19.1 Principle of Work and Energy

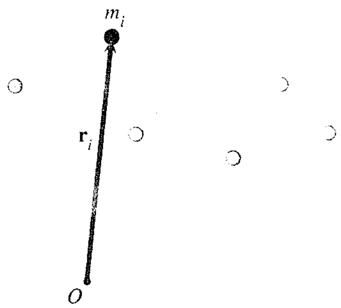


Figure 19.1
A system of particles. The vector $\mathbf{r}_i$ is the position vector of the ith particle.

We will show that the work done on a rigid body by external forces and couples as it moves between two positions is equal to the change in its kinetic energy. To obtain this result, we adopt the same approach used in Chapter 18 to obtain the equations of motion for a rigid body. We derive the principle of work and energy for a system of particles and use it to deduce the principle for a rigid body.

Let m_i be the mass of the ith particle of a system of N particles. Let $\mathbf{r}_i$ be the position of the ith particle relative to a point O that is fixed with respect to an inertial reference frame (Fig. 19.1). We denote the sum of the kinetic energies of the particles by

$$T = \sum_i \tfrac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i, \tag{19.1}$$

where $\mathbf{v}_i = d\mathbf{r}_i/dt$ is the velocity of the ith particle. Our objective is to relate the work done on the system of particles to the change in T. We begin with Newton's second law for the ith particle,

$$\sum_j \mathbf{f}_{ij} + \mathbf{f}_i^{\mathrm{E}} = \frac{d}{dt}(m_i \mathbf{v}_i),$$

where $\mathbf{f}_{ij}$ is the force exerted on the ith particle by the jth particle and $\mathbf{f}_i^{\mathrm{E}}$ is the external force on the ith particle. We take the dot product of this equation with $\mathbf{v}_i$ and sum from $i = 1$ to N:

$$\sum_i \sum_j \mathbf{f}_{ij} \cdot \mathbf{v}_i + \sum_i \mathbf{f}_i^{\mathrm{E}} \cdot \mathbf{v}_i = \sum_i \mathbf{v}_i \cdot \frac{d}{dt}(m_i \mathbf{v}_i). \tag{19.2}$$

We can express the term on the right side of this equation as the rate of change of the total kinetic energy:

$$\sum_i \mathbf{v}_i \cdot \frac{d}{dt}(m_i \mathbf{v}_i) = \frac{d}{dt} \sum_i \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \frac{dT}{dt}.$$

Multiplying Eq. (19.2) by dt yields

$$\sum_i \sum_j \mathbf{f}_{ij} \cdot d\mathbf{r}_i + \sum_i \mathbf{f}_i^{\mathrm{E}} \cdot d\mathbf{r}_i = dT.$$

We integrate this equation, obtaining

$$\sum_i \sum_j \int_{(\mathbf{r}_i)_1}^{(\mathbf{r}_i)_2} \mathbf{f}_{ij} \cdot d\mathbf{r}_i + \sum_i \int_{(\mathbf{r}_i)_1}^{(\mathbf{r}_i)_2} \mathbf{f}_i^{\mathrm{E}} \cdot d\mathbf{r}_i = T_2 - T_1. \tag{19.3}$$

The terms on the left side are the work done on the system by internal and external forces as the particles move from positions $(\mathbf{r}_i)_1$ to positions $(\mathbf{r}_i)_2$. We see that the work done by internal and external forces as a system of particles moves between two positions equals the change in the total kinetic energy of the system.

If the particles represent a rigid body, and we assume that the internal forces between each pair of particles are directed along the straight line between them, the work done by internal forces is zero. To show that this is true, we consider two particles of a rigid body designated 1 and 2 (Fig. 19.2).

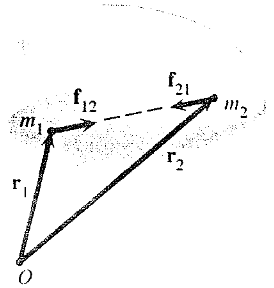


Figure 19.2
Particles 1 and 2 and the forces they exert on each other.

The sum of the forces the two particles exert on each other is zero ($\mathbf{f}_{12} + \mathbf{f}_{21} = \mathbf{0}$), so the rate at which the forces do work (the power) is

$$\mathbf{f}_{12} \cdot \mathbf{v}_1 + \mathbf{f}_{21} \cdot \mathbf{v}_2 = \mathbf{f}_{21} \cdot (\mathbf{v}_2 - \mathbf{v}_1).$$

We can show that $\mathbf{f}_{21}$ is perpendicular to $\mathbf{v}_2 - \mathbf{v}_1$, and therefore the rate at which work is done by the internal forces between these two particles is zero. Because the particles are points of a rigid body, we can express their relative velocity in terms of the rigid body's angular velocity $\boldsymbol{\omega}$ as

$$\mathbf{v}_2 - \mathbf{v}_1 = \boldsymbol{\omega} \times (\mathbf{r}_2 - \mathbf{r}_1). \tag{19.4}$$

This equation shows that the relative velocity $\mathbf{v}_2 - \mathbf{v}_1$ is perpendicular to $\mathbf{r}_2 - \mathbf{r}_1$, which is the position vector from particle 1 to particle 2. Since the force $\mathbf{f}_{21}$ is parallel to $\mathbf{r}_2 - \mathbf{r}_1$, it is perpendicular to $\mathbf{v}_2 - \mathbf{v}_1$. We can repeat this argument for each pair of particles of the rigid body, so the total rate at which work is done by internal forces is zero. This implies that the work done by internal forces as a rigid body moves between two positions is zero.

Therefore, in the case of a rigid body, the work done by internal forces in Eq. (19.3) vanishes. Denoting the work done by external forces by U_{12}, we obtain the principle of work and energy for a rigid body: *The work done by external forces and couples as a rigid body moves between two positions equals the change in the total kinetic energy of the body*:

$$U_{12} = T_2 - T_1. \tag{19.5}$$

We can also state this principle for a *system* of rigid bodies: *The work done by external and internal forces as a system of rigid bodies moves between two positions equals the change in the total kinetic energy of the system.*

19.2 Kinetic Energy

The kinetic energy of a rigid body can be expressed in terms of the velocity of the center of mass of the body and its angular velocity. We consider first general planar motion and then rotation about a fixed axis.

General Planar Motion

Let us model a rigid body as a system of particles, and let $\mathbf{R}_i$ be the position vector of the ith particle relative to the body's center of mass (Fig. 19.3). The position of the center of mass is

$$\mathbf{r} = \frac{\sum_i m_i \mathbf{r}_i}{m}, \tag{19.6}$$

where m is the mass of the rigid body. The position of the ith particle relative to O is related to its position relative to the center of mass by

$$\mathbf{r}_i = \mathbf{r} + \mathbf{R}_i, \tag{19.7}$$

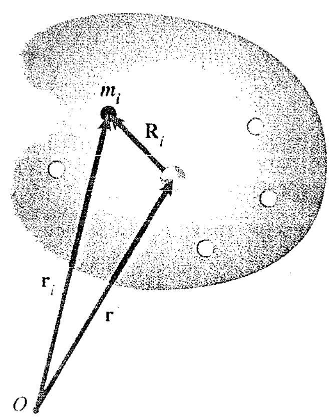


Figure 19.3
Representing a rigid body as a system of particles.

and the vectors $\mathbf{R}_i$ satisfy the relation

$$\sum_i m_i \mathbf{R}_i = \mathbf{0}. \tag{19.8}$$

The kinetic energy of the rigid body is the sum of the kinetic energies of its particles, given by Eq. (19.1):

$$T = \sum_i \tfrac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i. \tag{19.9}$$

By taking the derivative of Eq. (19.7) with respect to time, we obtain

$$\mathbf{v}_i = \mathbf{v} + \frac{d\mathbf{R}_i}{dt},$$

where $\mathbf{v}$ is the velocity of the center of mass. Substituting this expression into Eq. (19.9) and using Eq. (19.8), we obtain the kinetic energy of the rigid body in the form

$$T = \tfrac{1}{2} m v^2 + \sum_i \frac{1}{2} m_i \frac{d\mathbf{R}_i}{dt} \cdot \frac{d\mathbf{R}_i}{dt}, \tag{19.10}$$

where v is the magnitude of the velocity of the center of mass.

