

3.2 Model Formulation

By considering the challenges of last-mile distribution, the problem of delivering multiple lightweight relief packages to remote areas using one or more drones is examined. The model incorporates three types of geographical locations:

1. O , representing the Depot or central station from which drones depart. It is assumed that all drones originate from a single station, making O a singleton set. Additionally, drones are assumed to return empty to the warehouse.
2. L , representing demand locations where relief packages must be delivered.
3. R , representing recharging stations where drones can recharge their batteries.

Finally, the set N includes all locations in the system comprising O (depot), L (demand locations), and R (charging stations).

It is assumed that relief packages are available only at the warehouse. Therefore, drones depart from the warehouse with fully charged batteries, carrying a nonnegative number of packages and may stop at additional charging stations if necessary. The distance a drone must travel from the location i to j is associated with a symmetric length δ_{ij} (i.e., $\delta_{ij} = \delta_{ji}$). Each drone can carry a maximum of P_{max} relief packages and E_{max} represents the maximum energy supplied by a fully charged battery. A detailed description of the notations is provided in Table 2.

Objective Function

The objective function of the proposed model is designed to optimize travel time and travel time reliability simultaneously. Travel time is modeled as a fuzzy stochastic variable. Unlike travel time, this research treats reliability as a fuzzy variable.

To summarize, since the goal of this model is to minimize travel time while maximizing reliability, an initial fractional objective function may be set as follows:

$$Minf(x) = \frac{Travel\ Time(x)}{Reliability(x)} \quad (1)$$

Where x specifies the number of times the drone traverses each arc in the network. However, given the above considerations, the fractional objective function (1) is rewritten as follows:

$$Travel\ Time(x) = \sum_{(i,j) \in N} \tilde{t}_{ij}^s x_{ij} + t_0$$

$$Reliability(x) = \sum_{(i,j) \in N} \tilde{r}_{ij} x_{ij} + r_0$$

Where it is assumed that $\tilde{t}_{ij}^s \sim (\mu_{\tilde{t}_{ij}^s}, \sigma_{\tilde{t}_{ij}^s}^2)$ is a normally distributed random variable with a fuzzy mean and variance, representing the travel time of the drone from the location i to j , and $\tilde{r}_{ij}$ denotes the fuzzy reliability of the travel time for this route. The variable x_{ij} represents the number of times the drone travels from the location i to location j . The parameters r_0 and t_0 are constant values determined based on the problem conditions. Furthermore, without loss of generality, it is assumed that $\sum_{(i,j) \in N} \tilde{r}_{ij} x_{ij} + r_0 > 0$, which ensures the feasibility of the problem.

Constraints

In the proposed model, it is assumed that each demand location in L must be visited exactly once and by only one drone. Therefore, we define:

$$\sum_{i \in N \setminus \{j\}} x_{ij} = 1, \quad \forall j \in L, \quad (2)$$

Since drones must travel from the warehouse to demand points and eventually return to the warehouse, the flow of drones must be balanced within the network. In the proposed model, each drone can visit charging stations and warehouses as often as necessary, either to recharge the battery or load the relief supplies. However, for all network nodes, the number of entries to any node must equal the number of exits from that node. This means that any drone entering a point other than the warehouse must also exit from the same point.

$$\sum_{i \in N \setminus \{j\}} x_{ij} = \sum_{i \in N \setminus \{j\}} x_{ji}, \quad \forall j \in N \quad (3)$$

Constraint (3) ensures that no node is visited in an unbalanced manner and the movement paths of the drones within the network are logically and coherently organized.

At least one drone must be used to deliver relief packages to demand locations. Therefore, the number of drone trips between the warehouse and demand locations or charging stations must be greater than zero. Thus:

$$\sum_{j \in N \setminus \{O\}} x_{ij} > 0, \quad i \in O \quad (4)$$

Table 2: Description of Notations

Notations	Description
Indices	
N	Set of all locations (Depot O , demand locations L , and recharging stations R).
Parameters	
d_i	Demand for location $i \in L$.
P_{max}	The maximum payload capacity of the drone.
E_{max}	Maximum energy capacity of the drone's battery.
δ_{ij}	Distance between locations i and j .
e_{TL}^0	Base energy needed for takeoff and landing of an empty drone.
e_{TL}^+	Additional energy needed for takeoff and landing per package.

e_F^0	Energy required to fly per unit distance for an empty drone
e_F^+	Additional energy required to fly per unit distance per package
$\tilde{t}_{ij}^s$	Travel time for the drone to move from location i to location j .
$\tilde{r}_{ij}$	Travel time reliability on (i, j) .
λ^α and λ^β	Environmental impact on energy consumption baseline energy and load-dependent energy
Decision Variables	
x_{ij}	Number of times the drone moves from location i to location j .
y_{ij}	Number of packages carried by the drone when it leaves location i for j .
e_{ij}	Level of energy in the drone's battery when leaving the location i to j .

