 1 (Lyapunov).
The null solution of (3.8) is stable if there exist a LFC such that the Euler derivative is negative semi-definite along trajectories (3.8), that is

$$\dot{L} = \langle dL, f(x) \rangle = (dL)^T f(x) \leq 0 \tag{3.9}$$

Such L is called a Lyapunov function for (3.8).

Theorem 2 (Lyapunov).
The null solution of (3.8) is asymptotically stable if there exist a LFC such that the Euler derivative is strictly negative definite along trajectories (3.8), that is

$$\dot{L} < 0 \tag{3.10}$$

Corollary
Let L be a LFC and S be any level surface of L, that is

$$S(c_0) = \{x \in R^n : L(x) = c_0\}; \quad c_0 = const > 0$$

Then solutions of (3.8) are uniformly ultimately bounded with respect to S if

$$\dot{L} = \langle dL, f(x) \rangle < 0$$

for x outside of S.

Theorem 3 (LaSalle).
Given the system (3.8), suppose L is LFC and the Euler derivative

$$\dot{L} \leq 0$$

Then (3.8) is asymptotically stable if $\dot{L}$ does not vanish identically along any solution of (3.8) other then the null solution.

Quadratic form. A class of scalar functions that plays an important role in the stability analysis based on the second method of Liapunov is the quadratic form. An example is

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{12} & p_{22} & \cdots & p_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

Note that $\mathbf{x}$ is a real vector and $\mathbf{P}$ is a real symmetric matrix.

The positive definiteness of the quadratic form or the Hermitian form $V(\mathbf{x})$ can be determined by Sylvester's criterion, which states that the necessary and sufficient conditions that the quadratic form or Hermitian form $V(\mathbf{x})$ be positive definite are that all the successive principal minors of $\mathbf{P}$ be positive; that is,

$$p_{11} > 0, \qquad \begin{vmatrix} p_{11} & p_{12} \\ \bar{p}_{12} & p_{22} \end{vmatrix} > 0, \qquad \ldots, \qquad \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \bar{p}_{12} & p_{22} & \cdots & p_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \bar{p}_{1n} & \bar{p}_{2n} & \cdots & p_{nn} \end{vmatrix} > 0$$

(Note that $\bar{p}_{ij}$ is the complex conjugate of p_{ij}. For the quadratic form, $\bar{p}_{ij} = p_{ij}$.)

$V(\mathbf{x}) = \mathbf{x}^*\mathbf{P}\mathbf{x}$ is positive semidefinite if $\mathbf{P}$ is singular and all the principal minors are nonnegative.

$V(\mathbf{x})$ is negative definite if $-V(\mathbf{x})$ is positive definite. Similarly, $V(\mathbf{x})$ is negative semidefinite if $-V(\mathbf{x})$ is positive semidefinite.

Show that the following quadratic form is positive definite:

$$V(\mathbf{x}) = 10x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_1x_3$$

The quadratic form $V(\mathbf{x})$ can be written

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Applying Sylvester's criterion, we obtain

$$10 > 0, \qquad \begin{vmatrix} 10 & 1 \\ 1 & 4 \end{vmatrix} > 0, \qquad \begin{vmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{vmatrix} > 0$$

Since all the successive principal minors of the matrix $\mathbf{P}$ are positive, $V(\mathbf{x})$ is positive definite.

EXAMPLE 9–19 Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Clearly, the only equilibrium state is the origin, $\mathbf{x} = \mathbf{0}$. Determine the stability of this state.

Let us choose the following scalar function as a possible Liapunov function:

$$V(\mathbf{x}) = 2x_1^2 + x_2^2 = \text{positive definite}$$

Then $\dot{V}(\mathbf{x})$ becomes

$$\dot{V}(\mathbf{x}) = 4x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1x_2 - 2x_2^2$$

$\dot{V}(\mathbf{x})$ is indefinite. This implies that this particular $V(\mathbf{x})$ is not a Liapunov function, and therefore stability cannot be determined by its use. [Since the eigenvalues of the coefficient matrix are $(-1 + j\sqrt{3})/2$ and $(-1 - j\sqrt{3})/2$, clearly the origin of the system is stable. This means that we have not chosen a suitable Liapunov function.]

If we choose the following scalar function as a possible Liapunov function,

$$V(\mathbf{x}) = x_1^2 + x_2^2 = \text{positive definite}$$

then

$$\dot{V}(\mathbf{x}) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2x_2^2 \quad \leq 0$$

which is negative semidefinite. If $\dot{V}(\mathbf{x})$ is to vanish identically for $t \geq t_1$, then x_2 must be zero for all $t \geq t_1$. This requires that $\dot{x}_2 = 0$ for $t \geq t_1$. Since

$$\dot{x}_2 = -x_1 - x_2$$

x_1 must also be equal to zero for $t \geq t_1$. This means that $\dot{V}(x)$ vanishes identically only at the origin. Hence, by Theorem 9–2, the equilibrium state at the origin is asymptotically stable in the large.

To show that a different choice of a Liapunov function yields the same stability information, let us choose the following scalar function as another possible Liapunov function:

$$V(\mathbf{x}) = \tfrac{1}{2}[(x_1 + x_2)^2 + 2x_1^2 + x_2^2] = \text{positive definite}$$

Then $\dot{V}(\mathbf{x})$ becomes

$$\begin{aligned} \dot{V}(\mathbf{x}) &= (x_1 + x_2)(\dot{x}_1 + \dot{x}_2) + 2x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= (x_1 + x_2)(x_2 - x_1 - x_2) + 2x_1x_2 + x_2(-x_1 - x_2) \\ &= -(x_1^2 + x_2^2) \quad < 0 \end{aligned}$$

which is negative definite. Since $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, by Theorem 9–1, the equilibrium state at the origin is asymptotically stable in the large.

Since the stability theorems of the second method require positive definiteness of $V(\mathbf{x})$, we often (but not always) choose $V(\mathbf{x})$ to be a quadratic form or Hermitian form in $\mathbf{x}$. (Note that the simplest positive-definite function is a quadratic form or Hermitian form.) Then we examine if $\dot{V}(\mathbf{x})$ is at least negative semidefinite.

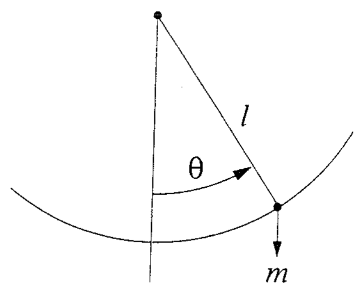


Figure 1.1: Pendulum.