Now, let L_0 be the axis through a fixed point O that is perpendicular to the plane of the motion, and let L be the parallel axis through the center of mass (Fig. 19.4a). Then, in terms of the coordinate system shown, we can express the angular velocity vector as $\boldsymbol{\omega} = \omega\mathbf{k}$. The velocity of the ith particle relative to the center of mass is $d\mathbf{R}_i/dt = \omega\mathbf{k} \times \mathbf{R}_i$, so we can write Eq. (19.10) as

$$T = \tfrac{1}{2} m v^2 + \tfrac{1}{2}\left[\sum_i m_i (\mathbf{k} \times \mathbf{R}_i) \cdot (\mathbf{k} \times \mathbf{R}_i)\right]\omega^2. \tag{19.11}$$

The magnitude of the vector $\mathbf{k} \times \mathbf{R}_i$ is the perpendicular distance r_i from L to the ith particle (Fig. 19.4b), so the term in brackets in Eq. (19.11) is the moment of inertia of the body about L:

$$\sum_i m_i (\mathbf{k} \times \mathbf{R}_i) \cdot (\mathbf{k} \times \mathbf{R}_i) = \sum_i m_i |\mathbf{k} \times \mathbf{R}_i|^2 = \sum_i m_i r_i^2 = I.$$

Thus, we obtain the kinetic energy of a rigid body in general planar motion in the form

$$T = \tfrac{1}{2} m v^2 + \tfrac{1}{2} I \omega^2. \tag{19.12}$$

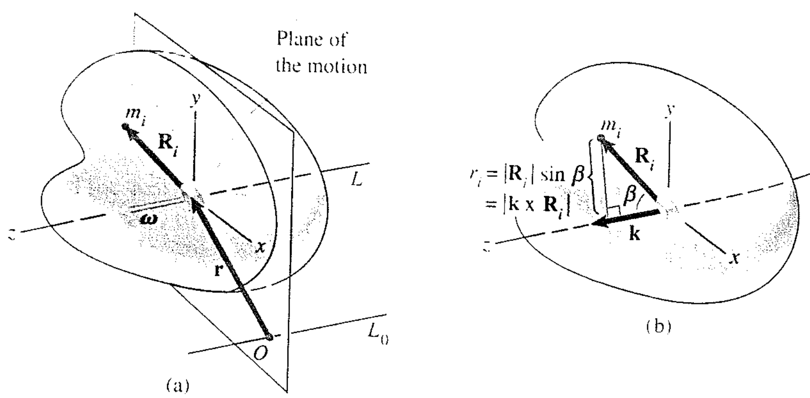


Figure 19.4
(a) A coordinate system with the z axis aligned with L.
(b) The magnitude of $\mathbf{k} \times \mathbf{R}_i$ is the perpendicular distance from L to m_i.

The kinetic energy consists of two terms: the *translational kinetic energy*, expressed in terms of the velocity of the center of mass, and the *rotational kinetic energy* (Fig. 19.5).

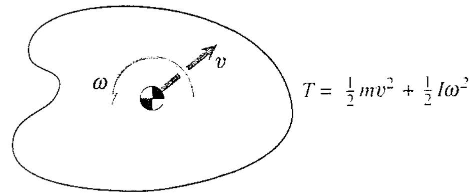


Figure 19.5
Kinetic energy in general planar motion.

Fixed-Axis Rotation

An object rotating about a fixed axis is in general planar motion, and its kinetic energy is given by Eq. (19.12). But in this case there is another expression for the kinetic energy that is often convenient. Suppose that a rigid body rotates with angular velocity ω about a fixed axis O. In terms of the distance d from O to the center of mass of the body, the velocity of the center of mass is $v = \omega d$ (Fig. 19.6a). From Eq. (19.12), the kinetic energy is

$$T = \tfrac{1}{2}m(\omega d)^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(I + d^2m)\omega^2.$$

According to the parallel-axis theorem, the moment of inertia about O is $I_0 = I + d^2m$, so we obtain the kinetic energy of a rigid body rotating about a fixed axis O in the form (Fig. 19.6b)

$$T = \tfrac{1}{2}I_0\omega^2. \tag{19.13}$$

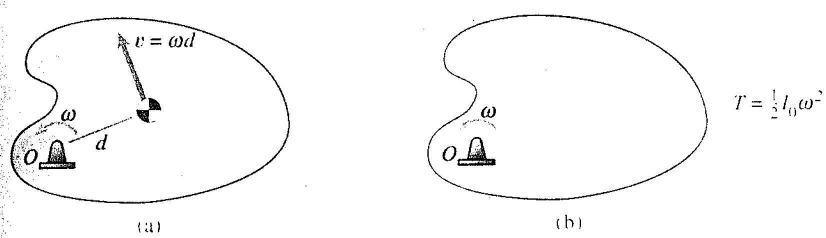


Figure 19.6
(a) Velocity of the center of mass.
(b) Kinetic energy of a rigid body rotating about a fixed axis.

19.3 Work and Potential Energy

The procedures for determining the work done by different types of forces and the expressions for the potential energies of forces discussed in Chapter 15 provide the essential tools for applying the principle of work and energy to a rigid body. The work done on a rigid body by a force $\mathbf{F}$ is given by

$$U_{12} = \int_{(\mathbf{r}_p)_1}^{(\mathbf{r}_p)_2} \mathbf{F} \cdot d\mathbf{r}_p, \tag{19.14}$$

where $\mathbf{r}_p$ is the position of the point of application of $\mathbf{F}$ (Fig. 19.7). If the point of application is stationary, or if its direction of motion is perpendicular to $\mathbf{F}$, no work is done.

A force $\mathbf{F}$ is conservative if a potential energy V exists such that

$$\mathbf{F} \cdot d\mathbf{r}_p = -dV. \tag{19.15}$$

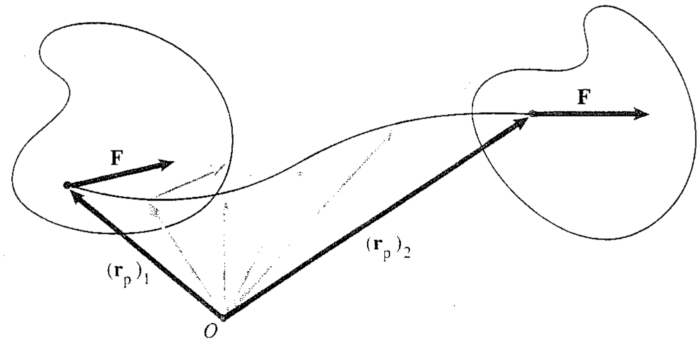


Figure 19.7
The work done by a force on a rigid body is determined by the path of the point of application of the force.

In terms of its potential energy, the work done by a conservative force **F** is

$$U_{12} = \int_{(\mathbf{r}_p)_1}^{(\mathbf{r}_p)_2} \mathbf{F} \cdot d\mathbf{r}_p = \int_{V_1}^{V_2} -dV = -(V_2 - V_1),$$

where V_1 and V_2 are the values of V at $(\mathbf{r}_p)_1$ and $(\mathbf{r}_p)_2$.

If a rigid body is subjected to a couple M (Fig. 19.8a), what work is done as the body moves between two positions? We can evaluate the work by representing the couple by forces (Fig. 19.8b) and determining the work done by the forces. If the rigid body rotates through an angle $d\theta$ in the direction of the couple (Fig. 19.8c), the work done by each force is $(\frac{1}{2} D\, d\theta)F$, so the total work is $DF\, d\theta = M\, d\theta$. Integrating this expression, we obtain the work done by a couple M as the rigid body rotates from θ_1 to θ_2 in the direction of M:

$$U_{12} = \int_{\theta_1}^{\theta_2} M\, d\theta. \qquad (19.16)$$

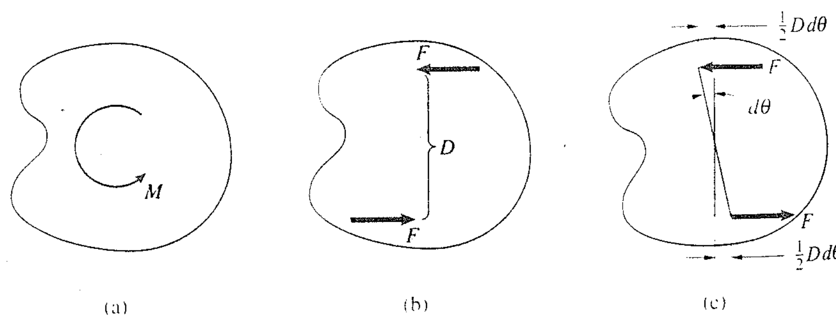


Figure 19.8
(a) A rigid body subjected to a couple.
(b) An equivalent couple consisting of two forces: $DF = M$.
(c) Determining the work done by the forces.