Constraint (4) ensures that the operation begins with a drone movement from the warehouse to a demand location. Without this constraint, no drone might initiate its journey, leaving the system inactive. Therefore, this constraint indicates that at least one drone must start its journey to a demand location to initiate the delivery process and activate the system.

Since each drone has limited payload and energy capacities, the number of packages a drone can carry from location i to location j cannot exceed the drone maximum capacity. Here, selecting the appropriate type of drone based on payload and energy capacity is one of the most critical challenges. Each type of drone has different payload and energy capacities, directly affecting travel time and operational reliability. Therefore, the right drone for each mission must be carefully selected to ensure the system provides optimal performance with minimum travel time and maximum reliability. The best way to address this multi-choice challenge is by considering the drone's maximum payload and energy capacity. Hence, the maximum payload and energy capacities of the drone are defined as multi-choice parameters such that $P_{max}^t, t = 1, 2, \dots, T$ represents the maximum payload capacity of a drone t , and $E_{max}^t, t = 1, 2, \dots, T$ represents its maximum energy capacity. T is the finite number of drones that will realize their delivery function. This can be expressed by constraints (5) to (7):

$$y_{ij} \leq (P_{max}^1, P_{max}^2, \dots, P_{max}^T).x_{ij}, \quad \forall i, j \in N \quad (5)$$

$$e_{ij} = (E_{max}^1, E_{max}^2, \dots, E_{max}^T).x_{ij}, \quad \forall i \in O \cup R, j \in N \setminus \{i\}, \quad (6)$$

$$e_{ij} \leq (E_{max}^1, E_{max}^2, \dots, E_{max}^T).x_{ij}, \quad \forall i \in L, j \in N \setminus \{i\}, \quad (7)$$

Where y_{ij} represents the number of packages carried by the drone when traveling from a location i to location j , and e_{ij} denotes the energy available in the drone battery when traveling from location i to location j . Constraint (6) specifies that the drone battery is always fully charged when the drone moves from the warehouse or charging station to another destination. However, constraint (7) indicates that the energy level in the battery at demand locations will always be less than or equal to the maximum energy level. Therefore, these two constraints are necessary to ensure the battery is fully charged at the start of the journey and to prevent excessive energy

consumption along the subsequent paths. Using multi-choice parameters here allows us to evaluate the issue in a single model instead of considering it across t different scenarios.

Accurate estimation of drone energy consumption ensures efficient operational decisions for drone distribution systems and all applications, including humanitarian missions. However, in many optimization models designing drone distribution systems, energy consumption is indirectly calculated only as a fixed limit to determine the flight range. Few studies have directly calculated energy based on a physical force-based energy consumption model or field measurements.

Various factors must be considered for the drone battery energy level. These include drone design (such as the number of rotors, rotor size, battery weight, battery energy capacity, and flight mechanism), environmental conditions (including air density, gravity, wind speed, and flight regulations), drone dynamics (including speed, vibrations, acceleration, type of rotation, and flight altitude), and finally, delivery operations (including payload weight and size, fleet size, type of delivery (throw/release or landing), and operational area), all of which significantly affect energy consumption. This research divides the drone battery energy level into two main components. The first component is base energy, which includes energy required for takeoff, flight without packages, and landing. This amount of energy is consumed even when no package is carried. The second component is payload-induced energy, which depends on the payload weight and the distance traveled, determining the additional energy required for transporting the load. Furthermore, environmental conditions affecting the drone energy level are considered in both components. The combination of these factors determines the total energy consumed by the drone. Given these considerations, the energy balance constraint for the drone at each demand location i in the set L of demand locations is defined as follows:

$$\sum_{j \in N \setminus \{i\}} e_{ji} - \sum_{j \in N \setminus \{i\}} e_{ij} = \sum_{j \in N \setminus \{i\}} (\alpha_{ji} x_{ji} + \beta_{ji} y_{ji}), \forall i \in L, \quad (8)$$