Pendulum Equation

Consider the simple pendulum shown in Figure 1.1, where l denotes the length of the rod and m denotes the mass of the bob. Assume the rod is rigid and has zero mass. Let θ denote the angle subtended by the rod and the vertical axis through the pivot point. The pendulum is free to swing in the vertical plane. The bob of the pendulum moves in a circle of radius l. To write the equation of motion of the pendulum, let us identify the forces acting on the bob. There is a downward gravitational force equal to mg, where g is the acceleration due to gravity. There is also a frictional force resisting the motion, which we assume to be proportional to the speed of the bob with a coefficient of friction k. Using Newton's second law of motion, we can write the equation of motion in the tangential direction as

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

Writing the equation of motion in the tangential direction has the advantage that the rod tension, which is in the normal direction, does not appear in the equation. Note that we could have arrived at the same equation by writing the moment equation about the pivot point. To obtain a state-space model of the pendulum, let us take the state variables as $x_1 = \theta$ and $x_2 = \dot{\theta}$. Then, the state equation is given by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{aligned}$$

Example 3.2. Consider again the pendulum equation, but this time with friction:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2\end{aligned}$$

Let us try again the energy as a Lyapunov function candidate.

$$\begin{aligned}\text{Lyp}\quad V(x) &= \left(\frac{g}{l}\right)(1-\cos x_1) + \tfrac{1}{2}x_2^2\\ \dot{V}(x) &= \left(\frac{g}{l}\right)\dot{x}_1\sin x_1 + x_2\dot{x}_2\\ &= -\left(\frac{k}{m}\right)x_2^2 \leq 0\end{aligned}$$

$\dot{V}(x)$ is negative semidefinite. It is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1; that is, $\dot{V}(x) = 0$ along the x_1-axis. Therefore, we can only conclude that the origin is stable. However, using the phase portrait of the pendulum equation, we have seen that when $k > 0$, the origin is asymptotically stable. The energy Lyapunov function fails to show this fact. Let us look for a Lyapunov function $V(x)$ that would have a negative definite $\dot{V}(x)$. Starting from the energy Lyapunov function, let us replace the term $\frac{1}{2}x_2^2$ by the more general quadratic form $\frac{1}{2}x^TPx$ for some 2×2 positive definite matrix P.

$$\begin{aligned}V(x) &= \tfrac{1}{2}x^TPx + \left(\frac{g}{l}\right)(1-\cos x_1)\\ &= \tfrac{1}{2}[x_1\ x_2]\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \left(\frac{g}{l}\right)(1-\cos x_1)\end{aligned}$$

For the quadratic form $\frac{1}{2}x^TPx$ to be positive definite, the elements of the matrix P must satisfy

$$p_{11} > 0;\ p_{22} > 0;\ p_{11}p_{22} - p_{12}^2 > 0$$

The derivative $\dot{V}(x)$ is given by

$$\begin{aligned}\dot{V}(x) &= \left[p_{11}x_1 + p_{12}x_2 + \left(\frac{g}{l}\right)\sin x_1\right]x_2\\ &\quad + (p_{12}x_1 + p_{22}x_2)\left[-\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2\right]\\ &= \left(\frac{g}{l}\right)(1-p_{22})x_2\sin x_1 - \left(\frac{g}{l}\right)p_{12}x_1\sin x_1\\ &\quad + \left[p_{11} - p_{12}\left(\frac{k}{m}\right)\right]x_1x_2 + \left[p_{12} - p_{22}\left(\frac{k}{m}\right)\right]x_2^2\end{aligned}$$

Now we want to choose p_{11}, p_{12}, and p_{22} such that $\dot{V}(x)$ is negative definite. Since the cross product terms $x_2\sin x_1$ and x_1x_2 are sign indefinite, we will cancel them by taking

$$p_{22} = 1;\ \ p_{11} = \left(\frac{k}{m}\right)p_{12}$$

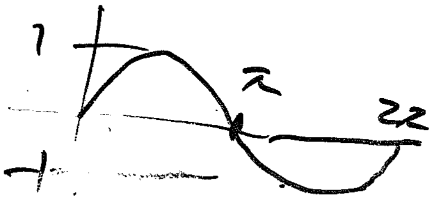
With these choices, p_{12} must satisfy

$$0 < p_{12} < \left(\frac{k}{m}\right)$$

for $V(x)$ to be positive definite. Let us take $p_{12} = 0.5(k/m)$. Then, $\dot{V}(x)$ is given by

$$\dot{V}(x) = -\tfrac{1}{2}\left(\frac{g}{l}\right)\left(\frac{k}{m}\right) x_1 \sin x_1 - \tfrac{1}{2}\left(\frac{k}{m}\right) x_2^2$$

The term $x_1 \sin x_1 > 0$ for all $0 < |x_1| < \pi$. Defining a domain D by $D = \{x \in R^2 \mid |x_1| < \pi\}$, we see that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D. Thus, we conclude that the origin is asymptotically stable. △



In searching for a Lyapunov function in Example 3.2 we approached the problem in a backward manner. We investigated an expression for $\dot{V}(x)$ and went back to choose the parameters of $V(x)$ so as to make $\dot{V}(x)$ negative definite. This is a useful idea in searching for a Lyapunov function. A procedure that exploits this idea is known as the *variable gradient method.* To describe this procedure, let $V(x)$ be a scalar function of x and $g(x) = \nabla V = (\partial V/\partial x)^T$. The derivative $\dot{V}(x)$ along the trajectories is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x) f(x)$$

The idea now is to try to choose $g(x)$ such that it would be the gradient of a positive definite function $V(x)$ and, at the same time, $\dot{V}(x)$ would be negative definite. It is not difficult to verify that $g(x)$ is the gradient of a scalar function if and only if the Jacobian matrix $[\partial g/\partial x]$ is symmetric, that is,

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\, i, j = 1, \ldots, n$$

Under this constraint, we start by choosing $g(x)$ such that $g^T(x) f(x)$ is negative definite. The function $V(x)$ is then computed from the integral

$$V(x) = \int_0^x g^T(y)\, dy = \int_0^x \sum_{i=1}^{n} g_i(y)\, dy_i$$

The integration is taken over any path joining the origin to x. Usually, this is done along the axes; that is,

$$\begin{aligned} V(x) \;&=\; \int_0^{x_1} g_1(y_1, 0, \ldots, 0)\, dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, \ldots, 0)\, dy_2 \\ &\quad + \cdots + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, y_n)\, dy_n \end{aligned}$$

By leaving some parameters of $g(x)$ undetermined, one would try to choose them to ensure that $V(x)$ is positive definite. The variable gradient method can be used to arrive at the Lyapunov function of Example 3.2. Instead of repeating the example, we illustrate the method on a slightly more general system.

