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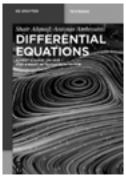

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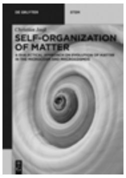

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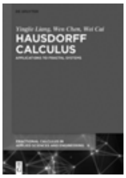

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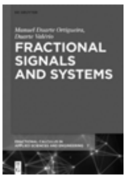

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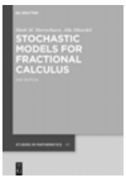

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Mathematics Subject Classification 2020
65M06, 65M12, 65M15

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ISBN 978-3-11-061517-3
e-ISBN (PDF) 978-3-11-061606-4
e-ISBN (EPUB) 978-3-11-061530-2

Library of Congress Control Number: 2020938024

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de.

Cover image: 00one / E+ / gettyimages.de
Typesetting: VTeX UAB, Lithuania
Printing and binding: CPI books GmbH, Leck

www.degruyter.com

Preface

As a generalization of classical calculus, fractional calculus has become an important branch of mathematics. It is popularly believed that this concept is stemmed from a letter by G. W. Leibniz (1646–1716) in the year 1695, where the one-half order of derivative was discussed. During the development of the past more than three centuries, numerous mathematicians made outstanding contributions on this field.

Now the fractional differential equations (FDEs) have become one of the important tools to model complex mechanics and physical behaviors and widespread applications have been found in anomalous diffusion, viscoelasticity, fluid flow, boundary layer effect of pipeline, electromagnetism, signal processing and control, quantum economy, fractal theory, etc., whereas, it is difficult to get the analytical solutions to the FDEs, even for the linear FDEs. Hence it becomes an important task to find some effective numerical simulations in current researches.

This book aims to make a systematic introduction to the finite difference method of FDEs. There are six chapters in this book.

Chapter 1 serves as a mathematical introduction to fractional calculus. It commences with four basic definitions of fractional derivatives. The analytical solutions to two kinds of fractional ordinary differential equations (FODEs) are given, from which, readers can have a general idea on the behaviors of solutions to FODEs. Several numerical approximation ways to fractional derivatives are introduced together with their numerical accuracy analysis. The applications of these formulae are also illustrated by solving the FODEs. This part is the important foundation of the following numerical solutions to fractional partial differential equations (FPDEs).

In Chapter 2, we study the finite difference methods for solving time-fractional subdiffusion equations. The time-fractional derivatives are approached by the G-L formula, the L1 approximation, the L2-1_σ approximation, the fast L1 approximation and the fast L2-1_σ approximation, respectively; The spatial derivatives are discretized by using the second-order central difference quotient or the compact approximation. For the 2D problem, several ADI difference schemes are derived. The unique solvability, stability and convergence for each scheme are proved.

Chapter 3 shows the finite difference methods for solving time-fractional wave equations. The time-fractional derivatives are discretized by the L1 approximation, the fast L1 approximation, the L2-1_σ approximation and the fast L2-1_σ approximation, respectively. For the 1D problem, two kinds of difference schemes are developed, among which one is of order two in space and the other is of order four in space. For the 2D problem, the ADI scheme and compact ADI scheme are both mentioned. The unique solvability, stability and convergence for each scheme are proved.

https://doi.org/10.1515/9783110616064-201

In Chapter 4, we introduce the finite difference methods for solving the space-fractional partial differential equations. For the 1D problem, the first-order method based on the shifted G-L formula, the second-order method based on the weighted-shifted G-L (WSGL) formula and the fourth-order method based on the WSGL formula are developed in turn. For the 2D problem, a fourth-order ADI method based on the WSGL formula is presented. The unique solvability, stability and convergence for each scheme are shown.

Chapter 5 considers the finite difference methods for solving a class of the time-space fractional differential equations. The time Caputo derivative is treated by the L2-1_σ approximation and the spatial Riesz derivatives are discretized by the second-order fractional central difference quotient and the fourth-order weighted fractional central difference quotient formula, respectively. The second-order and the fourth-order difference schemes in space are established, respectively. The unique solvability, stability and convergence for each scheme are proved.

In Chapter 6, the finite difference methods for solving a class of time distributed-order subdiffusion equations are concerned. The distributed integral is discretized using the composite trapezoid formula or composite Simpson formula and the Caputo time-fractional derivatives are approximated using the second-order WSGL formula. The second-order scheme in both time and distributed order, and another fourth-order scheme in both time and distributed order are constructed, respectively. In addition, for the 2D problem, a second-order ADI difference scheme and another fourth-order ADI difference scheme are developed, respectively. The unique solvability, stability and convergence for each scheme are analyzed.