If M is constant, the work is simply the product of the couple and the angular displacement:

$$U_{12} = M(\theta_2 - \theta_1). \quad \text{(constant couple)}.$$

A couple M is conservative if a potential energy V exists such that

$$M\, d\theta = -dV. \qquad (19.17)$$

We can express the work done by a conservative couple in terms of its potential energy:

$$U_{12} = \int_{\theta_1}^{\theta_2} M\, d\theta = \int_{V_1}^{V_2} -dV = -(V_2 - V_1).$$

For example, in Fig. 19.9, a torsional spring exerts a couple on a bar that is proportional to the bar's angle of rotation: $M = -k\theta$. From the relation

$$M\,d\theta = -k\theta\,d\theta = -dV,$$

we see that the potential energy must satisfy the equation

$$\frac{dV}{d\theta} = k\theta.$$

Integrating this equation, we find that the potential energy of the torsional spring is

$$V = \tfrac{1}{2}k\theta^2. \qquad (19.18)$$

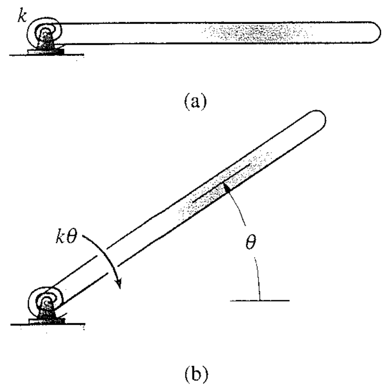


Figure 19.9
(a) A linear torsional spring connected to a bar.
(b) The spring exerts a couple of magnitude $k\theta$ in the direction opposite that of the bar's rotation.

If *all* the forces and couples that do work on a rigid body are conservative, we can express the total work done as the body moves between two positions 1 and 2 in terms of the total potential energy of the forces and couples:

$$U_{12} = V_1 - V_2.$$

Combining this relation with the principle of work and energy, Eq. (19.5), we conclude that the sum of the kinetic energy and the total potential energy is constant—energy is conserved:

$$T + V = \text{constant}. \qquad (19.19)$$

The results we have presented in Sections 19.1–19.3 can be used to relate changes in the translational and angular velocities of an object to a change in its position. This typically involves three steps:

1. *Identify the forces and couples that do work.* Use free-body diagrams to determine which external forces and couples do work.
2. *Apply the principle of work and energy or conservation of energy.* Either equate the total work done during a change in position to the change in the kinetic energy, or equate the sum of the kinetic and potential energies at two positions.
3. *Determine the kinematic relationships.* To complete the solution, it will often be necessary to obtain relations between velocities of points of rigid bodies and their angular velocities.

Study Questions

1. What is the principle of work and energy for a rigid body?
2. What is the kinetic energy of a rigid body in general planar motion?
3. How do you determine the work done by a couple acting on a rigid body in planar motion?
4. If all of the forces and couples that do work on a rigid body are conservative, what can you infer about the sum of the kinetic energy of the rigid body and the total potential energy?

Example 19.1

Applying Work and Energy to a Rolling Disk

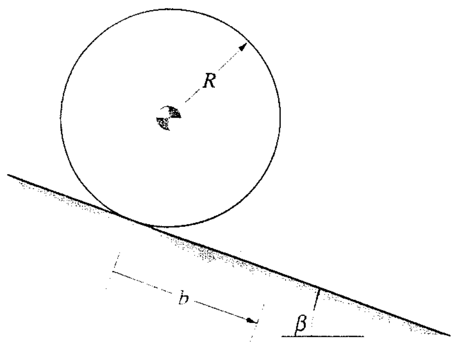


Figure 19.10

A disk of mass m and moment of inertia I is released from rest on an inclined surface (Fig. 19.10). Assuming that the disk rolls, what is the velocity of its center when it has moved a distance b?

Strategy

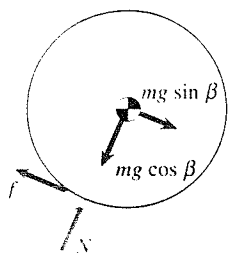
We can determine the velocity by equating the total work done as the disk rolls a distance b to the change in its kinetic energy.

Solution

Identify the Forces and Couples That Do Work We draw the free-body diagram of the disk in Fig. (a). The disk's weight does work as it rolls, but the normal force N and the friction force f do not. To explain why the friction force does no work, we can write the work done by a force $\mathbf{F}$ as

$$\int_{(\mathbf{r}_p)_1}^{(\mathbf{r}_p)_2} \mathbf{F}\cdot d\mathbf{r}_p = \int_{t_1}^{t_2} \mathbf{F}\cdot\frac{d\mathbf{r}_p}{dt}\,dt = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}_p\,dt,$$

where $\mathbf{v}_p$ is the velocity of the point of application of $\mathbf{F}$. Since the velocity of the point where f acts is zero as the disk rolls, the work done by f is zero.



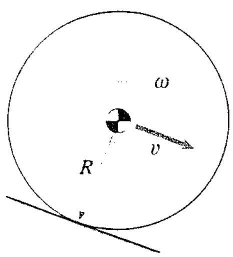
(a) Free-body diagram of the disk.

Apply Work and Energy We can determine the work done by the weight by multiplying the component of the weight in the direction of the motion of the center of the disk by the distance b:

$$U_{12} = (mg\sin\beta)b.$$

Letting v and ω be the velocity of the center and the angular velocity of the disk when it has moved a distance b (Fig. b), we equate the work to the change in the disk's kinetic energy:

$$mgb\sin\beta = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 - 0. \qquad (19.20)$$



(b) Velocity of the center and the angular velocity when the disk has moved a distance b.

Determine the Kinematic Relationship The angular velocity ω of the rolling disk is related to the velocity v by $\omega = v/R$. Substituting this relation into Eq. (19.20) and solving for v, we obtain

$$v = \sqrt{\frac{2gb\sin\beta}{1 + I/mR^2}}.$$

Discussion

Suppose that the surface is smooth, so that the disk slides instead of rolling. In this case, the disk has no angular velocity, so Eq. (19.20) becomes

$$mgb\sin\beta = \tfrac{1}{2}mv^2 - 0,$$

and the velocity of the center of the disk is

$$v = \sqrt{2gb\sin\beta}.$$

The velocity is greater when the disk slides. You can see why by comparing the two expressions for the principle of work and energy. The work done by the disk's weight is the same in each case. When the disk rolls, part of the work increases the disk's translational kinetic energy, and part increases its rotational kinetic energy. When the disk slides, all of the work increases its translational kinetic energy.

Example 19.2

Applying Work and Energy to a Motorcycle

Each wheel of the motorcycle in Fig. 19.11 has mass $m_W = 9\,\text{kg}$, radius $R = 330$ mm, and moment of inertia $I = 0.8$ kg-m^2. The combined mass of the rider and the motorcycle, not including the wheels, is $m_C = 142$ kg. The motorcycle starts from rest, and its engine exerts a constant couple $M = 140$ N-m on the rear wheel. Assume that the wheels do not slip. What horizontal distance b must the motorcycle travel to reach a velocity of 25 m/s?

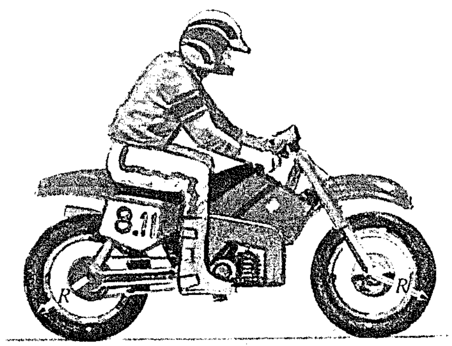


Figure 19.11

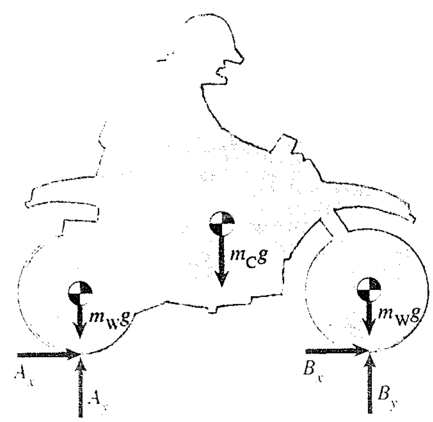
Strategy

We can apply the principle of work and energy to the system consisting of the rider and the motorcycle, including its wheels, to determine the distance b.

