A DC-motor basically works on the principle that a current carrying conductor in a magnetic field experiences a force $\mathbf{F} = \boldsymbol{\phi} \times \mathbf{i}$, where $\boldsymbol{\phi}$ is the magnetic flux and $\mathbf{i}$ is the current in the conductor. The motor itself consists of a fixed **stator** and a movable **rotor** that rotates inside the stator, as shown in Figure 7-2. If the stator produces a radial magnetic flux ϕ and the current in the rotor (also called the **armature**) is $\mathbf{i}$ then there will be a torque on the rotor causing it to rotate. The magnitude of this torque is

$$\tau_m = K_1 \phi i_a \tag{7.2.2}$$

where τ_m is the motor torque (N-m), ϕ is the magnetic flux (webers), i_a is the armature current (amperes), and K_1 is a physical constant. In addition, whenever a conductor moves in a magnetic field, a voltage V_b is generated across its terminals that is proportional to the velocity of the conductor in the field. This voltage, called the **back emf**, will tend to oppose the current flow in the conductor.

Thus, in addition to the torque τ_m in (7.2.2), we have the back emf relation

$$V_b = K_2 \phi \omega_m \tag{7.2.3}$$

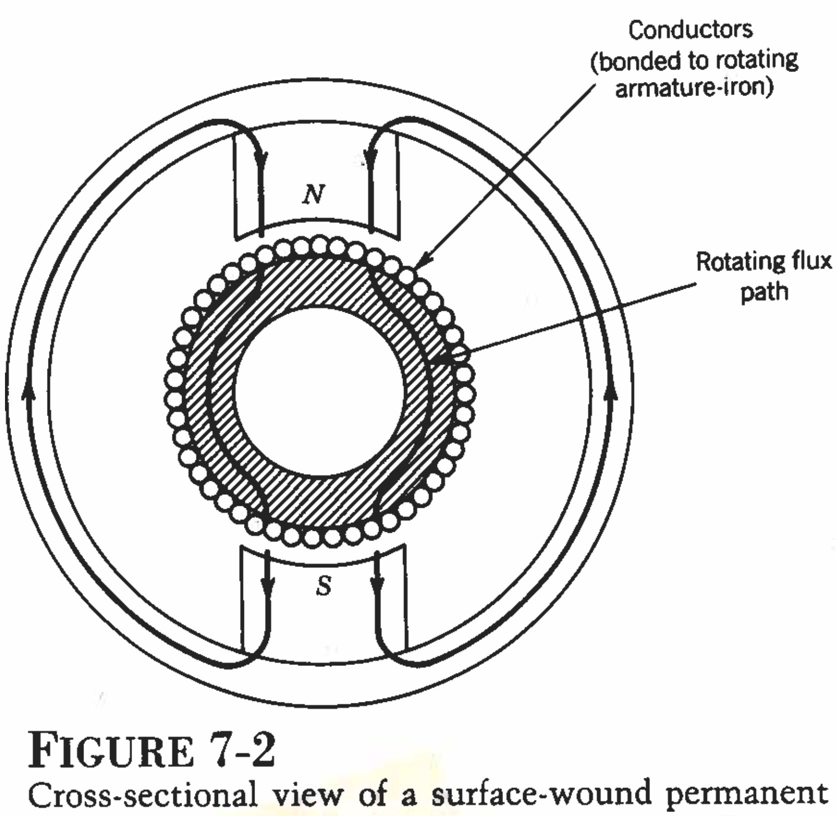

FIGURE 7-2
Cross-sectional view of a surface-wound permanent magnet DC motor.

where V_b denotes the back emf in Volts, ω_m is the angular velocity of the rotor (rad/sec), and K_2 is a proportionality constant.

DC-motors can be classified according to the way in which the magnetic field is produced and the armature is designed. Here we discuss only the so-called **permanent magnet** motors whose stator consists of a permanent magnet. In this case we can take the flux ϕ to be a constant. The torque on the rotor is then controlled by controlling the armature current i_a.