There are abundant results on the numerical method for FDEs in recent 20 years. In the last section of each chapter, we give a brief overview and only a limit part among them is listed in the references of this book, which are the resource or the referred materials of this book.

The main part of this book is based on the research results from the authors and their research group. The authors express their heartfelt thanks to all the collaborators.

The authors are also very grateful to Wanrong Cao, Rui Du, Ruilian Du, Xuping Wang, Renjun Qi and Xuanru Lu, who have read the manuscript and provided many valuable suggestions.

The writing of this monograph was supported by the National Natural Science Foundation of China under Grant No. 11671081 and the Natural Science Foundation of Jiangsu Province under Grant No. BK20191375.

As the authors' limited scientific research work experience, they sincerely hope that scholars and colleagues will not hesitate to correct the shortcomings and omissions in the book. Thank you for sending emails to us.

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March. 28, 2020

Contents

1 Fractional derivatives and numerical approximations

In this chapter, several common definitions and properties of fractional derivatives will be introduced. The analytical solution as well its behaviors of two kinds of fractional ordinary differential equations (FODEs) will be analyzed. Several numerical approximations to the fractional derivatives and their accuracy will be introduced and applications into solving the FODEs will also be illustrated. These contents will be the important foundation of the following numerical research on fractional partial differential equations (FPDEs).

1.1 Definitions and properties of fractional derivatives

1.1.1 Fractional integral

Definition 1.1.1. Suppose α is a positive real number and the function $f(t)$ is defined on the interval $[a,b]$. The α-th order fractional integral of function $f(t)$ is defined as

$$ {}_aD_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau, $$

where $t \in [a,b]$ and $\Gamma(z)$ is the *Gamma* function defined by

$$ \Gamma(z) = \int_0^\infty \mathrm{e}^{-t}t^{z-1}\mathrm{d}t, \quad \mathrm{Re}(z) > 0. $$

A direct calculation gives

$$ {}_aD_t^{-\alpha}(t-a)^p = \frac{\Gamma(1+p)}{\Gamma(1+p+\alpha)}(t-a)^{p+\alpha}, \quad p > -1. $$

1.1.2 Grünwald–Letnikov fractional derivative

Definition 1.1.2. Suppose α is a positive real number, $n-1 \leqslant \alpha < n$ with n a positive integer and the function $f(t)$ is defined on the interval $[a,b]$. The α-th order **Grünwald–Letnikov (G-L)** fractional derivative of function $f(t)$ is defined as

$$ {}_aD_t^{\alpha}f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[(t-a)/h]} (-1)^j \binom{\alpha}{j} f(t-jh), $$

https://doi.org/10.1515/9783110616064-001

where $t \in [a, b]$, $[z]$ is the maximum integer no more than z and $\binom{\alpha}{j}$ is the binomial coefficient defined by

$$\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}.$$

Suppose the functions $f^{(k)}(t), k = 0, 1, 2, \ldots, n$ are all continuous on $[a, b]$ and n is the minimum integer satisfying $\alpha < n$. It can be proved that

$${}_aD_t^\alpha f(t) = \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^{j-\alpha}}{\Gamma(1+j-\alpha)} + \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha-n+1}}.$$

1.1.3 Riemann–Liouville fractional derivative

Definition 1.1.3. Suppose α is a positive real number, $n-1 \leqslant \alpha < n$ with n a positive integer and the function $f(t)$ is defined on the interval $[a, b]$. The α-th order **Riemann–Liouville (R-L)** fractional derivative of function $f(t)$ is defined as

$${}_a\mathbf{D}_t^\alpha f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha-n+1}}\right),$$

where $t \in [a, b]$.

It is clear that

$${}_a\mathbf{D}_t^\alpha f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[{}_aD_t^{-(n-\alpha)} f(t)\right]$$

and a direct calculation gives

$${}_a\mathbf{D}_t^\alpha (t-a)^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)}(t-a)^{p-\alpha}, \quad p > -1.$$

It can be proved that

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left({}_a\mathbf{D}_t^\alpha f(t)\right) = {}_a\mathbf{D}_t^{m+\alpha} f(t), \quad \alpha > 0,\ m > 0,\ m \in \mathbb{Z}^+.$$

Note that there is an equivalence relation between the R-L fractional derivative and the G-L fractional derivative: For a positive real number α, suppose $n-1 \leqslant \alpha < n$. If the function $f(t)$, defined on $[a, b]$, has continuous derivatives up to the $(n-1)$-th order and $f^{(n)}(t)$ is integrable on $[a, b]$, then the α-th order R-L fractional derivative of function $f(t)$ is equivalent to the α-th order G-L fractional derivative.