Solution

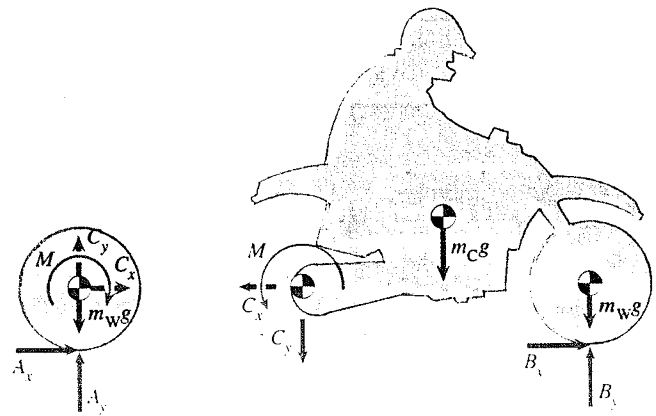
Determining the distance b requires three steps.

Identify the Forces and Couples That Do Work We draw the free-body diagram of the system in Fig. (a). The weights do no work because the motion is horizontal, and the forces exerted on the wheels by the road do no work because the velocity of their point of application is zero. (See Example 19.1.) Thus, no work is done by external forces and couples! However, work



(a) Free-body diagram of the system.

is done by the couple M exerted on the rear wheel by the engine (Fig. b). Although this is an internal couple for the system we are considering—the wheel exerts an opposite couple on the body of the motorcycle—net work is done because the wheel rotates whereas the body does not.



(b) Isolating the rear wheel.

Apply Work and Energy If the motorcycle moves a horizontal distance b, the wheels turn through an angle b/R rad, and the work done by the constant couple M is

$$U_{12} = M(\theta_2 - \theta_1) = M\left(\frac{b}{R}\right).$$

Let v be the motorcycle's velocity and ω the angular velocity of the wheels when the motorcycle has moved a distance b. The work equals the change in the total kinetic energy:

$$M\left(\frac{b}{R}\right) = \tfrac{1}{2}m_C v^2 + 2\left(\tfrac{1}{2}m_W v^2 + \tfrac{1}{2}I\omega^2\right) - 0. \qquad (19.21)$$

Determine Kinematic Relationship The angular velocity of the rolling wheels is related to the velocity v by $\omega = v/R$. Substituting this relation into Eq. (19.21) and solving for b, we obtain

$$\begin{aligned} b &= \left(\frac{1}{2}m_C + m_W + \frac{I}{R^2}\right)\frac{Rv^2}{M} \\ &= \left[\frac{1}{2}(142) + (9) + \frac{(0.8)}{(0.33)^2}\right]\frac{(0.33)(25)^2}{(140)} \\ &= 129 \text{ m}. \end{aligned}$$

Discussion

Although we drew separate free-body diagrams of the motorcycle and its rear wheel to clarify the work done by the couple exerted by the engine, notice that we treated the motorcycle, including its wheels, as a single system in applying the principle of work and energy. By doing so, we did not need to consider the work done by the internal forces between the motorcycle's body and its wheels. When applying the principle of work and energy to a system of rigid bodies, you will usually find it simplest to express the principle for the system as a whole. This is in contrast to determining the motion of a system of rigid bodies by using the equations of motion, which usually requires that you draw free-body diagrams of each rigid body and apply the equations to them individually.

Example 19.3

Applying Conservation of Energy to a Linkage

The slender bars AB and BC of the linkage in Fig. 19.12 have mass m and length l, and the collar C has mass m_C. A torsional spring at A exerts a clockwise couple $k\theta$ on bar AB. The system is released from rest in the position $\theta = 0$ and allowed to fall. Neglecting friction, determine the angular velocity $\omega = d\theta/dt$ of bar AB as a function of θ.

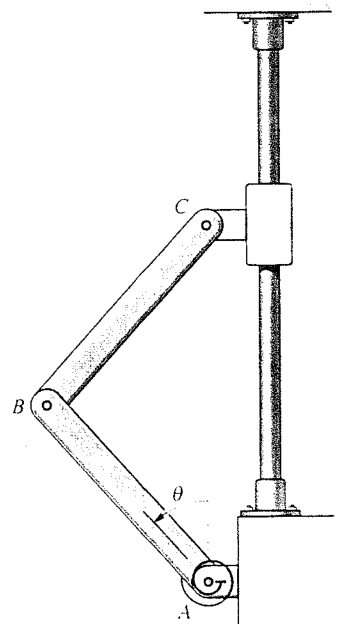
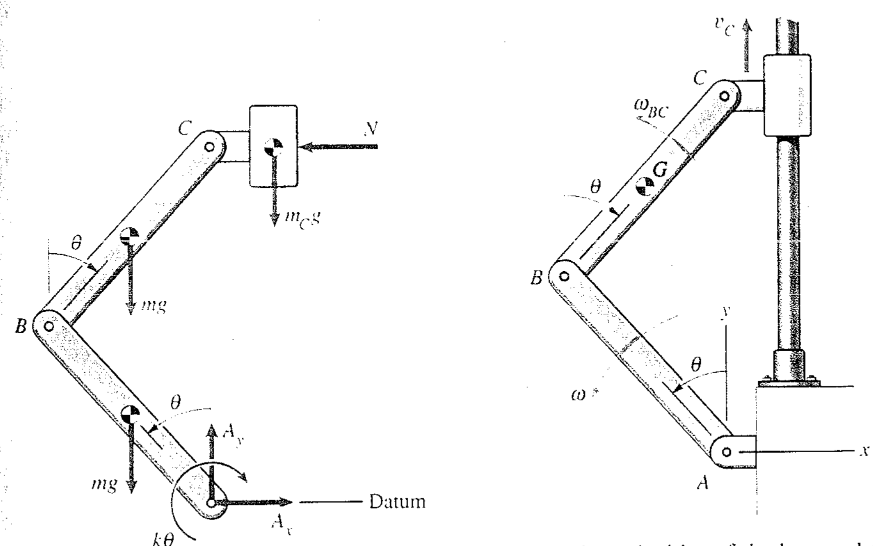


Figure 19.12

Solution

Identify the Forces and Couples That Do Work We draw the free-body diagram of the system in Fig. (a). The forces and couples that do work—the weights of the bars and collar and the couple exerted by the torsional spring—are conservative. We can use conservation of energy and the kinematic relationships between the angular velocities of the bars and the velocity of the collar to determine ω as a function of θ.

Apply Conservation of Energy We denote the center of mass of bar BC by G and the angular velocity of bar BC by ω_{BC} (Fig. b). The moment of inertia



(a) Free-body diagram of the system.

(b) Angular velocities of the bars and the velocity of the collar.

of each bar about its center of mass is $I = \frac{1}{12}ml^2$. Since bar AB rotates about the fixed point A, we can write its kinetic energy as

$$T_{\text{bar } AB} = \tfrac{1}{2}I_A\omega^2 = \tfrac{1}{2}\left[I + \left(\tfrac{1}{2}l\right)^2 m\right]\omega^2 = \tfrac{1}{6}ml^2\omega^2.$$

The kinetic energy of bar BC is

$$T_{\text{bar } BC} = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I\omega_{BC}^2 = \tfrac{1}{2}mv_G^2 + \tfrac{1}{24}ml^2\omega_{BC}^2.$$

The kinetic energy of the collar C is

$$T_{\text{collar}} = \tfrac{1}{2}m_C v_C^2.$$

Using the datum in Fig. (a), we obtain the potential energies of the weights:

$$V_{\text{bar } AB} + V_{\text{bar } BC} + V_{\text{collar}} = mg\left(\tfrac{1}{2}l\cos\theta\right) + mg\left(\tfrac{3}{2}l\cos\theta\right) + m_C g(2l\cos\theta).$$

The potential energy of the torsional spring is given by Eq. (19.18):

$$V_{\text{spring}} = \tfrac{1}{2}k\theta^2.$$

We now have all the ingredients to apply conservation of energy. We equate the sum of the kinetic and potential energies at the position $\theta = 0$ to the sum of the kinetic and potential energies at an arbitrary value of θ:

$$T_1 + V_1 = T_2 + V_2:$$

$$0 + 2mgl + 2m_C gl = \tfrac{1}{6}ml^2\omega^2 + \tfrac{1}{2}mv_G^2 + \tfrac{1}{24}ml^2\omega_{BC}^2 + \tfrac{1}{2}m_C v_C^2$$
$$+ 2mgl\cos\theta + 2m_C gl\cos\theta + \tfrac{1}{2}k\theta^2.$$

To determine ω from this equation, we must express the velocities v_G, v_C, and ω_{BC} in terms of ω.