Example 3.3. Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - a x_2 \end{aligned}$$

where $a > 0$, $h(\cdot)$ is locally Lipschitz, $h(0) = 0$ and $y h(y) > 0$ for all $y \neq 0$, $y \in (-b, c)$ for some positive constants b and c. The pendulum equation is a special case of this system. To apply the variable gradient method, we want to choose a second-order vector $g(x)$ that satisfies

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

$$\dot{V}(x) = g_1(x) x_2 - g_2(x)[h(x_1) + a x_2] < 0, \quad \text{for } x \neq 0$$

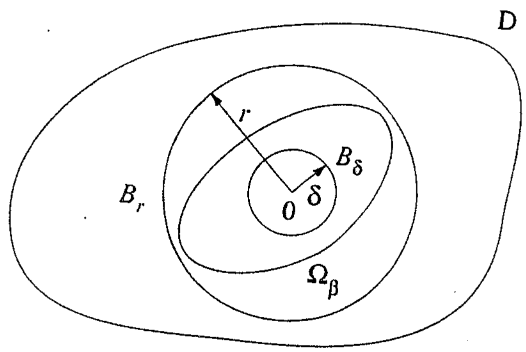


Figure 3.1: Geometric representation of sets in the proof of Theorem 3.1.

Theorem 3.1 *Let $x = 0$ be an equilibrium point for (3.1). Let $V : D \to R$ be a continuously differentiable function on a neighborhood D of $x = 0$, such that*

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \tag{3.2}$$

$$\dot{V}(x) \le 0 \text{ in } D \tag{3.3}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\} \tag{3.4}$$

then $x = 0$ is asymptotically stable. ◇

Proof: Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ such that

$$B_r = \{x \in R^n \mid \|x\| \le r\} \subset D$$

Let $\alpha = \min_{\|x\|=r} V(x)$. Then, $\alpha > 0$ by (3.2). Take $\beta \in (0, \alpha)$, and let

$$\Omega_\beta = \{x \in B_r \mid V(x) \le \beta\}$$

Then, Ω_β is entirely inside B_r;[2] see Figure 3.1. The set Ω_β has the property

[2]This fact can be shown by contradiction. Suppose Ω_β is not entirely inside B_r, then there is a point $p \in \Omega_\beta$ that lies on the boundary of B_r. At this point, $V(p) \ge \alpha > \beta$, but, for all $x \in \Omega_\beta$, $V(x) \le \beta$, a contradiction.

that any trajectory starting in Ω_β at $t = 0$, stays in Ω_β for all $t \geq 0$. This follows from (3.3) since

$$\dot{V}(x(t)) \leq 0 \;\Rightarrow\; V(x(t)) \leq V(x(0)) \leq \beta, \;\forall\, t \geq 0$$

Since Ω_β is a compact set,[3] we conclude by Theorem 2.4 that equation (3.1) has a unique solution defined for all $t \geq 0$, whenever $x(0) \in \Omega_\beta$. Since $V(x)$ is continuous and $V(0) = 0$, there is $\delta > 0$ such that

$$\|x\| \leq \delta \;\Rightarrow\; V(x) < \beta$$

Then

$$B_\delta \subset \Omega_\beta \subset B_r$$

and

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

Therefore,

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \epsilon, \quad \forall\, t \geq 0$$

which shows that the equilibrium point $x = 0$ is stable. Now, assume that (3.4) holds as well. To show asymptotic stability we need to show that $x(t) \to 0$ as $t \to \infty$; that is, for every $a > 0$, there is $T > 0$ such that $\|x(t)\| < a$, for all $t > T$. By repetition of previous arguments we know that for every $a > 0$, we can choose $b > 0$ such that $\Omega_b \subset B_a$. Therefore, it is sufficient to show that $V(x(t)) \to 0$ as $t \to \infty$. Since $V(x(t))$ is monotonically decreasing and bounded from below by zero,

$$V(x(t)) \to c \geq 0 \text{ as } t \to \infty$$

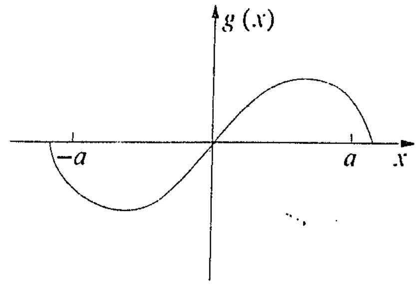
To show that $c = 0$, we use a contradiction argument. Suppose $c > 0$. By continuity of $V(x)$, there is $d > 0$ such that $B_d \subset \Omega_c$. The limit $V(x(t)) \to c > 0$ implies that the trajectory $x(t)$ lies outside the ball B_d for all $t \geq 0$. Let $-\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x)$. Then, $-\gamma < 0$ by (3.4). It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \leq V(x(0)) - \gamma t$$

Since the right-hand side will eventually become negative, the inequality contradicts the assumption that $c > 0$. $\square$

Consider the first-order differential equation

$$\dot{x} = -g(x)$$



where $g(x)$ is locally Lipschitz on $(-a, a)$ and satisfies

$$g(0) = 0; \quad xg(x) > 0, \quad \forall\, x \neq 0, \ x \in (-a, a)$$

The system has an isolated equilibrium point at the origin. It is not difficult in this simple example to see that the origin is asymptotically stable, because solutions starting on either side of the origin will have to move toward the origin due to the sign of the derivative $\dot{x}$. To arrive at the same conclusion using Lyapunov's theorem, consider the function

$$V(x) = \int_0^x g(y)\, dy$$

Over the domain $D = (-a, a)$, $V(x)$ is continuously differentiable, $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. Thus, $V(x)$ is a valid Lyapunov function candidate. To see whether or not $V(x)$ is indeed a Lyapunov function, we calculate its derivative along the trajectories of the system.

$$\dot{V}(x) = \frac{\partial V}{\partial x}[-g(x)] = -g^2(x) < 0, \ \forall\, x \in D - \{0\}$$

Thus, by Theorem 3.1 we conclude that the origin is asymptotically stable.