1.1.4 Caputo fractional derivative

Definition 1.1.4. Suppose α is a positive real number, $n-1 < \alpha \leqslant n$ with n a positive integer and the function $f(t)$ is defined on the interval $[a,b]$. The α-th order **Caputo** fractional derivative of function $f(t)$ is defined as

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha-n+1}}, $$

where $t \in [a,b]$.

It is easy to know that

$$ {}_a^C D_t^\alpha f(t) = {}_a D_t^{-(n-\alpha)}[f^{(n)}(t)] $$

and a direct calculation gives

$$ {}_a^C D_t^\alpha (t-a)^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)}(t-a)^{p-\alpha}, \quad p > n-1 \geqslant 0. $$

Suppose the function $f(t)$, defined on $[a,b]$, has continuous derivatives up to the $(n+1)$-th order, then

$$ \begin{aligned} &\lim_{\alpha\to n-0} {}_a^C D_t^\alpha f(t) \\ &= \lim_{\alpha\to n-0}\left[\frac{f^{(n)}(a)(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)}\int_a^t (t-\tau)^{n-\alpha} f^{(n+1)}(\tau)\mathrm{d}\tau\right] \\ &= f^{(n)}(a) + \int_a^t f^{(n+1)}(\tau)\mathrm{d}\tau = f^{(n)}(t). \end{aligned} $$

Note that there is also an equivalence relation between the Caputo fractional derivative and the R-L fractional derivative: For a positive real number α, suppose $0 \leqslant n-1 < \alpha < n$. If the function $f(t)$, defined on $[a,b]$, has continuous derivatives up to the $(n-1)$-th order and $f^{(n)}(t)$ is integral on $[a,b]$, then

$$ {}_a\mathbf{D}_t^\alpha f(t) = {}_a^C D_t^\alpha f(t) + \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^{j-\alpha}}{\Gamma(1+j-\alpha)}, \quad a \leqslant t \leqslant b. $$

Particularly, when $\alpha \in (0,1)$, it reads

$$ {}_a\mathbf{D}_t^\alpha f(t) = {}_a^C D_t^\alpha f(t) + \frac{f(a)(t-a)^{-\alpha}}{\Gamma(1-\alpha)}. $$

It can be found that if the function $f(t)$ satisfies

$$ f^{(j)}(a) = 0, \quad j = 0,1,\ldots,n-1, $$

the α-th order Caputo fractional derivative and the α-th order R-L fractional derivative are equivalent.

It can be known from the definitions of fractional derivatives above that the value of fractional derivatives at one point t is related with all of the function values on the left-hand side of this point. Hence, they are also called the left G-L fractional derivative, the left R-L fractional derivative and the left Caputo fractional derivative, respectively.

Similarly, the right G-L fractional derivative, the right R-L fractional derivative and the right Caputo fractional derivative can be defined respectively as

$$
\begin{aligned}
{}_tD_b^\alpha f(t) &= \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[(b-t)/h]} (-1)^j \binom{\alpha}{j} f(t+jh), \\
{}_t\mathbf{D}_b^\alpha f(t) &= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{1}{\Gamma(n-\alpha)} \int_t^b \frac{f(\tau)\mathrm{d}\tau}{(\tau-t)^{\alpha-n+1}} \right), \\
{}_t^CD_b^\alpha f(t) &= (-1)^n \frac{1}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(\tau)\mathrm{d}\tau}{(\tau-t)^{\alpha-n+1}}.
\end{aligned}
$$

When $a = -\infty$, the left G-L fractional derivative is defined as

$$
{}_{-\infty}D_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(t-jh);
$$

When $b = \infty$, the right G-L fractional derivative is defined as

$$
{}_tD_\infty^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(t+jh).
$$

1.1.5 Riesz fractional derivative

Definition 1.1.5. Suppose α is a positive real number, $n-1 \leqslant \alpha < n$ with n a positive integer, $\alpha \neq 2k+1$, $k = 0, 1, \ldots$, and the function $f(x)$ is defined on $[a, b]$. Then the α-th order **Riesz** fractional derivative of function $f(x)$ is defined as

$$
\frac{\partial^\alpha f(x)}{\partial |x|^\alpha} = -\frac{1}{2\cos(\frac{\alpha\pi}{2})}\left[{}_a\mathbf{D}_x^\alpha f(x) + {}_x\mathbf{D}_b^\alpha f(x) \right],
$$

where $x \in [a, b]$.