Consider the schematic diagram of Figure 7-3 where

$$\begin{aligned}
V(t) &= \text{armature voltage} \\
L &= \text{armature inductance} \\
R &= \text{armature resistance} \\
V_b &= \text{back emf} \\
i_a &= \text{armature current} \\
\theta_m &= \text{rotor position (radians)} \\
\tau_m &= \text{generated torque} \\
\phi &= \text{magnetic flux due to stator.}
\end{aligned}$$

The differential equation for the armature current is then

$$L\frac{di_a}{dt} + Ri_a = V - V_b \qquad (7.2.4)$$

Since the flux ϕ is constant the torque developed by the motor is

$$\tau_m = K_1\phi i_a = K_m i_a \qquad (7.2.5)$$

where K_m is the torque constant in N-m/amp. From (7.2.3) we have

$$V_b = K_2\phi\omega_m = K_b\omega_m = K_b\frac{d\theta_m}{dt} \qquad (7.2.6)$$

where K_b is the back emf constant.

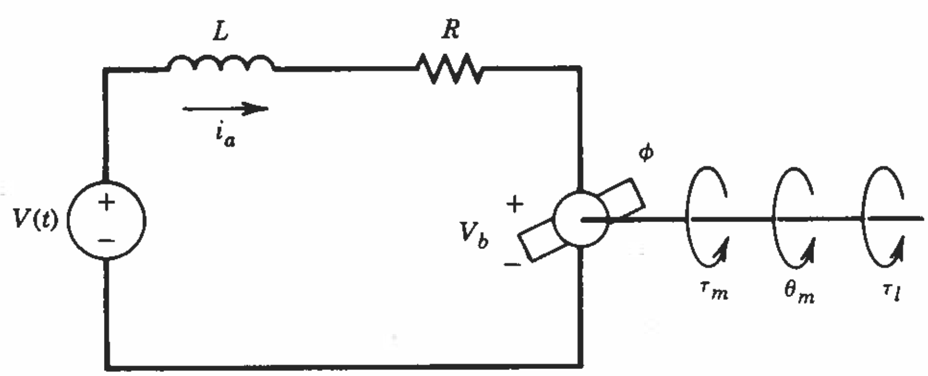

FIGURE 7-3
Circuit diagram for armature controlled DC motor.

Assume the system is stabilized, then θ_m will be kept small. This justifies the approximations

$$\sin(\theta_m/n) = \theta_m/n.$$

Then, one has

$$J\ddot{\theta}_m + B\dot{\theta}_m + C/n\,\theta_m = K_m i_a$$

$$L\frac{di_a}{dt} + R i_a = V - K_b\frac{d\theta_m}{dt}$$

The state variable of the system can be defined as θ_m, w_m, and i_a. Then, the state equation can be written as

$$\begin{bmatrix} \frac{di_a}{dt} \\ \frac{dw_m}{dt} \\ \frac{d\theta_m}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{K_b}{L} & 0 \\ \frac{K_m}{J} & -\frac{B}{J} & -\frac{C}{Jn} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ w_m \\ \theta_m \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} V$$

$K_m i_a = T_m$

torque constant in N-m/amp

$$R i_a = V - K_b \frac{d\theta_m}{dt}$$

armature Resistance

K_b is back emf constant

q : denote the angle of the link

θ_m = denote the angle of the motor shaft

$$q = \frac{1}{n} \theta_m$$

n : 1 the gear ratio

$$i_a = \frac{1}{R} V - \frac{K_b}{R} \cdot n \cdot \frac{dq}{dt}$$

$$\boxed{T_m = K_m i_a = \frac{K_m}{R} V - \frac{K_m K_b}{R} \cdot n \frac{dq}{dt}}$$

$$\boxed{T = n T_m = \frac{n K_m}{R} V - \frac{K_m K_b}{R} n^2 \frac{dq}{dt}}$$