Determine Kinematic Relationships We can determine the velocity of point B in terms of ω and then express the velocity of point C in terms of the velocity of point B and the angular velocity ω_{BC}.

The velocity of B is

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$
$$= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ -l\sin\theta & l\cos\theta & 0 \end{vmatrix}$$
$$= -l\omega\cos\theta\,\mathbf{i} - l\omega\sin\theta\,\mathbf{j}.$$

The velocity of C, expressed in terms of the velocity of B, is

$$v_C\mathbf{j} = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$
$$= -l\omega\cos\theta\,\mathbf{i} - l\omega\sin\theta\,\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ l\sin\theta & l\cos\theta & 0 \end{vmatrix}.$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components, we obtain

$$\omega_{BC} = -\omega, \qquad v_C = -2l\omega\sin\theta.$$

(The minus signs indicate that the directions of the velocities are opposite to the directions we assumed in Fig. b.) Now that we know the angular velocity of bar BC in terms of ω, we can determine the velocity of its center of mass in terms of ω by expressing it in terms of $\mathbf{v}_B$:

$$\mathbf{v}_G = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{G/B}$$

$$= -l\omega\cos\theta\mathbf{i} - l\omega\sin\theta\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega \\ \frac{1}{2}l\sin\theta & \frac{1}{2}l\cos\theta & 0 \end{vmatrix}$$

$$= -\tfrac{1}{2}l\omega\cos\theta\mathbf{i} - \tfrac{3}{2}l\omega\sin\theta\mathbf{j}.$$

Substituting these expressions for ω_{BC}, v_C, and $\mathbf{v}_G$ into our equation of conservation of energy and solving for ω, we obtain

$$\omega = \left[\frac{2gl(m + m_C)(1 - \cos\theta) - \frac{1}{2}k\theta^2}{\frac{1}{3}ml^2 + (m + 2m_C)l^2\sin^2\theta}\right]^{1/2}.$$

19.4 Power

The work done on a rigid body by a force $\mathbf{F}$ during an infinitesimal displacement $d\mathbf{r}_p$ of its point of application is

$$\mathbf{F} \cdot d\mathbf{r}_p.$$

We obtain the power P transmitted to the rigid body—the rate at which work is done on it—by dividing this expression by the interval of time dt during which the displacement takes place. We obtain

$$P = \mathbf{F} \cdot \mathbf{v}_p, \qquad (19.22)$$

where $\mathbf{v}_p$ is the velocity of the point of application of $\mathbf{F}$.

Similarly, the work done on a rigid body in planar motion by a couple M during an infinitesimal rotation $d\theta$ in the direction of M is

$$M\,d\theta.$$

Dividing this expression by dt, we find that the power transmitted to the rigid body is the product of the couple and the angular velocity:

$$P = M\omega. \qquad (19.23)$$

The total work done on a rigid body during an interval of time equals the change in kinetic energy of the body, so the total power transmitted equals the rate of change of the body's kinetic energy:

$$P = \frac{dT}{dt}.$$

The average with respect to time of the power during an interval of time from t_1 to t_2 is

$$P_{av} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} P\,dt = \frac{1}{t_2 - t_1}\int_{T_1}^{T_2} dT = \frac{T_2 - T_1}{t_2 - t_1}.$$

This expression shows that we can determine the average power transferred to or from a rigid body during an interval of time by dividing the change in kinetic energy of the body, or the total work done, by the interval of time:

$$P_{av} = \frac{T_2 - T_1}{t_2 - t_1} = \frac{U_{12}}{t_2 - t_1}. \qquad (19.24)$$

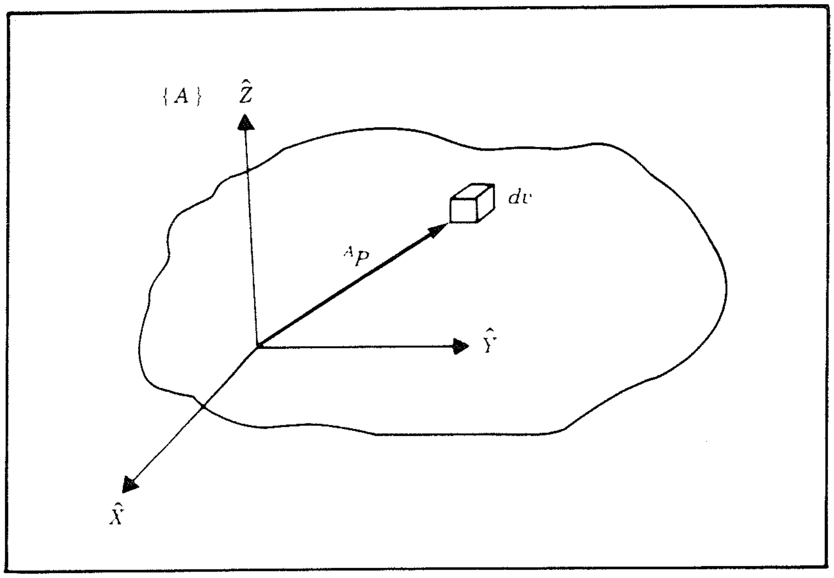


FIGURE 6.1 The inertia tensor of an object describes the object's mass distribution. Here a vector ^AP locates the differential volume element, dv.

The inertia tensor relative to frame $\{A\}$ is expressed in the matrix form as the 3×3 matrix:

$$^AI = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}, \tag{6.16}$$

where the scalar elements are given by

$$\begin{aligned} I_{xx} &= \iiint_V (y^2 + z^2)\rho dv, \\ I_{yy} &= \iiint_V (x^2 + z^2)\rho dv, \\ I_{zz} &= \iiint_V (x^2 + y^2)\rho dv, \\ I_{xy} &= \iiint_V xy\ \rho dv, \\ I_{xz} &= \iiint_V xz\ \rho dv, \\ I_{yz} &= \iiint_V yz\ \rho dv, \end{aligned} \tag{6.17}$$

where the rigid body is composed of differential volume elements, dv, containing material of density ρ. Each volume element is located with a vector, $^AP = [x\ y\ z]^T$, as shown in Fig. 6.1.

The elements I_{xx}, I_{yy}, and I_{zz} are called the **mass moments of inertia**.

EXAMPLE 6.1

Find the inertia tensor for the rectangular body of uniform density ρ with respect to the coordinate system shown in Fig. 6.2.

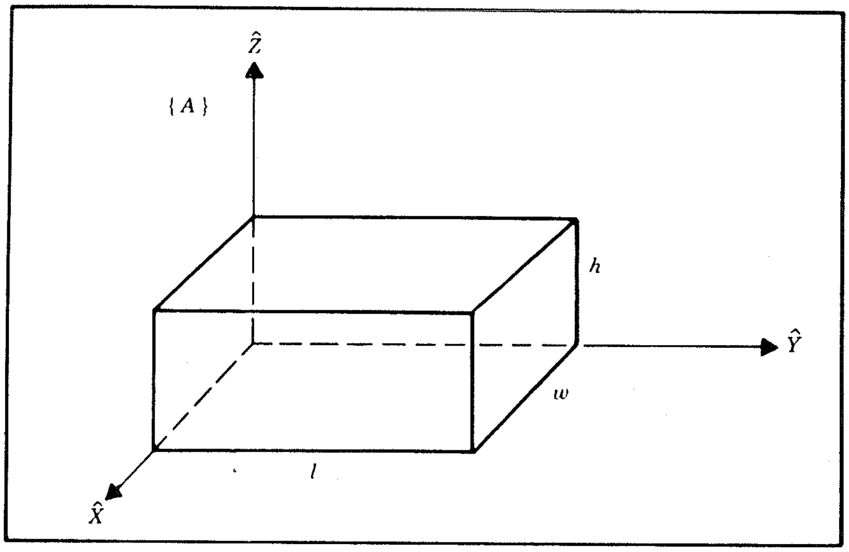


FIGURE 6.2 A body of uniform density.