It can be known from the definition above that the Riesz fractional derivative can be regarded as a weighted sum of the left R-L fractional derivative and the right one. And the value of a Riesz fractional derivative at one point x is related with values of function $f(x)$ on the both hand sides of x.

1.1.6 Behaviors of fractional derivatives at the lower limit of integrals

Here, we will describe some behaviors of the fractional derivative ${}_a\mathbf{D}_t^\alpha f(t)$ at the lower limit of the integral.

Suppose that for some small and positive constant ϵ, the function $f(t)$ is analytical at least on the interval $[a, a+\epsilon]$. By Taylor expansion, the function $f(t)$ can be expressed as

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(t-a)^k, \quad t \in [a, a+\epsilon]. \tag{1.1}$$

Computing the R-L fractional derivative of each term in (1.1), one obtains

$${}_a\mathbf{D}_t^\alpha f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}. \tag{1.2}$$

For the function $f(t)$ in the form of (1.1) with $f(a) \neq 0$, it follows from (1.2) that

$${}_a\mathbf{D}_t^\alpha f(t) \sim \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \quad t \to a+0.$$

If the function $f(t)$ has the integrable singularity at $t = a$, that is, the function $f(t)$ can be expressed as

$$f(t) = (t-a)^q g(t), \quad \text{where } g(a) \neq 0,\ q > -1,$$

and the function $g(t)$ can be expressed into the form of Taylor series, then one has

$$f(t) = (t-a)^q \sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!}(t-a)^k = \sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!}(t-a)^{q+k}.$$

Computing the R-L fractional derivative term by term, it follows

$${}_a\mathbf{D}_t^\alpha f(t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!} \frac{\Gamma(q+k+1)}{\Gamma(q+k-\alpha+1)}(t-a)^{q+k-\alpha},$$

which implies that

$${}_a\mathbf{D}_t^\alpha f(t) \sim \frac{g(a)\Gamma(q+1)}{\Gamma(q-\alpha+1)}(t-a)^{q-\alpha}, \quad t \to a+0.$$

1.2 The Fourier transform of fractional derivatives

Definition 1.2.1. Suppose the function $g(t)$, defined on $\mathcal{R} = (-\infty, \infty)$, is piecewise continuous and absolutely integrable. The **Fourier** transform of function $g(t)$, denoted by $G(\omega)$, is defined as

$$G(\omega) \equiv \mathcal{F}[g(t); \omega] = \int_{-\infty}^{\infty} g(t)e^{i\omega t}dt.$$

Moreover, the function $g(t)$ can be recovered by the **Fourier** inverse transform as

$$g(t) = \mathcal{F}^{-1}[G(\omega); t] \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}\omega.$$

If the function $g(t)$ is differentiable on $\mathcal{R}$, $g'(t)$ is piecewise continuous; the functions $g(t)$ and $g'(t)$ are both absolutely integrable; when $|t| \to \infty$, $g(t) \to 0$, then

$$\mathcal{F}[g'(t); \omega] = (-\mathrm{i}\omega)\mathcal{F}[g(t); \omega].$$

If the function $g(t)$ has the continuous derivatives up to order $n-1$ and the piecewise continuous derivative of order n on $\mathcal{R}$; the functions $g(t), g'(t), \ldots, g^{(n)}(t)$ are all absolutely integrable; the functions $g(t), g'(t), \ldots, g^{(n-1)}(t) \to 0$ when $|t| \to \infty$, then

$$\mathcal{F}[g^{(n)}(t); \omega] = (-\mathrm{i}\omega)^n\mathcal{F}[g(t); \omega].$$

Suppose $\alpha > 0$. Let

$$h_+^\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \leqslant 0, \end{cases}$$

then it follows

$$\mathcal{F}[h_+^\alpha(t); \omega] = (-\mathrm{i}\omega)^{-\alpha}.$$

Now we consider the case of fractional derivatives at lower limit $a = -\infty$ when the function $f(t)$ and its derivatives up to some necessary order approach to zero as $t \to -\infty$. Suppose $n-1 \leqslant \alpha < n$. It can be easily obtained from the integration by parts that the G-L fractional derivative, the R-L fractional derivative and the Caputo fractional derivative are in the same form of

$$\left.\begin{array}{l} {}_{-\infty}D_t^\alpha f(t) \\ {}_{-\infty}\mathbf{D}_t^\alpha f(t) \\ {}_{-\infty}^{\;\;C}D_t^\alpha f(t) \end{array}\right\} = \frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^{t} \frac{f^{(n)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-n}}$$

$$= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^{t} \frac{f(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-n}}\right) \equiv D^\alpha f(t).$$

Assume that:

1. the functions $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are existent and continuous, the function $f^{(n)}(t)$ is piecewise continuous and they are all absolutely integrable;
2. the functions $f(t), f'(t), \ldots, f^{(n-1)}(t) \to 0$ when $|t| \to \infty$,

then it holds

$$\mathcal{F}[D^\alpha f(t); \omega] = (-\mathrm{i}\omega)^\alpha\mathcal{F}[f(t); \omega].$$

1.3 Fractional ordinary differential equations

In this section, two types of fractional ordinary differential equations (FODEs) will be analytically solved.