$\triangle$

LaSalle Case

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -g(x_1) - h(x_2)\end{aligned}$$

where $g(\cdot)$ and $h(\cdot)$ are locally Lipschitz and satisfy

$$g(0) = 0, \quad yg(y) > 0, \quad \forall\, y \neq 0, \ y \in (-a, a)$$

$$h(0) = 0, \quad yh(y) > 0, \quad \forall\, y \neq 0, \ y \in (-a, a)$$

The system has an isolated equilibrium point at the origin. Depending upon the functions $g(\cdot)$ and $h(\cdot)$ it might have other equilibrium points. The equation of this system can be viewed as a generalized pendulum equation with $h(x_2)$ as the friction term. Therefore, a Lyapunov function candidate may be taken as the energy-like function

$$V(x) = \int_0^{x_1} g(y)\, dy \ + \ \tfrac{1}{2}x_2^2$$

Let $D = \{x \in R^2 \mid -a < x_i < a\}$. $V(x)$ is positive definite in D. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = g(x_1)x_2 + x_2[-g(x_1) - h(x_2)] = -x_2 h(x_2) \leq 0$$

Thus, $\dot{V}(x)$ is negative semidefinite. To characterize the set $S = \{x \in D \mid \dot{V}(x) = 0\}$, note that

$$\dot{V}(x) = 0 \ \Rightarrow \ x_2 h(x_2) = 0 \ \Rightarrow \ x_2 = 0, \quad \text{since } -a < x_2 < a$$

Hence, $S = \{x \in D \mid x_2 = 0\}$. Suppose that $x(t)$ is a trajectory that belongs to S for all t.

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_1(t) \equiv 0 \Rightarrow x_1(t) \equiv c, \quad \text{where } c \in (-a, a)$$

Also,

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow g(c) = 0 \Rightarrow c = 0$$

Therefore, the only solution that can stay in S for all t is the trivial solution $x(t) = 0$. Thus, the origin is asymptotically stable. $\triangle$

Lyapunov stability for linear Time-invariant system

$$\dot{x} = Ax \qquad x \in R^n$$

Choose Lyapunov function candidate as

$$V = x^T P x$$

$P \in R^{n \times n}$ is a positive definite matrix

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x}$$

$$= (P^T \dot{x})^T x + x^T P \dot{x}$$

$$= (P^T A x)^T x + x^T P A x$$

$$= x^T A^T P x + x^T P A x$$

$$= x^T (A^T P + P A) x$$

Define

$$\underline{A^T P + P A = -Q}$$

Then

$$\dot{V} = -x^T Q x < 0$$

where Equation

$$A^T P + P A = -Q$$

is defined as <u>Lyapunov Equation</u>

There are two possible ways:

(1) One can pick a particular matrix P and study the properties of matrix Q resulting from Lyapunov equation

(2) One can pick Q and study the matrix P resulting from Lyapunov equation.

We can now state one of the main results for the Lyapunov equation

Theorem: ① A is a Hurwitz matrix

② There exists some positive definite matrix Q such that Lyapunov equation has a corresponding unique solution for P. and this P is positive definite

③ For every positive definite matrix Q the solution P is positive definite.

the above three statements are equivalent.

~~Thus~~.

Consider the motion of a space vehicle about the principal axes of inertia. The Euler equations are

$$A\dot{\omega}_x - (B - C)\omega_y\omega_z = T_x$$

$$B\dot{\omega}_y - (C - A)\omega_z\omega_x = T_y$$

$$C\dot{\omega}_z - (A - B)\omega_x\omega_y = T_z$$

where A, B, and C denote the moments of inertia about the principal axes; ω_x, ω_y, and ω_z denote the angular velocities about the principal axes; and T_x, T_y, and T_z are the control torques.

Assume that the space vehicle is tumbling in orbit. It is desired to stop the tumbling by applying control torques, which are assumed to be

$$T_x = k_1 A\omega_x$$

$$T_y = k_2 B\omega_y$$

$$T_z = k_3 C\omega_z$$

Determine sufficient conditions for asymptotically stable operation of the system.

Solution. Let us choose the state variables as

$$x_1 = \omega_x, \qquad x_2 = \omega_y, \qquad x_3 = \omega_z$$

Then the system equations become

$$\dot{x}_1 - \left(\frac{B}{A} - \frac{C}{A}\right)x_2x_3 = k_1x_1$$

$$\dot{x}_2 - \left(\frac{C}{B} - \frac{A}{B}\right)x_3x_1 = k_2x_2$$

$$\dot{x}_3 - \left(\frac{A}{C} - \frac{B}{C}\right)x_1x_2 = k_3x_3$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} k_1 & \frac{B}{A}x_3 & -\frac{C}{A}x_2 \\ -\frac{A}{B}x_3 & k_2 & \frac{C}{B}x_1 \\ \frac{A}{C}x_2 & -\frac{B}{C}x_1 & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The equilibrium state is the origin, or $\mathbf{x} = \mathbf{0}$. If we choose

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} = \mathbf{x}^T \begin{bmatrix} A^2 & 0 & 0 \\ 0 & B^2 & 0 \\ 0 & 0 & C^2 \end{bmatrix} \mathbf{x}$$

$$= A^2x_1^2 + B^2x_2^2 + C^2x_3^2$$

$$= \text{positive definite}$$

then the time derivative of $V(\mathbf{x})$ is

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}}$$

$$= \mathbf{x}^T \begin{bmatrix} k_1 & -\frac{A}{B}x_3 & \frac{A}{C}x_2 \\ \frac{B}{A}x_2 & k_2 & -\frac{B}{C}x_1 \\ -\frac{C}{A}x_2 & \frac{C}{B}x_1 & k_3 \end{bmatrix} \begin{bmatrix} A^2 & 0 & 0 \\ 0 & B^2 & 0 \\ 0 & 0 & C^2 \end{bmatrix} \mathbf{x}$$

$$+ \mathbf{x}^T \begin{bmatrix} A^2 & 0 & 0 \\ 0 & B^2 & 0 \\ 0 & 0 & C^2 \end{bmatrix} \begin{bmatrix} k_1 & \frac{B}{A}x_3 & -\frac{C}{A}x_2 \\ -\frac{A}{B}x_3 & k_2 & \frac{C}{B}x_1 \\ \frac{A}{C}x_2 & -\frac{B}{C}x_1 & k_3 \end{bmatrix} \mathbf{x}$$

$$= \mathbf{x}^T \begin{bmatrix} 2k_1A^2 & 0 & 0 \\ 0 & 2k_2B^2 & 0 \\ 0 & 0 & 2k_3C^2 \end{bmatrix} \mathbf{x} = -\mathbf{x}^T\mathbf{Q}\mathbf{x}$$

For asymptotic stability, the sufficient condition is that $\mathbf{Q}$ be positive definite. Hence we require

$$k_1 < 0, \qquad k_2 < 0, \qquad k_3 < 0$$

If the k_i are negative, then noting that $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, we see that the equilibrium state is asymptotically stable in the large.

Summary:

The favorable aspect for the stability theorems

① They enable one to draw conclusions about the stability status of a system without solving the system equation

The unfavorable aspect:

They represent only sufficient condition for ~~[illegible]~~ the various forms of stability. Thus, if a particular Lyapunov function candidate V fails to satisfy the hypothesis on $\dot{V}$, then no conclusion can be drawn, and one has to begin a new with another Lyapunov function candidate.