First, we compute I_{xx}. Using volume element $dv = dx\,dy\,dz$, we get

$$
\begin{aligned}
I_{xx} &= \int_0^h \int_0^l \int_0^w (y^2 + z^2)\rho dx dy dz \\
&= \int_0^h \int_0^l (y^2 + z^2) w\rho dy dz \\
&= \int_0^h \left(\frac{l^3}{3} + z^2 l \right) w\rho dz \\
&= \left(\frac{hl^3 w}{3} + \frac{h^3 l w}{3} \right) \rho \\
&= \frac{m}{3} \left(l^2 + h^2 \right),
\end{aligned}
\tag{6.18}
$$

where m is the total mass of the body. Permuting the terms, we can get I_{yy} and I_{zz} by inspection:

$$
I_{yy} = \frac{m}{3} \left(w^2 + h^2 \right) \tag{6.19}
$$

and

$$I_{zz} = \frac{m}{3}\left(l^2 + w^2\right). \tag{6.20}$$

We next compute I_{xy}:

$$\begin{aligned}
I_{xy} &= \int_0^h \int_0^l \int_0^w xy\rho dx dy dz \\
&= \int_0^h \int_0^l \frac{w^2}{2} y\rho dy dz \\
&= \int_0^h \frac{w^2 l^2}{4} \rho dz \\
&= \frac{m}{4} wl.
\end{aligned} \tag{6.21}$$

Permuting the terms, we get

$$I_{xz} = \frac{m}{4} hw \tag{6.22}$$

and

$$I_{yz} = \frac{m}{4} hl. \tag{6.23}$$

Hence the inertia tensor for this object is

$$^{A}I = \begin{bmatrix} \frac{m}{3}\left(l^2 + h^2\right) & -\frac{m}{4} wl & -\frac{m}{4} hw \\ -\frac{m}{4} wl & \frac{m}{3}\left(w^2 + h^2\right) & -\frac{m}{4} hl \\ -\frac{m}{4} hw & -\frac{m}{4} hl & \frac{m}{3}\left(l^2 + w^2\right) \end{bmatrix}. \quad \blacksquare \tag{6.24}$$

.Comments

Theory

Part III
State-space representation of mechanical dynamic systems

3.1 Kinetic and potential energy. Momentum.

The kinetic energy of a particle of the mass m is

$$K = \frac{1}{2} m\dot{r}^2 = \frac{1}{2} mv^2 \qquad (3.1)$$

where $\boldsymbol{r}$ is the position vector.

The work done by a force $\boldsymbol{f}$ acting on a particle is

$$\delta W = \boldsymbol{f} \cdot \delta \boldsymbol{r}$$

$$W = \int_c \boldsymbol{f} \cdot d\boldsymbol{r} \qquad (3.2)$$

Definition 1.

If the work done by a force on a particle when it is moved between any two points is independent of the path taken, then the force field is said to be *conservative.*

Definition 2.

The *potential energy* of a particle situated at a point in a conservative field is the work that would be done by the force on the particle if it were moved from that point to some standard ("zero-level") position.

$$V(\boldsymbol{r}) = \int_{\boldsymbol{r}}^{\boldsymbol{r}_0} \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} \qquad (3.3)$$

where $\boldsymbol{r}_0$ is the vector of the zero-level position.

Then, if a particle move between points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ we have

$$W = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{f} \cdot d\boldsymbol{r} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_0} \boldsymbol{f} \cdot d\boldsymbol{r} - \int_{\boldsymbol{r}_2}^{\boldsymbol{r}_0} \boldsymbol{f} \cdot d\boldsymbol{r} = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2) \qquad (3.4)$$

Since this result applies to any arbitrary points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ we have for conservative forces

$$\boldsymbol{f} = -grad\ V = -\left(\frac{\partial V}{\partial x}\boldsymbol{i} + \frac{\partial V}{\partial y}\boldsymbol{j} + \frac{\partial V}{\partial z}\boldsymbol{k} \right) \qquad (3.5)$$

In any conservative field the force is equal to minus the gradient of its associated potential energy.

Comments

Theory

Proposition 1.
The sum of the kinetic and potential energy of a particle is constant through the motion if, and only if, the only forces which do work are conservative.

Definitions 4.
Linear momentum is $\boldsymbol{p} = m\dot{\boldsymbol{r}} = m\boldsymbol{v}$ (3.6)

Angular momentum is $\boldsymbol{h} = \boldsymbol{r} \times m\dot{\boldsymbol{r}} = \boldsymbol{r} \times \boldsymbol{p}$ (3.7)

3.2 System of particles

Definition 5.
The centroid (or "mass center") of a system of particles is the point denoted by the vector $\bar{\boldsymbol{r}}$ relative to the origin, where

$$m\bar{\boldsymbol{r}} = \sum_{i=1}^{N} m_i \boldsymbol{r}_i \,; \quad m = \sum_{i=1}^{N} m_i \tag{3.8}$$

Proposition 2.
The kinetic energy of a system of particles relative to a given frame of reference is equal to the sum of two parts, (a) the kinetic energy of the system calculated relative to a frame with origin at the centroid and axes parallel to the given frame, and (b) one-half the product of the total mass of the system and square of the velocity of the centroid

$$K = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\rho}_i^2 + \frac{1}{2} m \dot{\bar{\boldsymbol{r}}}^2 \tag{3.9}$$

Potential energy is $$V = \sum_{i=1}^{N} V_i$$

Proposition 3.
The total angular momentum of a system relative to a given frame of references the sum of two parts, (a) the total angular momentum of the system about its centroid using axes parallel to those of the given frame, and (b) the angular momentum of a hypothetical particles with mass equal to that of the system moving with the centroid relative to the origin

$$\boldsymbol{h} = \sum_{i=1}^{N} \rho_i \times m\dot{\rho}_i + \bar{\boldsymbol{r}} \times m\dot{\bar{\boldsymbol{r}}} \tag{3.10}$$

Comments

Theory

3.3 Generalized (Lagrangian) coordinates.

Definition 4
The generalized coordinates (q_i) are any set of n parameters which completely determine the configuration of the system at any time

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, .. q_n); \qquad n < 3N \qquad (3.11)$$

Differentiating (3.11)

$$\dot{\boldsymbol{r}}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \dot{q}_j \qquad (3.12)$$

we can write for the kinetic energy

$$K = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \dot{q}_j \right)^2 = \frac{1}{2}\sum_{i=1}^{n} \sum_{k=1}^{n} a_{jk} \dot{q}_j \dot{q}_k \qquad (3.13)$$

where

$$a_{jk} = \frac{1}{2}\sum_{i=1}^{N} m_i \frac{\partial \boldsymbol{r}_i}{\partial q_j} \frac{\partial \boldsymbol{r}_i}{\partial q_k}; \qquad (j,k = 1,2,\dots,n)$$

$$a_{jk} = a_{kj}.$$

Definition 5.
Virtual displacement is any set of infinitesimal displacement $\delta\boldsymbol{r}_1, \delta\boldsymbol{r}_2, \dots, \delta\boldsymbol{r}_n$ that is consistent with the constraints.

$$\delta\boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j \qquad (3.14)$$

Definition 6.
Virtual work is the work done by the forces acting on the particles of a system when they are subjected to a virtual displacement.

$$\delta W = \sum_{i=1}^{N} \boldsymbol{f}_i \cdot \delta\boldsymbol{r}_i \qquad (3.15)$$

Introducing generalized forces

$$\delta W = \sum_{j=1}^{n} Q_j \delta q_j \qquad (3.16)$$

Comments

Theory

and using (3.14) we obtain

$$\sum_{j=1}^{n} Q_j \delta q_j = \sum_{i=1}^{N} \boldsymbol{f}_i \cdot \delta \boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{f}_i \cdot \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{N} \boldsymbol{f}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) \delta q_j$$

Then

$$Q_j = \sum_{i=1}^{N} \boldsymbol{f}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \qquad (3.17)$$

Note.

The common and convenient way to determine Q_j is to make a small changes in only one generalized coordinate q_j and calculate corresponding $\delta \boldsymbol{r}_i$ and δW.