1.3.1 Solutions to the FODEs in Riemann-Liouville type

Problem 1.3.1. Suppose $0 < \alpha < 1$. Consider the single-term FODEs in R-L type as follows:

$$\begin{cases} {}_0\mathbf{D}_t^{\alpha} y(t) = f(t), \quad t > 0, & (1.3) \\ y(0) = 0, & (1.4) \end{cases}$$

where the function $f(t)$ is supposed to be in the Taylor series form of

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n,$$

with the convergence radius R, $R > 0$.

Here, for the problem (1.3)–(1.4), we aim to seek the solution in the form of

$$y(t) = t^{\alpha} \sum_{n=0}^{\infty} y_n t^n = \sum_{n=0}^{\infty} y_n t^{n+\alpha}. \tag{1.5}$$

By means of

$${}_0\mathbf{D}_t^{\alpha} t^{\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)} t^{\nu-\alpha},$$

the substitution of (1.5) into (1.3) gives

$$\sum_{n=0}^{\infty} y_n \frac{\Gamma(1+n+\alpha)}{\Gamma(1+n)} t^n = f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n.$$

Comparing the coefficients of two series above arrives at

$$y_n = \frac{f^{(n)}(0)}{\Gamma(1+n+\alpha)}, \quad n = 0, 1, 2, \ldots.$$

Hence, it follows

$$y(t) = t^{\alpha} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(1+n+\alpha)} t^n,$$

which can be rearranged as

$$y(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(1+n+\alpha)} t^{n+\alpha}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{\Gamma(n+1)}{\Gamma(1+n+\alpha)} t^{n+\alpha} \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} {}_0D_t^{-\alpha} t^n \\
&= {}_0D_t^{-\alpha} \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n \right) \\
&= {}_0D_t^{-\alpha} f(t).
\end{aligned}$$

If the function $f(t)$ has the following form:

$$f(t) = t^q g(t), \quad g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n, \quad g(0) \neq 0, q > -1,$$

the method above is still valid. To seek the solution in the form of

$$y(t) = t^{q+\alpha} \sum_{n=0}^{\infty} y_n t^n$$

subjected to the zero initial condition (1.4), similarly, it can be obtained that

$$y_n = \frac{\Gamma(1+n+q)}{\Gamma(1+n+\alpha+q)} \cdot \frac{g^{(n)}(0)}{n!}, \quad n = 0, 1, 2, \ldots.$$

Suppose $q + \alpha \geqslant 0$ and denote $m = [q + \alpha]$, then $y(t)$ has the continuous derivative of order m.

Problem 1.3.2. Suppose $0 < \alpha < 1$. Consider the problem

$$\begin{cases} {}_0\mathbf{D}_t^{\alpha} y(t) = f(t), \quad t > 0, & (1.6) \\ y(0) = A, \quad A \neq 0. & (1.7) \end{cases}$$

It can be found that a necessary condition to ensure a solution in the form of

$$y(t) = \sum_{n=0}^{\infty} y_n t^n \tag{1.8}$$

is that

$$f(t) \sim \frac{At^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \to 0.$$

Suppose

$$f(t) = \frac{At^{-\alpha}}{\Gamma(1-\alpha)} + t^{1-\alpha} \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n, \tag{1.9}$$

where the coefficient $\{g^{(n)}(0)\}$ is known.