The role of Lyapunov theory today can be as follows.

Originally Lyapunov stability theory was developed as a means of testing the stability status of a given system. Nowdays, however, it is increasingly being used to guarantee stability.

In the following example, we first choose a Lyapunov function candidate, and then choose the control law to ensure that the hypotheses of a particular stability theorem are satisfied. In this way, the problem searching a Lyapunov function is alleviated.

Consider a nonlinear system

$$\dot{x} = f(x) + g(x)u \qquad f(0) = 0$$

where $x \in R^n$, $u \in R$

we can pick a function $V(x)$ as a Lyapunov candidate. The objective is to find u to guarantee that for all $x \in R^n$,

$$\frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial x} g(x) u \leq -w(x)$$

where $w(x)$ is a positive definite function

A particular control law can be designed

$$u = \begin{cases} -\dfrac{\dfrac{\partial V}{\partial x} f + \sqrt{\left(\dfrac{\partial V}{\partial x} f\right)^2 + \left(\dfrac{\partial V}{\partial x} g\right)^4}}{\dfrac{\partial V}{\partial x} g}, & \dfrac{\partial V}{\partial x} g \neq 0 \\[2ex] 0, & \dfrac{\partial V}{\partial x} g = 0 \end{cases}$$

Example:

Consider the follow scalar system

$$\dot{x} = x^3 + x^2 u \qquad \dot{x} = f(x) + g(x)u$$

To design a control law u so that $f(0) = 0$

$x \to 0$.

Solution:

$$f(x) = x^3 \qquad g(x) = x^2$$

Choose $V(x) = \frac{1}{2}x^2$, using above control law.

we have

$$u = -\frac{\frac{\partial V}{\partial x}f + \sqrt{(\frac{\partial V}{\partial x}f)^2 + (\frac{\partial V}{\partial x}g)^4}}{\frac{\partial V}{\partial x}g}$$

Since $\frac{\partial V}{\partial x} = x$

$$u = -\frac{x x^3 + \sqrt{(x^4)^2 + (x x^2)^4}}{x x^2}$$

$$= -\frac{x^4 + x^4\sqrt{1+x^4}}{x^3}$$

$$\boxed{= -x(1 + \sqrt{1+x^4})}$$

We can verify that

$$\dot{V} = -x^4\sqrt{1+x^4} < 0$$

Consider the system

$$\dot{x} = f(t, x) \qquad (3.1)$$

where $f \in R^n$, $x \in R^n$,

* Definition:

The system (3.1) is said to be autonomous if function f does not explicitly depend on t; it is said to be nonautonomous otherwise.

* Lyapunov Theorem for nonautonomous system

Theorem 4: *Let $x = 0$ be an equilibrium point for (3.1)* *Let $V : [0, \infty) \times D \to R$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -\alpha_3(\|x\|)$$

$\forall\, t \ge 0, \forall\, x \in D$, *where* $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, *and* $\alpha_3(\cdot)$ *are class* $\mathcal{K}$ *functions*. *Then $x = 0$ is uniformly asymptotically stable.*

Notice: V(t, x) includes t, In this case, we must use above result.

A continuous function :[0,)[0,), is said to belong to class K if it is strictly increasing (0)=0 and (t) as t.

radially unbounded: $V(x, t) \to \infty$ as $|x| \to \infty$ uniformly in t

Consider a system

$$\frac{d}{dt}\left(\frac{1}{\sin t + 2}\right) = -\frac{\frac{d}{dt}(\sin t)}{(\sin t + 2)^2}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - (2+\sin t)x_1$$

Try the Lyapunov function candidate

$$V(t, x_1, x_2) = x_1^2 + \frac{x_2^2}{2+\sin t}$$

$$\underbrace{x_1^2 + \frac{x_2^2}{3}}_{\alpha_1} \le V(t, x_1, x_2) \le \underbrace{x_1^2 + x_2^2}_{\alpha_2}$$

$$\frac{dV}{dt} = \dot{V}(t, x_1, x_2)$$
$$= -x_2^2 \frac{\cos t}{(2+\sin t)^2} + 2x_1\dot{x}_1 + \frac{2x_2}{2+\sin t}\dot{x}_2$$
$$= -x_2^2 \frac{\cos t}{(2+\sin t)^2} + 2x_1 x_2 + \frac{2x_2}{2+\sin t}\cdot\left[-x_2 - 2(2+\sin t)x_1\right]$$
$$= -\frac{\cos t + 2(2+\sin t)}{(2+\sin t)^2} x_2^2$$
$$= \underbrace{\boxed{-\frac{4 + 2\sin t + \cos t}{(2+\sin t)^2} x_2^2}}_{\alpha_3(x)} < 0 \quad \forall t \ge 0 \quad \forall x_1, x_2.$$

Consider the system

$$\dot{X} = f(t, x) \qquad (3.1)$$

where $f \in R^n$, $x \in R^n$,

* Definition:

The system (3.1) is said to be autonomous if function f does not explicitly depend on t; it is said to be nonautonomous otherwise.

* Lyapunov Theorem for nonautonomous system

Theorem 4: *Let $x = 0$ be an equilibrium point for (3.1)* *Let $V : [0, \infty) \times D \to R$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -\alpha_3(\|x\|)$$

$\forall\, t \ge 0$, $\forall\, x \in D$, where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, and $\alpha_3(\cdot)$ are class $\mathcal{K}$ functions. Then $x = 0$ is uniformly asymptotically stable.

A function $\phi : R_+ \to R_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\phi(0) = 0$.

Theorem 1: Given a continuous system

$$\dot{x}(t) = f\big(x(t), t\big)$$

where $x(t)$ is an $n \times 1$ vector, let $V(x, t)$ be the associated Lyapunov function with the following properties:

$$\lambda_1 \|x(t)\|^2 \leq V(x, t) \leq \lambda_2 \|x(t)\|^2 \quad \forall\ (x, t) \in \mathbb{R}^n \times \mathbb{R}$$
$$\dot{V}(x, t) \leq -\lambda_3 \|x(t)\|^2 + \epsilon \quad \forall\ (x, t) \in \mathbb{R}^n \times \mathbb{R}$$

where λ_1, λ_2, λ_3, and ϵ are positive scalar constants. From the above properties of the Lyapunov function, the state $x(t)$ is GUUB in the sense that

$$\|x(t)\| \leq \left[\frac{\lambda_2}{\lambda_1}\|x(0)\|^2 e^{-\lambda t} + \frac{\epsilon}{\lambda_1 \lambda}\left[1 - e^{-\lambda t}\right]\right]^{1/2}$$

where $\lambda = \lambda_3/\lambda_2$, and e is used to denote the natural logarithm exponential. Global uniform ultimate boundedness.