3.4 The Lagrange (Euler - Lagrange) equation.

The dynamic behavior of a holonomic system with n generalized coordinates (n degree-of-freedom) can be determined by equations

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} = Q_i\,; \qquad (i = 1,2,\ldots,n) \qquad (3.18)$$

If all the forces are conservative, then

$$Q_i = -\frac{\partial V}{\partial q_i}\,; \qquad (i = 1,2,\ldots,n) \qquad (3.19)$$

and

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} = -\frac{\partial V}{\partial q_i} \qquad (3.20)$$

Introducing the *Lagrangian* by

$$L(q,\dot{q}) = K(q,\dot{q}) - V(q) \qquad (3.21)$$

we have

$$\frac{\partial K}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} \quad \text{and} \quad \frac{\partial L}{\partial q_i} = \frac{\partial K}{\partial q_i} - \frac{\partial V}{\partial q_i}$$

Substituting into (3.20) we obtain the Lagrange equation for conservative systems

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad (3.22)$$

In general case of conservative and non-conservative forces the Lagrange equation is

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i} \qquad (i = 1,2,\ldots,n) \qquad (3.23)$$

(i) *Example 6.1.1 Single-Link Manipulator*

Consider the single-link arm shown in Figure 6-1, consisting of a rigid link coupled through a gear train to a DC-motor. Let θ_ℓ and θ_m denote the angle of the link and the angle of the motor shaft, respectively. Then $\theta_\ell = \frac{1}{n}\theta_m$ where $n:1$ is the gear ratio. The kinetic energy of the system is given by

$$K = {}^1\!/\!_2 J_m \dot{\theta}_m^2 + {}^1\!/\!_2 J_\ell \dot{\theta}_\ell^2 \qquad (6.1.30)$$
$$= {}^1\!/\!_2 (J_m + J_\ell/n^2)\dot{\theta}_m^2$$

where J_m, J_ℓ are the rotational inertias of the motor and link, respectively. The potential energy is given as

$$V = Mg\ell(1 - \cos(\theta_\ell)) = Mg\ell(1 - cos(\theta_m/n)) \qquad (6.1.31)$$

where M is the total mass of the link and ℓ is the distance from the joint axis to the link center of mass.

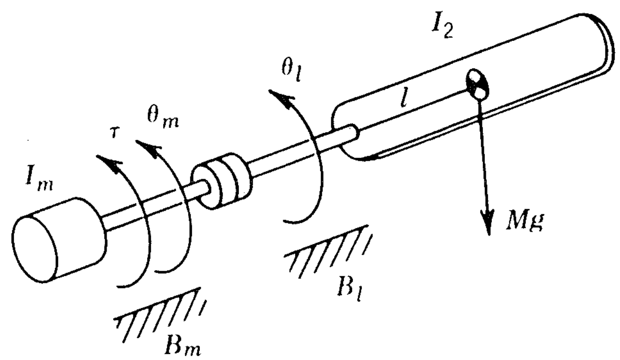


FIGURE 6-1
Single-link robot.

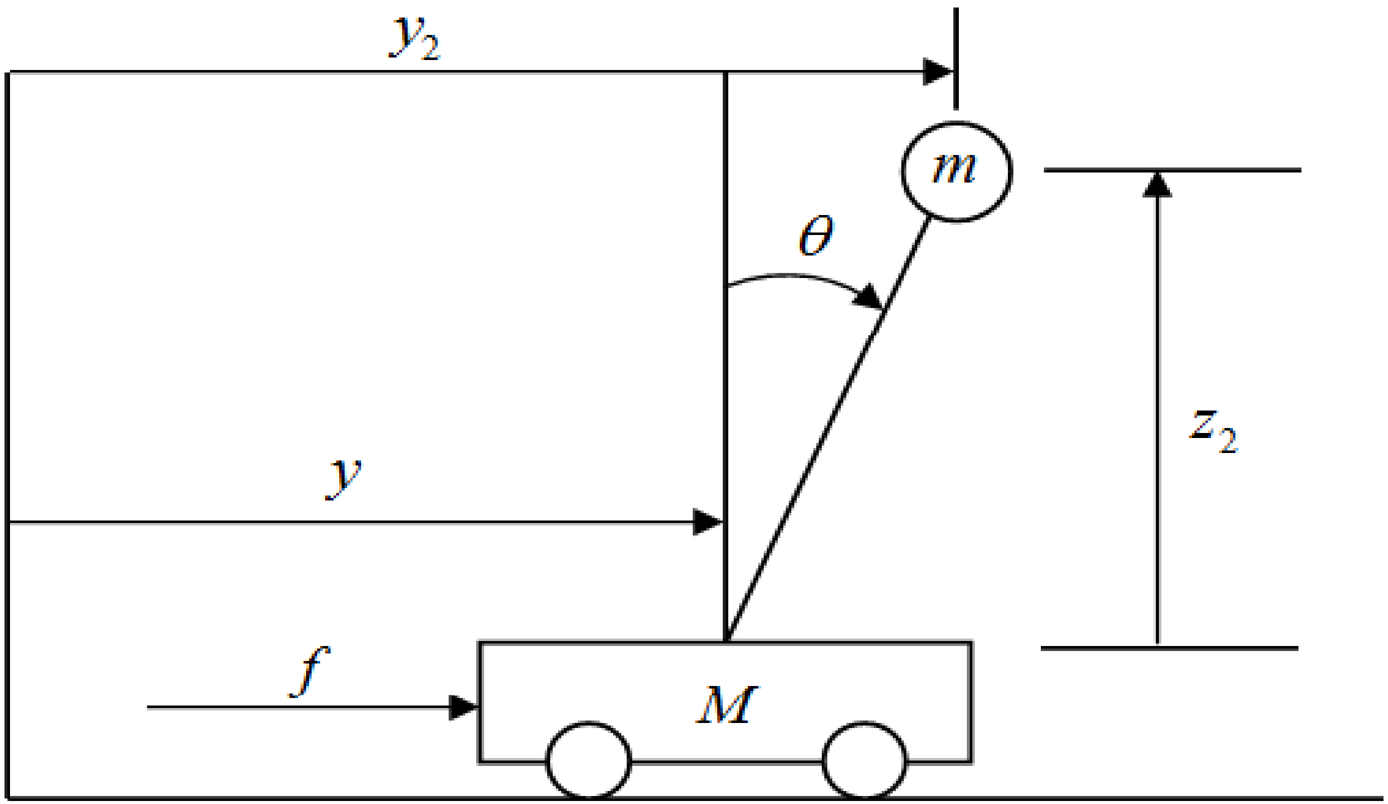
Therefore the Lagrangian L is given by

$$L = {}^1\!/\!_2 (J_m + J_\ell/n^2)\dot{\theta}_m^2 - Mg\ell(1 - \cos(\theta_m/n)) \qquad (6.1.32)$$

Substituting this expression into the Euler–Lagrange equations yields the equation of motion

$$(J_m + J_\ell/n)\ddot{\theta}_m + \frac{Mg\ell}{n} sin(\theta_m/n) = \tau \qquad (6.1.33)$$

Example 2



Kinetic energy of the car:

$$T_1 = \frac{1}{2} M\dot{y}^2$$

Kinetic energy of the bob:

$$T_2 = \frac{1}{2} m(\dot{y}_2^2 + \dot{z}_2^2)$$

$$y_2 = y + l\sin\theta \qquad \dot{y}_2 = \dot{y} + l\dot{\theta}\cos\theta$$

$$z_2 = l\cos\theta \qquad \dot{z}_2 = -l\dot{\theta}\sin\theta$$

$$T = T_1 + T_2 = \frac{1}{2} M\dot{y}^2 + \frac{1}{2} m\left[\left(\dot{y} + l\dot{\theta}\, cos\,\theta\right)^2 + l^2\dot{\theta}^2 sin^2\theta\right]$$

$$= \frac{1}{2} M\dot{y}^2 + \frac{1}{2} m\left[\dot{y}^2 + 2\dot{y}\dot{\theta} l cos\theta + l^2\dot{\theta}^2\right]$$

$$V = mgz_2 = mgl\cos\theta$$

Lagrangian function:

$$L = T - V = \tfrac{1}{2}(M+m)\dot{y}^2 + ml\cos\theta\,\dot{y}\dot{\theta} + \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = f$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{y}} = (M+m)\dot{y} + ml\cos\theta\dot{\theta}$$

$$\frac{\partial L}{\partial y} - 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml\cos\theta\dot{y} + ml^2\dot{\theta}$$