Substituting (1.8) and (1.9) into (1.6) produces

$$\sum_{n=0}^{\infty} y_n \frac{\Gamma(1+n)}{\Gamma(1+n-\alpha)} t^{n-\alpha} = f(t) = \frac{At^{-\alpha}}{\Gamma(1-\alpha)} + t^{1-\alpha} \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n.$$

Comparing the coefficients gives

$$y_0 = A, \quad y_n = \frac{\Gamma(1+n-\alpha)}{\Gamma(1+n)} \cdot \frac{g^{(n-1)}(0)}{(n-1)!}, \quad n = 1, 2, \dots.$$

Hence,

$$y(t) = A + \sum_{n=1}^{\infty} \frac{\Gamma(1+n-\alpha)}{\Gamma(1+n)} \cdot \frac{g^{(n-1)}(0)}{(n-1)!} t^n.$$

Let

$$y(t) = z(t) + A,$$

then the problem (1.6)–(1.7) is converted to the following initial value problem with respect to $z(t)$:

$$\begin{cases} {}_0\mathbf{D}_t^{\alpha} z(t) = \hat{f}(t), \quad t > 0, & (1.10) \\ z(0) = 0, & (1.11) \end{cases}$$

where

$$\hat{f}(t) = f(t) - \frac{At^{-\alpha}}{\Gamma(1-\alpha)} = t^{1-\alpha} \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n.$$

Solving (1.10)–(1.11) arrives at

$$z(t) = \sum_{n=1}^{\infty} \frac{\Gamma(1+n-\alpha)}{\Gamma(1+n)} \cdot \frac{g^{(n-1)}(0)}{(n-1)!} t^n.$$

Definition 1.3.1. [54] Suppose $\mu \in \mathcal{R}$. Define

$$C_\mu = \{f \mid f(t) \text{ is a real-valued function and } f(t) = t^p g(t)$$
$$\text{with } g(t) \text{ properly smooth}, t > 0, g(0) \neq 0, p \geqslant \mu\}.$$

Denote $N_0 = N \cup \{0\}$ and suppose $m \in N_0$. We call $f \in C_\mu^m$ if $f^{(m)} \in C_\mu$.

For a reasonable large μ, when $f \in C_\mu$, the existence of the solution to problem (1.3)–(1.4) is ensured and the bigger μ is, the smoother the solution is.

1.3.2 Solutions to the FODEs in Caputo type

Problem 1.3.3. Consider the single-term FODEs in Caputo type as follows:

$$\begin{cases} {}_0^C D_t^\alpha y(t) = f(t), \quad t > 0, & (1.12) \\ y(0) = A, & (1.13) \end{cases}$$

where $0 < \alpha < 1$ and the function $f(t)$ can be expanded into the following series:

$$f(t) = t^q \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n, \quad g^{(0)}(0) = g(0) \neq 0, \tag{1.14}$$

with the convergence radius R, $R > 0$ and $q + \alpha > 0$.

Suppose that the problem (1.12)–(1.13) has the solution in the form of

$$y(t) = A + t^p \sum_{n=0}^{\infty} y_n t^n, \quad y_0 \neq 0,\ p > 0. \tag{1.15}$$

Substituting (1.15) and (1.14) into (1.12) leads to

$$\sum_{n=0}^{\infty} y_n \frac{\Gamma(1+n+p)}{\Gamma(1+n+p-\alpha)} t^{n+p-\alpha} = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^{n+q}.$$

Comparing the coefficients gives

$$p = q + \alpha, \quad y_n = \frac{\Gamma(1+n+q)}{\Gamma(1+n+q+\alpha)} \cdot \frac{g^{(n)}(0)}{n!}, \quad n = 0, 1, 2, \ldots.$$

Hence,

$$y(t) = A + t^{q+\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(1+n+q)}{\Gamma(1+n+q+\alpha)} \cdot \frac{g^{(n)}(0)}{n!} t^n. \tag{1.16}$$

If we let

$$y(t) = z(t) + A,$$

then the function $z(t)$ satisfies from (1.12)–(1.13) that

$$\begin{cases} {}_0^C D_t^\alpha z(t) = f(t), \quad t > 0, & (1.17) \\ z(0) = 0. & (1.18) \end{cases}$$

Comparing (1.12) and (1.17), one can get the same differential equation for $y(t)$ and $z(t)$.

We can conclude from (1.16) that:

1. The initial value $A = 0$ or $A \neq 0$ does not affect the smoothness of solution $y(t)$.

2. The solution $y(t)$ is smooth if $q + \alpha$ is a positive integer.
3. The solution $y(t)$ is continuous at $t = 0 + 0$ if $q + \alpha > 0$.
4. The solution $y \in C^0[0, R]$ if $q + \alpha \in (0, 1)$.
5. The solution $y \in C^1[0, R]$ if $q + \alpha \in [1, 2)$.
6. The solution $y \in C^2[0, R]$ if $q + \alpha \in [2, 3)$.
7. The solution $y \in C^3[0, R]$ if $q + \alpha \in [3, 4)$.
8. The solution $y \in C^4[0, R]$ if $q + \alpha \in [4, 5)$.