where α_{ji} represents the base energy required for moving the drone between locations i and j . This amount includes the energy required for the takeoff and landing of an empty drone (e_{TL}^0), the energy required for flying one unit distance for an empty drone (e_F^0), and the impact of environmental conditions λ^α :

$$\alpha_{ji} = e_{TL}^0 + \delta_{ji} e_F^0 + \lambda^\alpha$$

β_{ji} represents the additional energy required for carrying packages between location i and j , which includes the extra energy required for takeoff and landing with the additional package (e_{TL}^+), the extra energy required for flying one unit distance with the package (e_F^+), and the impact of environmental conditions λ^β :

$$\beta_{ji} = e_{TL}^+ + \delta_{ji} e_F^+ + \lambda^\beta$$

This constraint represents the energy consumed when moving from any location to a demand location. In addition, the impact of environmental conditions on energy consumption is considered

to make the model more realistic. To calculate λ^α and λ^β , an FIS is used, with environmental conditions such as wind speed W , temperature T , and altitude A as its inputs. These inputs are then transformed into the outputs λ^α and λ^β by using fuzzy rules. Each input variable is modeled as a fuzzy set F_x with corresponding linguistic terms:

$$F_W = \{low, medium, high\}$$

$$F_T = \{cold, moderate, hot\}$$

$$F_A = \{low, medium, high\}$$

Each input x is mapped to a membership function $\mu_{F_x}(x)$ that determines its degree of belonging to each linguistic term. The fuzzy inference system applies a rule base R_k , consisting of expert-defined rules of the form:

$$R_k: \text{If } w \in F_W^k \text{ and } t \in F_T^k \text{ and } a \in F_A^k \text{ then } \lambda^k \in F_\lambda^k,$$

Where k is the index of the fuzzy rule in the set of rules R and F_λ^k represents the fuzzy set for the environmental impact factor λ^k , with linguistic values $\{low, medium, high\}$. The aggregation of rule outputs λ is performed using the max-min composition:

$$\mu_\lambda(\lambda^k) = \max_k \min(\mu_{F_W^k}(w), \mu_{F_T^k}(t), \mu_{F_A^k}(a))$$

Finally, the defuzzification process, using the centroid method, computes a crisp value for $\lambda^{\alpha,\beta}$:

$$\lambda^\alpha = \frac{\sum_k \lambda^k \cdot \mu_\lambda(\lambda^k)}{\sum_k \mu_\lambda(\lambda^k)} \quad (9)$$

$$\lambda^\beta = \lambda^\alpha(1 + \xi) \quad (10)$$

where

$$\xi = \frac{e_F^+ - e_F^0}{e_F^0} \quad (11)$$

FIS is applied to model energy consumption under varying environmental conditions. The input variables (wind speed, temperature, and altitude) are represented as fuzzy sets with linguistic values (low, medium, high). Triangular membership functions are used to map numerical values to their corresponding degrees of membership. The system output, representing the environmental impact factor on energy (λ), is also modeled using three fuzzy membership functions (low, medium, high). These functions form the basis for evaluating fuzzy rules and computing the final λ value through aggregation and defuzzification. On the other hand, the battery energy level of the drone at the moment of departure from location

i to reach the next destination j must be sufficient. In other words, the drone should not start a path unless the remaining energy in the battery is enough to cover that path. Therefore:

$$e_{ij} \geq (\alpha_{ij}x_{ij} + \beta_{ij}y_{ij}), \quad \forall i \in N, \forall j \in N \setminus \{i\}, \quad (12)$$

According to the model's assumption, drones should not carry any package when returning to the warehouse. This assumption is made because the warehouse is the only loading point.

$$y_{ji} = 0, \quad \forall i \in O, j \in N \setminus \{O\} \quad (13)$$

This constraint ensures that no package is carried on the return paths to the warehouse. The number of packages a drone carries changes when it enters and exits a location. These changes must be adjusted according to the demand of the demand locations and the nature of the charging stations. Therefore:

$$y_{ji} - y_{ij} = \begin{cases} d_i, & \forall i \in L, j \in N \\ 0, & \forall i \in R, j \in N \end{cases} \quad (14)$$