Comments on "Adaptive PD Controller for Robot Manipulators"

Rafael Kelly

Abstract—An alternative analysis without invoking the LaSalle invariance principle of the PD controller with desired gravity compensation for robots presented in Section III of Tomei's paper [IEEE Trans. Robotics Automat., vol. 7, no. 4, pp. 565–570, Aug. 1991] is established in this work. We use this result to straightforwardly develop a new PD controller with adaptive desired gravity compensation, and we give conditions to assure closed-loop stability and global zero position error convergence.

I. Introduction

Following the notation of [1], in the absence of friction and other disturbances, the dynamics of a serial n-link rigid robot manipulator can be written as

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + e(q) = u \tag{1}$$

where q is the $n \times 1$ vector of joint displacements, u is the $n \times 1$ vector of applied joint torques, $B(q)$ is the $n \times n$ symmetric positive definite manipulator inertia matrix, $C(q,\dot{q})\dot{q}$ is the $n \times 1$ vector of centripetal and Coriolis torques, and $e(q)$ is the $n \times 1$ vector of gravitational torques. We assume that the links are jointed together with revolute joints.

We digress momentarily to present the following technical result. Let us define $\tilde{q} = q_0 - q$. Given two $n \times 1$ vectors q_0 and $\tilde{q}$, define function $f(q_0, \tilde{q})$ as

$$f(q_0, \tilde{q}) = U(q_0 - \tilde{q}) - U(q_0) + e(q_0)^T\tilde{q} + \frac{1}{\varepsilon_1}\tilde{q}^T K_P \tilde{q} \tag{2}$$

where $K_P = K_P^T$ is a $n \times n$ positive definite matrix, $U(q) = U(q_0 - \tilde{q})$ is the potential energy of the robot, and ε_1 is a positive constant. For all constant $n \times 1$ vectors q_0, function $f(q_0, \tilde{q})$ is a positive definite function in $\tilde{q}$ provided that

$$\lambda_m(K_P) > \frac{\varepsilon_1}{2} M_1 \geq \frac{\varepsilon_1}{2}\left\|\frac{\partial e(q)}{\partial q}\right\|$$

where $\lambda_m(K_P)$ indicates the smallest eigenvalue of K_P. As pointed out in [1], the positive constant M_1 always exists for all $q \in \Re^n$.

II. PD with Desired Gravity Compensation

In this section, we present an alternative analysis of the PD with the desired gravity compensation controller presented in Section III of [1].

The PD with desired gravity compensation control law is given by (6) in [1]

$$u = -K_P(q - q_0) - K_D\dot{q} + e(q_0) \tag{3}$$

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where K_P and K_D are the $n \times n$ symmetric positive definite proportional and derivative gain matrices, and q_0 is the $n \times 1$ desired joint position vector (constant). Matrix K_P is selected so that

$$\lambda_m(K_P) > M_1. \tag{4}$$

As shown in [1], with this choice of K_P, for all positive definite matrices K_D, and for all constant desired joint position vectors q_0, the control aim $\lim_{t\to\infty} q(t) = q_0$ is attained, and the closed-loop system is globally asymptotically stable. These claims are proven in [1] by invoking the LaSalle invariance principle.

In the following, we provide an alternative analysis with a different Lyapunov function that obviates the LaSalle's theorem.

Closed-Loop Analysis

Let us define the joint position error vector as $\tilde{q} = q_0 - q$. The closed-loop equation is obtained by combining the robot model (1) and control law (3):

$$\frac{d}{dt}\begin{bmatrix}\tilde{q}\\ \dot{q}\end{bmatrix} = \begin{bmatrix} -\dot{q} \\ B(q)^{-1}[K_P\tilde{q} - K_D\dot{q} - C(q,\dot{q})\dot{q} + e(q_0) - e(q)]\end{bmatrix} \tag{5}$$

whose origin $[\tilde{q}^T \;\; \dot{q}^T] = 0$ is the unique equilibrium point because K_P has been selected so that $\lambda_m(K_P) > M_1$.

To carry out the stability analysis, we propose the following Lyapunov function candidate:

$$\begin{aligned} v(\tilde{q},\dot{q}) &= \frac{1}{2}\begin{bmatrix}\tilde{q}\\ \dot{q}\end{bmatrix}^T \overbrace{\begin{bmatrix} \frac{2}{\varepsilon_2}K_P & -\frac{\varepsilon_0}{1+\|\tilde{q}\|}B(q) \\ -\frac{\varepsilon_0}{1+\|\tilde{q}\|}B(q) & B(q)\end{bmatrix}}^{P} \begin{bmatrix}\tilde{q}\\ \dot{q}\end{bmatrix} \\ &\quad + \underbrace{U(q) - U(q_0) + e(q_0)^T\tilde{q} + \frac{1}{\varepsilon_1}\tilde{q}^T K_P\tilde{q}}_{f(q_0,\tilde{q})} \\ &= \frac{1}{2}\dot{q}^T B(q)\dot{q} + U(q) - U(q_0) + e(q_0)^T\tilde{q} \\ &\quad + \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\tilde{q}^T K_P\tilde{q} - \frac{\varepsilon_0}{1+\|\tilde{q}\|}\tilde{q}^T B(q)\dot{q} \end{aligned} \tag{6}$$

where $f(q_0, \tilde{q})$ was defined in (2), and constants $\varepsilon_0 > 0$, $\varepsilon_1 > 2$ and $\varepsilon_2 > 2$ are selected as

$$\frac{2\lambda_m(K_P)}{M_1} > \varepsilon_1 > 2 \tag{7}$$

$$\varepsilon_2 = \frac{2\varepsilon_1}{\varepsilon_1 - 2} > 2 \tag{8}$$

$$\sqrt{\frac{2\lambda_m(K_P)}{\varepsilon_2\lambda_M(B)}} > \varepsilon_0. \tag{9}$$

Condition (7) assures that $f(q_0, \tilde{q})$ is a positive definite function while condition (9) guarantees that matrix P is positive definite; thus, $v(\tilde{q}, \dot{q})$ is a positive definite function. Finally, condition (8) implies that $1/\varepsilon_1 + 1/\varepsilon_2 = 1/2$.