For a reasonable large μ, when $f \in C_\mu$, the existence of the solution to problem (1.12)–(1.13) is ensured and the bigger μ is, the smoother the solution $y(t)$ is.

1.4 G-L approximations of Riemann-Liouville fractional derivatives

In this section, the G-L approximation of ${}_{-\infty}\mathbf{D}_t^\alpha f(t)$ will be considered, where $0 \leqslant n-1 \leqslant \alpha < n$.

The **shifted Grünwald–Letnikov (G-L) formula** is defined as

$$A_{h,p}^\alpha f(t) = h^{-\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} f(t - (k-p)h), \tag{1.19}$$

where p is a constant, usually called the displacement,

$$g_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

When $p = 0$, (1.19) is called the **standard G-L formula**.

In fact, the term $\{g_k^{(\alpha)}\}$ here is the coefficient of power series of function $(1-z)^\alpha$, namely,

$$(1-z)^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k = \sum_{k=0}^{\infty} g_k^{(\alpha)} z^k, \quad -1 < z \leqslant 1.$$

The following recursive relation is true:

$$g_0^{(\alpha)} = 1, \quad g_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{k}\right) g_{k-1}^{(\alpha)}, \quad k = 1, 2, \ldots. \tag{1.20}$$

Several lemmas below will display some properties of the coefficient $\{g_k^{(\alpha)}\}$.

Lemma 1.4.1. *The coefficient $\{g_k^{(\alpha)}\}$ in (1.19) satisfies:*
(I) *When $\alpha = 0$,*

$$g_0^{(\alpha)} = 1, \quad g_1^{(\alpha)} = g_2^{(\alpha)} = \cdots = 0;$$

(II) *When* $0 < \alpha < 1$,

$$g_0^{(\alpha)} = 1, \quad g_1^{(\alpha)} = -\alpha, \quad g_2^{(\alpha)} < g_3^{(\alpha)} < \cdots < 0,$$
$$\sum_{k=0}^{\infty} g_k^{(\alpha)} = 0, \quad \sum_{k=0}^{m} g_k^{(\alpha)} > 0, \quad m \geqslant 1;$$

(III) *When* $\alpha = 1$,

$$g_0^{(\alpha)} = 1, \quad g_1^{(\alpha)} = -1, \quad g_2^{(\alpha)} = g_3^{(\alpha)} = \cdots = 0;$$

(IV) *When* $1 < \alpha < 2$,

$$g_0^{(\alpha)} = 1, \quad g_1^{(\alpha)} = -\alpha, \quad g_2^{(\alpha)} > g_3^{(\alpha)} > \cdots > 0,$$
$$\sum_{k=0}^{\infty} g_k^{(\alpha)} = 0, \quad \sum_{k=0}^{m} g_k^{(\alpha)} < 0, \quad m \geqslant 1;$$

(V) *When* $\alpha = 2$,

$$g_0^{(\alpha)} = 1, \quad g_1^{(\alpha)} = -2, \quad g_2^{(\alpha)} = 1, \quad g_3^{(\alpha)} = g_4^{(\alpha)} = \cdots = 0.$$

Lemma 1.4.2 (Inequalities on exponential functions).

$$\text{(I)} \quad 1 - x < \mathrm{e}^{-x}, \quad 0 < x \leqslant 1;$$
$$\text{(II)} \quad 1 - x > \mathrm{e}^{-x-x^2}, \quad 0 < x \leqslant \frac{2}{3}.$$

Lemma 1.4.3. [13] *When* $0 < \alpha < 1$, *it holds*

$$\frac{\alpha(1-\alpha)2^{\alpha}}{5\,k^{\alpha+1}} < \left|g_k^{(\alpha)}\right| \leqslant \frac{\alpha\,2^{\alpha+1}}{(k+1)^{\alpha+1}}, \quad k \geqslant 1;$$
$$\frac{1-\alpha}{5}\left(\frac{2}{k}\right)^{\alpha} < \sum_{n=k}^{\infty}\left|g_n^{(\alpha)}\right| < 2\left(\frac{2}{k}\right)^{\alpha}, \quad k \geqslant 1.$$