Constraint (14) states that for each demand location, the difference between the number of packages a drone carries when it enters the area and the amount it holds when it exits is equal to the demand of that location. Also, it states that at charging stations, the number of packages a drone has when it enters is equal to the number it has when it exits. In other words, the package amount does not change. However, this constraint alone is not sufficient, as the model still considers charging stations as intermediate nodes. Therefore, the following two constraints have been defined to prevent such behavior in the model.

$$y_{ij} - y_{jp} \leq M(2 - x_{ij} - x_{jp}) \quad (15)$$

$$y_{jp} - y_{ij} \leq M(2 - x_{ij} - x_{jp}) \quad (16)$$

Let M be a large enough positive constant to ensure that the constraint becomes non-binding when at least one of the related arcs is not employed. The value of M can be used to set an upper limit on the number of packages that can be delivered between any two nodes.

Finally, in order to prevent direct flights of drones between depots and charging stations, constraints have been defined to eliminate the possibility of such routes in the model. Therefore:

$$\sum_{j \in O} y_{ij} = 0, \quad \forall i \in O \quad (17)$$

$$\sum_{j \in R} y_{ij} = 0, \quad \forall i \in R \quad (18)$$

4- Deterministic Approach

The objective function of the problem includes fuzzy stochastic variables and constraints (5) to (7) and involve multi-choice parameters, therefore the model cannot be directly solved using conventional methods. In such cases, one approach to overcoming fuzzy stochastic variables and multi-choice parameters is to transform the model into its equivalent deterministic form.

4-1- Probability-Credibility Approach

In addressing uncertainty from fuzzy stochastic variables, three major approaches have been widely used in the literature: probability-possibility, probability-necessity, and probability-credibility. This paper adopts the probability-credibility approach because credibility measurement is defined based on both possibility and necessity measures. Accordingly, the probability-possibility form of the objective function (1) is expressed as follows:

$$\begin{aligned} & \text{Min } f_0 \\ & \text{Pr} \left[\text{Cr} \left[\frac{\sum_{(i,j) \in N} \tilde{t}_{ij}^s x_{ij} + t_0}{\sum_{(i,j) \in N} \tilde{r}_{ij} x_{ij} + r_0} \leq f_0 \right] \geq \delta_0 \right] \geq \gamma_0 \end{aligned} \quad (19)$$

Where $\text{Pr}(\cdot)$ and $\text{Cr}(\cdot)$ represent the probability measure and the fuzzy credibility measure, respectively. Additionally, δ_0 and γ_0 are predefined confidence levels determined by the decision-maker or the client. f_0 is the target variable to the objective function. In this approach, the decision-maker aims to achieve at least f_0 such that the fractional objective function (1) remains less than or equal to f_0 with confidence levels $\delta, \gamma \in [0,1]$. Higher value of δ , known in the literature as the fuzzy credibility degree, increases the number of real-world scenarios in which the obtained solution holds. Therefore, a risk-seeking decision-maker may choose lower values of δ , whereas a risk-averse decision-maker may opt for higher values. On the other hand, a lower δ reduces operational costs. Consequently, selecting a higher δ implies an additional cost for obtaining a more stable solution. Given the importance of post-disaster relief operations in this research, the credibility degree is set to be greater than 0.5 to ensure stable solutions for the problem.

A review of the existing literature indicates no specific guidelines for selecting the shape of fuzzy numbers for fuzzy parameters in a given problem. In decision-making problems, fuzzy numbers are typically chosen in an *LR* form (triangular or trapezoidal) to maintain the linearity of the problem. However, trapezoidal fuzzy numbers are used when the data includes an interval in which all values are equally valid (having the highest membership degree). This condition rarely occurs in real-world scenarios. Based on this, in this research, fuzzy parameters are defined as *LR*-type triangular fuzzy numbers as follows:

$$\begin{aligned} \tilde{t}_{ij}^s &= (t_{ij}^s, t_{ij}^\alpha, t_{ij}^\beta)_{LR} \\ \tilde{r}_{ij} &= (r_{ij}, r_{ij}^\alpha, r_{ij}^\beta)_{LR} \end{aligned}$$

where t_{ij}^s and r_{ij} represent the central values of the fuzzy number, while t_{ij}^α and r_{ij}^α denote the left spreads, and t_{ij}^β and r_{ij}^β denote the right spreads. Unlike deterministic numbers, which have exact values, a fuzzy number represents the membership degree to a specific value within the range $[0,1]$. The membership function determines this degree of membership between 0 and 1. This concept allows for incorporating uncertainty and imprecision in mathematical models. For the proposed model, the membership function of the fuzzy numbers is defined as follows:

$$\mu_{\tilde{t}_{ij}^s}(x) = \begin{cases} L\left(\frac{t_{ij}^s - x}{t_{ij}^\alpha}\right), & x \leq t_{ij}^s \\ R\left(\frac{x - t_{ij}^s}{t_{ij}^\beta}\right), & x \geq t_{ij}^s \end{cases} \quad (20)$$

$$\mu_{r_{ij}}(x) = \begin{cases} L\left(\frac{r_{ij} - x}{r_{ij}^\alpha}\right), & x \leq r_{ij} \\ R\left(\frac{x - r_{ij}}{r_{ij}^\beta}\right), & x \geq r_{ij} \end{cases} \quad (21)$$

where $L(\cdot)$ and $R(\cdot)$ are the left and right non-increasing continuous functions from $[0,1]$ to $[0,1]$, such that $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$.

Now, using the concepts of fuzzy credibility measure and expressions (20) and (21) for $\delta_0 > 0.5$, we have (Tavana 2012):

$$\begin{aligned} & Cr \left[\frac{\sum_{(i,j) \in N} \tilde{t}_{ij}^s x_{ij} + t_0}{\sum_{(i,j) \in N} \tilde{r}_{ij} x_{ij} + r_0} \leq f_0 \right] \geq \delta_0 \\ \Leftrightarrow & \sum_{(i,j) \in N} \left(t_{ij}^s + t_{ij}^\beta L^{-1}(2(1 - \delta_0)) \right) x_{ij} + t_0 \\ & \leq f_0 \left(\sum_{(i,j) \in N} \left(r_{ij} + r_{ij}^\alpha L^{-1}(2(1 - \delta_0)) \right) x_{ij} + r_0 \right) \end{aligned} \quad (22)$$

Since t_{ij}^s in (22) is a random variable with a normal distribution, by assuming that $m_{t_{ij}^s}$ and $\sigma_{t_{ij}^s}$ are its mean and standard deviation, respectively, we can express the following for $\gamma_0 \in [0,1]$ by using probability theory concepts (Bavandi and Nasseri 2022):

$$\frac{\sum_{(i,j) \in N} \left(m_{t_{ij}^s} + t_{ij}^\beta L^{-1}(2(1 - \delta_0)) \right) x_{ij} + t_0 + \Phi^{-1}(\gamma_0) \sqrt{\sum_{(i,j) \in N} \sigma_{t_{ij}^s}^2 x_{ij}^2}}{\sum_{(i,j) \in N} \left(r_{ij} + r_{ij}^\alpha L^{-1}(2(1 - \delta_0)) \right) x_{ij} + r_0} \leq f_0 \quad (23)$$

Expression (23) represents the deterministic form of (19).

4-2 Lagrange Interpolation Polynomial Approach

Constraints (5) to (7) include multi-choice parameters, making solving this problem infeasible. Therefore, as a first step, multi-choice parameters are transformed using interpolation polynomials. The interpolation polynomial is constructed by introducing a non-negative integer variable corresponding to each multi-choice parameter. In fact, if the corresponding parameter has n options, each integer variable exactly covers n knot points. In the proposed model, there are two multi-choice parameters, P_{max} and E_{max} , so the integer variables u and v are introduced for these

two parameters, respectively. Since P_{max} and E_{max} , have T options, u and v , therefore, cover exactly T knot points. Each node corresponds exactly to one functional value of a multi-choice parameter, as shown in Table 3. The Lagrange interpolation polynomials corresponding to these two multi-choice parameters are denoted by $g_{P_{max}}(u)$ and $g_{E_{max}}(v)$.

Table 3: Data Related to the Multi-Choice Parameters P_{max} and E_{max}

u & v	0	1	2	...	$T - 1$
$g_{P_{max}}(u)$	$P_{max}^{(1)}$	$P_{max}^{(2)}$	$P_{max}^{(3)}$	...	$P_{max}^{(T)}$
$g_{E_{max}}(v)$	$E_{max}^{(1)}$	$E_{max}^{(2)}$	$E_{max}^{(3)}$	...	$E_{max}^{(T)}$