The term $1/2[\tilde{q}^T \;\; \dot{q}^T]^T P[\tilde{q}^T \;\; \dot{q}^T]$ with $\varepsilon_2 = 2$ in Lyapunov function candidate $v(\tilde{q}, \dot{q})$, has been used in [3] to analyze the so-called fixed nonlinear inverse dynamics controller. On the other hand, term $f(q_0, \tilde{q})$ with $\varepsilon_1 = 2$, which depends on potential energy, s considered in [2] and more recently in [4] to analyze the PD with gravity compensation and PID controllers, respectively.

The Lyapunov function candidate (6) with $\varepsilon_0 = 0$, $\varepsilon_1 = 2$, and $\varepsilon_2 = 2$ together with the LaSalle invariance principle were used in [5] and in [1] to prove asymptotic stability of PD with desired gravity compensation.

The Lyapunov function candidate (6) is also different from the one proposed in [6], which provides an unified framework to Lyapunov studies of a large class of PD-type controllers, in the sense that the former includes potential energy terms and the normalized-type term $\varepsilon_0/(1 + \|\tilde{q}\|)$.

As a remark, a Lyapunov function similar to (6) can be obtained from (12) and (22) in [1] to prove asymptotic stability without invoking the LaSalle invariance principle.

After some simplifications and using the fact that, for a proper definition of $C(q, \dot{q})$, matrix $(1/2)\dot{B}(q) - C(q, \dot{q})$ is skew-symmetric and $\dot{B}(q) = C(q, \dot{q}) + C(q, \dot{q})^T$, the time derivative $\dot{v}(\tilde{q}, \dot{q})$ along the trajectories of the closed-loop equation (12) can be written as

$$\dot{v}(\tilde{q}, \dot{q}) = -\dot{q}^T K_D \dot{q} + \varepsilon \dot{q}^T B(q)\dot{q} - \varepsilon \tilde{q}^T K_P \tilde{q} + \varepsilon \tilde{q}^T K_D \dot{q} - \varepsilon \dot{q}^T C(q, \dot{q})\tilde{q} - \varepsilon \tilde{q}^T[e(q_0) - e(q)] - \dot{\varepsilon} \tilde{q}^T B(q)\dot{q}. \quad (10)$$

where $\varepsilon = \varepsilon_0/(1 + \|\tilde{q}\|)$.

Now we provide upper bounds on the following terms:

$$-\varepsilon \dot{q}^T C(q, \dot{q})\tilde{q} \le \varepsilon_0 k_C \, \| \dot{q} \|^2 \quad (11)$$

$$-\varepsilon \tilde{q}^T [e(q_0) - e(q)] \le \varepsilon M_1 \, \| \tilde{q} \|^2 \quad (12)$$

$$-\dot{\varepsilon} \tilde{q}^T B(q)\dot{q} \le \varepsilon_0 \lambda_M(B) \, \| \dot{q} \|^2 \quad (13)$$

where we used (4) and (7) from [1].

From inequalities (11), (12), and (13), it now follows that time derivative $\dot{v}(\tilde{q}, \dot{q})$ in (10) gives (14), found at the bottom of this page.

Matrix Q is positive definite provided that

$$\lambda_m(K_P) > M_1 \quad (15)$$

$$\frac{2\lambda_m(K_D)(\lambda_m(K_P) - M_1)}{\lambda_M^2(K_D)} > \varepsilon_0 \quad (16)$$

and $\delta > 0$ if the following inequality is satisfied:

$$\frac{\lambda_m(K_D)}{2(k_C + 2\lambda_M(B))} > \varepsilon_0. \quad (17)$$

It is important to remark that constant ε_0 is only necessary for analysis purposes. Selecting ε_0 in such a way as to simultaneously satisfy (16) and (17), we have $\lambda_m(Q) > 0$. Thus, from (14) we obtain

$$\dot{v}(\tilde{q}, \dot{q}) \le -\varepsilon_0 \lambda_m(Q) \frac{\| \tilde{q} \|^2}{1 + \| \tilde{q} \|} - \frac{\delta}{2} \| \dot{q} \|^2 \quad (18)$$

which is a negative definite function. Finally, by invoking Lyapunov's direct method, we conclude that the origin $\begin{bmatrix}\tilde{q}^T & \dot{q}^T\end{bmatrix} = 0$ is a globally asymptotically stable equilibrium of the closed-loop equation; hence, $\lim_{t\to\infty}(q_0 - q(t)) = 0$, as desired.

III. PD with Adaptive Desired Gravity Compensation

When the payload changes during operation of the robot, or the gravitational torque vector $e(q)$ parameters cannot be evaluated accurately, the PD with desired gravity compensation controller (3) causes an offset in final positioning [5].

To compensate the offset in final positioning, [1] proposes the *adaptive PD controller*:

$$u = K_P \tilde{q} - K_D \dot{q} + E(q)\hat{p}$$

$$\hat{p}(t) = \beta \int_0^t E(q)^T \left[\frac{2\tilde{q}(\sigma)}{1 + 2\tilde{q}(\sigma)^T \tilde{q}(\sigma)} - \gamma \dot{q}(\sigma)\right] d\sigma + \hat{p}(0)$$

where the gravity vector $e(q)$ has been expressed as $e(q) = E(q)p$, p is the $m \times 1$ unknown parameter vector, which is assumed constant, $E(q)$ is an $n \times m$ known matrix, and β and γ are suitably selected positive constants.

Another solution to point-to-point robot control with parametric uncertainty is to consider an adaptive version of the PD controller with desired gravity compensation (3) given by

$$u = K_P \tilde{q} - K_D \dot{q} + E(q_0)\hat{p} \quad (19)$$

$$\hat{p}(t) = \Gamma E(q_0)^T \int_0^t \left[\frac{\varepsilon_0}{1 + \| \tilde{q}(\sigma) \|}\tilde{q}(\sigma) - \dot{q}(\sigma)\right] d\sigma + \hat{p}(0) \quad (20)$$

where K_P and K_D are again the $n \times n$ symmetric positive definite proportional and derivative matrices, $\hat{p}$ is the so-called $m \times 1$ adaptive parameter vector, Γ is the $m \times m$ symmetric positive definite adaptation gain matrix, $\hat{p}(0)$ is any $m \times 1$ vector usually selected in practice as the "best" *a priori* approximation available of the unknown vector p, and ε_0 is a positive constant.

Among the design matrices K_P, K_D, Γ, vector $\hat{p}(0)$, and constant ε_0, only K_P and ε_0 must be carefully chosen using *a priori* weak knowledge on the robot dynamic model. Specifically, we assume to know the constants $\lambda_M(B)$, k_C, and M_1.