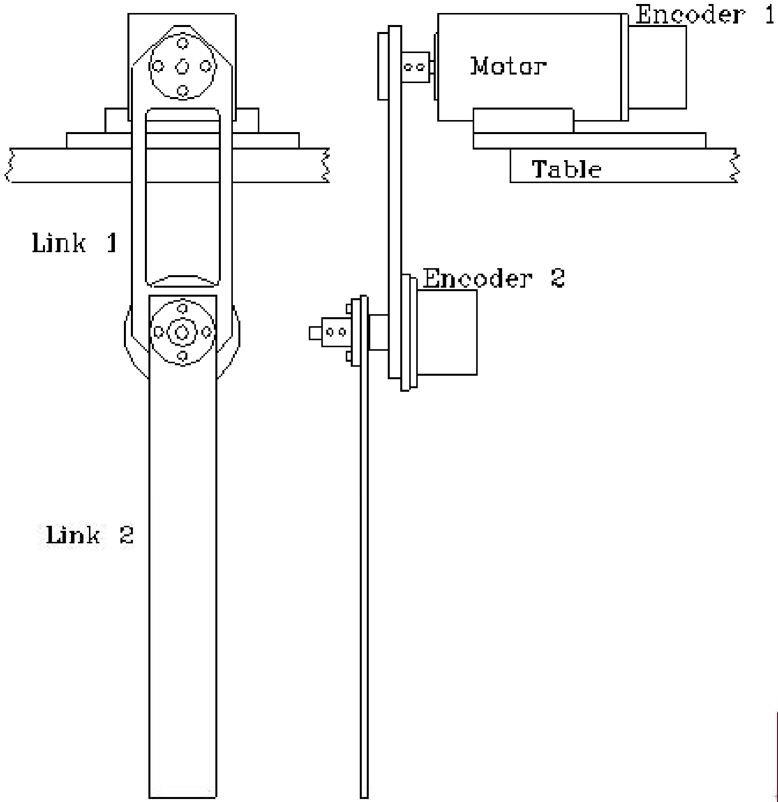
$$\frac{\partial L}{\partial \theta} = mgl\sin\theta$$

$$(M+m)\ddot{y} + ml\cos\theta\ddot{\theta} - ml\dot{\theta}^2\sin\theta = f$$

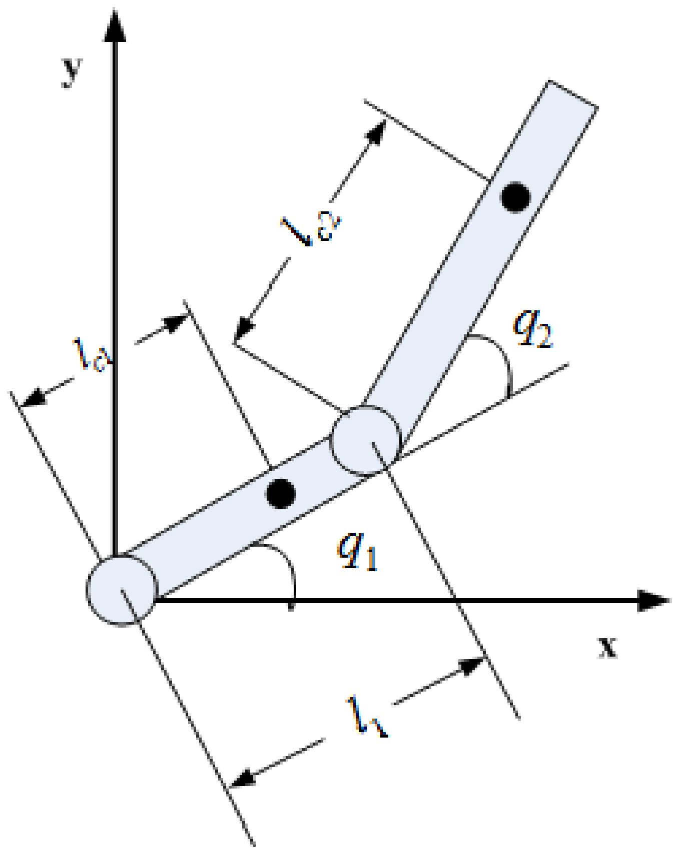
$$ml\cos\theta\ddot{y} - ml\sin\theta\dot{y}\,\dot{\theta} + ml^2\ddot{\theta} - mgl\sin\theta = 0$$

Pendubot

The Pendubot is an underactuated mechanical system with two degrees of freedom, which has been invented and designed by M.W. Spong and D.J. Block.



Modeling the Pendubot System by Using Lagrange's Equation



$$T_1 = \frac{1}{2} m_1 l_{c1}^2 \dot{q}_1^{\,2} + \frac{1}{2} I_1 \dot{q}_1^{\,2} \qquad V_1 = m_1 l_{c1} g \ \sin(q_1)$$

$$T_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} I_2 (q_1 + q_2) \quad V_2 = m_2 g \left[l_1 \sin(q_1) + l_{c2} \sin(q_1 + q_2) \right]$$

$$x_2 = l_1 \cos(q_1) + l_{c2} \cos(q_1 + q_2)$$

$$y_2 = l_1 \sin(q_1) + l_{c2} \sin(q_1 + q_2)$$

$$\dot{x}_2 = -l_1 \sin(q_1) \dot{q}_1 - l_{c2} \sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2)$$

$$\dot{y}_2 = l_1 \cos(q_1) \dot{q}_1 + l_{c2} \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2)$$

$$T_2 = \frac{1}{2} m_2 [l_1^2 \dot{q}_1^{\,2} + l_{c2}^2 \left(\dot{q}_1^{\,2} + \dot{q}_2^{\,2} + 2\dot{q}_1 \dot{q}_2 \right) + 2 l_1 l_{c2} \cos(q_2) \left(\dot{q}_1^{\,2} + \dot{q}_1 \dot{q}_2 \right)]$$

$$+ \frac{1}{2} I_2 (q_1 + q_2)^2$$

Lagrangian function:

$$L = T_1 + T_2 - V_1 - V_2$$

$$\frac{1}{2} m_1 l_{c1}^2 \dot{q}_1{}^2 + \frac{1}{2} I_1 \dot{q}_1{}^2 + \frac{1}{2} m_2 [l_1^2 \dot{q}_1{}^2 + l_{c2}^2 \left(\dot{q}_1{}^2 + \dot{q}_2{}^2 + 2\dot{q}_1\dot{q}_2\right) + 2 l_1 l_{c2} \cos\left(q_2\right)\left(\dot{q}_1{}^2 + \dot{q}_1\dot{q}_2\right)]$$

$$+\frac{1}{2} I_2 (q_1 + q_2)^2 - m_1 l_{c1} g \ \sin(q_1) - m_2 g \left[l_1 \sin\left(q_1\right) + l_{c2} \sin\left(q_1 + q_2\right)\right]$$

Lagrange's equations:

$$\tau_1 = \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1}$$

$$= \left\{m_1 l_{c1}^2 + m_2\left[l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos\left(q_2\right)\right] + I_1 + I_2\right\}\ddot{q}_1$$

$$+\left\{m_2\left[l_{c2}^2 + l_1 l_{c2}\cos\left(q_2\right)\right] + I_2\right\}\ddot{q}_2 - m_2 l_1 l_{c2}\sin\left(q_2\right)\dot{q}_2^{\,2}$$

$$-2m_2 l_1 l_{c2}\sin\left(q_2\right)\dot{q}_1\dot{q}_2 + \left(m_1 l_{c1} + m_2 l_1\right)\mathrm{g}\cos\left(q_1\right) + m_2 l_{c2}\mathrm{g}\sin(q_1 + q_2)$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2}$$

$$= \left[m_2 l_{c2}^2 + m_2 l_1 l_{c2}\cos\left(q_2\right) + I_2\right]\ddot{q}_1 + \left(m_2 l_{c2}^2 + I_2\right)\ddot{q}_2$$

$$+m_2 l_1 l_{c2}\sin\left(q_2\right)\dot{q}_1^{\,2} + m_2 l_{c2} g\cos\left(q_1 + q_2\right)$$

$$D(q)\ddot{q}+c(q,\dot{q})\dot{q}+G(q)=\tau$$

$$D(q)=\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

$$C(q,\dot{q})=\begin{bmatrix} h\dot{q}_2 & h\dot{q}_2+h\dot{q}_1 \\ -h\dot{q}_2 & 0 \end{bmatrix}$$

$$G(q)=\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$

$$\tau=\begin{bmatrix} \tau_1 \\ 0 \end{bmatrix}$$

$$d_{11}=m_1 l_{c1}^2+m_2[l_1^2+l_{c2}^2+2l_1 l_{1c2}\cos(q_2)]+I_1+I_2$$

$$d_{12}=d_{21}=m_2[l_{c2}^2+2l_1 l_{1c2}\cos(q_2)]+I_2$$

$$d_{22}=m_2 l_{c2}^2+I_2$$

$$h=-m_2 l_1 l_{c2}\sin(q_2)$$

$$\varphi_1=(m_1 l_{c1}+m_2 l_1)g\cos(q_1)+m_2 l_{c2} g\cos(q_1+q_2)$$

$$\varphi_2=m_2 l_{c2} g\cos(q_1+q_2)$$