Based on the information provided in Table 3 and the Lagrange interpolation formula, the interpolating polynomials for the multi-choice parameters of the problem are obtained as follows:

$$\begin{aligned}
g_{P_{max}}(u) = & \frac{(u-1)(u-2)\dots(u-T+1)}{(-1)^{(T-1)}(T-1)!} P_{max}^{(1)} \\
& + \frac{u(u-2)(u-3)\dots(u-T+1)}{(-1)^{(T-2)}1!(T-2)!} P_{max}^{(2)} \\
& \vdots \\
& + \frac{u(u-1)(u-2)\dots(u-T+2)}{(T-1)!} P_{max}^{(T)}
\end{aligned} \tag{24}$$

$$\begin{aligned}
g_{E_{max}}(v) = & \frac{(v-1)(v-2)\dots(v-T+1)}{(-1)^{(T-1)}(T-1)!} P_{max}^{(1)} \\
& + \frac{v(v-2)(v-3)\dots(v-T+1)}{(-1)^{(T-2)}1!(T-2)!} P_{max}^{(2)} \\
& \vdots \\
& + \frac{v(v-1)(v-2)\dots(v-T+2)}{(T-1)!} P_{max}^{(T)}
\end{aligned} \tag{25}$$

Using (24) and (25), and given that minimizing f_0 is equivalent to minimizing the left-hand side of (23), the deterministic model of the problem is obtained as follows:

$$Min \frac{\sum_{(i,j) \in N} \left(m_{t_{ij}^S} + t_{ij}^{\beta} L^{-1} (2(1-\delta_0)) \right) x_{ij} + t_0 + \Phi^{-1}(\gamma_0) \sqrt{\sum_{(i,j) \in N} \sigma_{t_{ij}^S}^2 x_{ij}^2}}{\sum_{(i,j) \in N} \left(r_{ij} + r_{ij}^{\alpha} L^{-1} (2(1-\delta_0)) \right) x_{ij} + r_0} \tag{26}$$

s.t. (2)-(4), (8)-(12)

$$\sum_{k=1}^K y_{ij}^k \leq g_{P_{max}}(u) \cdot x_{ij}, \quad \forall i, j \in N \tag{27}$$

$$e_{ij} = g_{E_{max}}(v) \cdot x_{ij}, \quad \forall i \in O \cup R, j \in N \setminus \{i\}, \tag{28}$$

$$e_{ij} \leq g_{E_{max}}(v) \cdot x_{ij}, \quad \forall i \in L, j \in N \setminus \{i\}, \tag{29}$$

$$x_{ij} \in \mathbb{Z}^+, y_{ij}^k \in \mathbb{Z}^+, e_{ij} \geq 0$$

The latest model is a deterministic fractional model that can be solved using the bisection method (Yano 2012) or the Dinkelbach algorithm (Dinkelbach 1967). In this research, we employ

Dinkelbach's algorithm to solve the above nonlinear fractional programming problem. The Dinkelbach algorithm is summarized as follows:

Algorithm 1: Dinkelbach's Algorithm

Step 1: Consider the fractional objective function (26). For convenience, we set:

$$N(x) = \sum_{(i,j) \in N} \left(m_{t_{ij}^s} + t_{ij}^\beta L^{-1}(2(1 - \delta_0)) \right) x_{ij} + t_0 + \Phi^{-1}(\gamma_0) \sqrt{\sum_{(i,j) \in N} \sigma_{t_{ij}^s}^2 x_{ij}^2}$$

$$D(x) = \sum_{(i,j) \in N} \left(r_{ij} + r_{ij}^\alpha L^{-1}(2(1 - \delta_0)) \right) x_{ij} + r_0$$

For all $x \in X$, where X is the feasible set, it follows that $N(x) \geq 0$ and $D(x) > 0$. Let x^1 denote the initial solution in the feasible set X . set

$$\varphi^1 = \frac{N(x^1)}{D(x^1)},$$

and $k = 1$.

Step 2: Solve the problem

$$\begin{aligned} \text{Min } & N(x) - \varphi^k D(x) \\ \text{s. t. } & x \in X \end{aligned}$$

Let x^{k+1} be the optimal solution of this subproblem.

Step 3: Check the convergence condition:

$$\eta = N(x^{k+1}) - \varphi^k D(x^{k+1})$$

If $0 \leq \eta \leq \varepsilon$, then x^{k+1} is the optimal solution and stop. Otherwise, go to step 4.

Step 4: Update

$$\varphi^{k+1} = \frac{N(x^{k+1})}{D(x^{k+1})}$$

Set $k \leftarrow k + 1$, and return to step 2.