The symmetric positive definite matrix K_P and the positive constant ε_0 must be chosen in such a way to verify

- $\lambda_m(K_P) > M_1$
- $\sqrt{\frac{2\lambda_m(K_P)}{\varepsilon_2 \lambda_M(B)}} > \varepsilon_0$
- $\frac{2\lambda_m(K_D)(\lambda_m(K_P) - M_1)}{\lambda_M^2(K_D)} > \varepsilon_0$ and
- $\frac{\lambda_m(K_D)}{2(k_C + 2\lambda_M(B))} > \varepsilon_0$

where ε_2 is given in (8).

Closed-Loop Analysis

Let us denote the unknown parameter error vector $\tilde{p} = p - \hat{p}$. The closed-loop equation is obtained by combining the robot model (1)

$$\dot{v}(\tilde{q}, \dot{q}) \le -\varepsilon \begin{bmatrix} \| \tilde{q} \| \\ \| \dot{q} \| \end{bmatrix}^T \overbrace{\begin{bmatrix} \lambda_m(K_P) - M_1 & -\frac{1}{2}\lambda_M(K_D) \\ -\frac{1}{2}\lambda_M(K_D) & \frac{1}{2\varepsilon_0}\lambda_m(K_D) \end{bmatrix}}^{Q} \begin{bmatrix} \| \tilde{q} \| \\ \| \dot{q} \| \end{bmatrix} - \frac{1}{2} \underbrace{[\lambda_m(K_D) - 2\varepsilon_0(k_C + 2\lambda_M(B))]}_{\delta} \| \dot{q} \|^2 . \quad (14)$$

and the adaptive control law (19) and (20):

$$\frac{d}{dt}\begin{bmatrix}\tilde{q}\\ \dot{q}\\ \tilde{p}\end{bmatrix} = \begin{bmatrix}-\dot{q}\\ B(q)^{-1}[K_P\tilde{q} - K_D\dot{q} - C(q,\dot{q})\dot{q} + e(q_0) - e(q) + E(q_0)\tilde{p}]\\ \Gamma\, E(q_0)^T\left[\frac{\varepsilon_0}{1+\|\tilde{q}\|}\tilde{q} - \dot{q}\right]\end{bmatrix} \tag{21}$$

whose origin $[\tilde{q}^T \;\; \dot{q}^T \;\; \tilde{p}^T] = 0$ is an equilibrium point. To carry out the origin stability analysis, we consider the following Lyapunov function candidate:

$$v(\tilde{q},\dot{q},\tilde{p}) = \frac{1}{2}\begin{bmatrix}\tilde{q}\\ \dot{q}\\ \tilde{p}\end{bmatrix}^T \begin{bmatrix}\frac{2}{\varepsilon_2}K_P & -\frac{\varepsilon_0}{1+\|\tilde{q}\|}B(q) & 0\\ -\frac{\varepsilon_0}{1+\|\tilde{q}\|}B(q) & B(q) & 0\\ 0 & 0 & \Gamma^{-1}\end{bmatrix}\begin{bmatrix}\tilde{q}\\ \dot{q}\\ \tilde{p}\end{bmatrix} + U(q) - U(q_0) + e(q_0)^T\tilde{q} + \frac{1}{\varepsilon_1}\tilde{q}^T K_P\tilde{q} \tag{22}$$

where ε_1 and ε_2 are given in (7) and (8), respectively. Due to arguments presented in previous section, (22) is a positive definite function.

Taking the time derivative of the Lyapunov function candidate (22) along the trajectories of the closed-loop equation (21) and after some tedious but straightforward simplifications, we have

$$\begin{aligned}\dot{v}(\tilde{q},\dot{q},\tilde{p}) = &-\dot{q}^T K_D\dot{q} + \varepsilon\dot{q}^T B(q)\dot{q} - \varepsilon\tilde{q}^T K_P\tilde{q} + \varepsilon\tilde{q}^T K_D\dot{q}\\ &-\varepsilon\dot{q}^T C(q,\dot{q})\tilde{q} - \varepsilon\tilde{q}^T[e(q_0) - e(q)] - \dot{\varepsilon}\tilde{q}^T B(q)\dot{q}.\end{aligned} \tag{23}$$

Notice that right-hand side in (23) and (10) are the same. Thus, by using inequality (18) we obtain

$$\dot{v}(\tilde{q},\dot{q},\tilde{\theta}) \le -\varepsilon_0\lambda_m(Q)\frac{\|\tilde{q}\|^2}{1+\|\tilde{q}\|} - \frac{\delta}{2}\|\dot{q}\|^2 \tag{24}$$

which is a nonpositive function because of the choice of K_P and ε_0. It follows that $v(\tilde{q},\dot{q},\tilde{p})$ is bounded; hence, $\tilde{q}$, $\dot{q}$, and $\tilde{p}$ are in turn bounded vectors. Using Lyapunov's direct method we conclude immediately stability of the origin. To prove that the control objective is attained, i.e., $\lim_{t\to\infty} q(t) = q_0$, we invoke standard adaptive control arguments. To this end, by integrating both sides of inequality (24), we conclude that $\sqrt{\varepsilon}\tilde{q}$ and $\dot{q}$ are square integrable functions. Since $\tilde{q}$ was proven to be bounded, it follows that $\sqrt{\varepsilon}$ is bounded away from zero. Thus, $\tilde{q}$ is also a square integrable function. But a square integrable function whose derivative ($\dot{\tilde{q}} = -\dot{q}$) is bounded must tend to zero; hence, $\lim_{t\to\infty}(q_0 - q(t)) = 0$, as desired.

IV. Concluding Remarks

This work presented an alternative analysis of the PD controller with desired gravity compensation for set-point control of robot manipulators developed in Section III of [1]. By using a suitable Lyapunov function, and without invoking LaSalle's theorem, we showed, as in [1], that closed-loop global asymptotic stability can be attained. Also, we have proposed a PD controller with adaptive desired gravity compensation where, under adequate conditions, the closed-loop system is stable and global zero position error convergence is assured.

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Correction to "Robust Adaptive Controller Designs for Robot Manipulator Systems"

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There are two important typing errors in the above paper.[1] First, in (8d) where it is written

$$m(t) > -\lambda_{\min}(Q)\|e_p\|^2/\{\lambda_{\min}(U)\|u_z\|^2\}$$

the $\lambda_{\min}(U)$ should read

$$\lambda_{\max}(U)$$

(this typing error has somehow been propagated throughout many of our papers). We thank our many colleagues who brought this fact to our attention, and we thank S.-J. Xu, M. Darouach, and J. Schaefers, who brought it to the attention of the IEEE Transactions on Robotics and Automation.

Second, the second term in the denominator of (16)

$$\lambda_{\max}(U)/2\|u_z\|^2$$

should read

$$\lambda_{\max}(U)\|u_z\|^2/2.$$

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