Proof. (I) It follows from (1.20) and Lemma 1.4.2 (I) that

$$\begin{aligned}
\left|g_k^{(\alpha)}\right| &= \left(1 - \frac{\alpha+1}{k}\right)\left|g_{k-1}^{(\alpha)}\right| \\
&< \mathrm{e}^{-\frac{\alpha+1}{k}}\left|g_{k-1}^{(\alpha)}\right| \\
&< \mathrm{e}^{-\frac{\alpha+1}{k}}\mathrm{e}^{-\frac{\alpha+1}{k-1}}\left|g_{k-2}^{(\alpha)}\right| \\
&< \cdots < \mathrm{e}^{-\frac{\alpha+1}{k}}\mathrm{e}^{-\frac{\alpha+1}{k-1}}\cdots\mathrm{e}^{-\frac{\alpha+1}{2}}\left|g_1^{(\alpha)}\right| \\
&= \alpha\mathrm{e}^{-(\alpha+1)\sum_{n=2}^{k}\frac{1}{n}}, \quad k \geqslant 2.
\end{aligned}$$

Noticing that the function $1/x$ is monotone decreasing when $x > 0$, we have

$$\sum_{n=2}^{k} \frac{1}{n} \geqslant \sum_{n=2}^{k} \int_{n}^{n+1} \frac{1}{x} \mathrm{d}x = \int_{2}^{k+1} \frac{1}{x} \mathrm{d}x = \ln\left(\frac{k+1}{2}\right),$$

thus,

$$\left|g_k^{(\alpha)}\right| < \alpha \mathrm{e}^{-(\alpha+1)\ln(\frac{k+1}{2})} = \frac{\alpha 2^{\alpha+1}}{(k+1)^{\alpha+1}}, \quad k \geqslant 2.$$

In addition, it is obvious that

$$\left|g_1^{(\alpha)}\right| = \alpha = \frac{\alpha 2^{\alpha+1}}{(1+1)^{\alpha+1}}.$$

Therefore,

$$\left|g_k^{(\alpha)}\right| \leqslant \frac{\alpha 2^{\alpha+1}}{(k+1)^{\alpha+1}}, \quad k \geqslant 1. \tag{1.21}$$

(II) With the help of (1.20) and Lemma 1.4.2 (II), we have

$$\begin{aligned}
\left|g_k^{(\alpha)}\right| &= \left(1 - \frac{\alpha+1}{k}\right)\left|g_{k-1}^{(\alpha)}\right| \\
&> \mathrm{e}^{-\frac{\alpha+1}{k} - (\frac{\alpha+1}{k})^2}\left|g_{k-1}^{(\alpha)}\right| \\
&> \mathrm{e}^{-\frac{\alpha+1}{k} - (\frac{\alpha+1}{k})^2}\mathrm{e}^{-\frac{\alpha+1}{k-1} - (\frac{\alpha+1}{k-1})^2}\left|g_{k-2}^{(\alpha)}\right| \\
&> \cdots > \mathrm{e}^{-\frac{\alpha+1}{k} - (\frac{\alpha+1}{k})^2}\mathrm{e}^{-\frac{\alpha+1}{k-1} - (\frac{\alpha+1}{k-1})^2} \cdots \mathrm{e}^{-\frac{\alpha+1}{3} - (\frac{\alpha+1}{3})^2}\left|g_2^{(\alpha)}\right| \\
&= \frac{\alpha(1-\alpha)}{2}\mathrm{e}^{-(\alpha+1)\sum_{n=3}^{k}\frac{1}{n}}\mathrm{e}^{-(\alpha+1)^2\sum_{n=3}^{k}\frac{1}{n^2}}, \quad k \geqslant 3.
\end{aligned}$$

In view of

$$\sum_{n=3}^{k} \frac{1}{n^2} \leqslant \sum_{n=3}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \left(1 + \frac{1}{4}\right) = \frac{\pi^2}{6} - \frac{5}{4},$$

it follows

$$\mathrm{e}^{-(\alpha+1)^2\sum_{n=3}^{k}\frac{1}{n^2}} \geqslant \mathrm{e}^{-(\alpha+1)^2(\frac{\pi^2}{6} - \frac{5}{4})} > \frac{1}{5}.$$

Consequently,

$$\left|g_k^{(\alpha)}\right| > \frac{\alpha(1-\alpha)}{10}\mathrm{e}^{-(\alpha+1)\sum_{n=3}^{k}\frac{1}{n}}, \quad k \geqslant 3.$$

Noticing that the function $1/x$ is monotone decreasing when $x > 0$, it holds

$$\sum_{n=3}^{k} \frac{1}{n} \leqslant \sum_{n=3}^{k} \int_{n-1}^{n} \frac{1}{x} \mathrm{d}x = \int_{2}^{k} \frac{1}{x} \mathrm{d}x = \ln \frac{k}{2}.$$

Therefore,

$$|g_k^{(\alpha)}| > \frac{\alpha(1-\alpha)}{10} \mathrm{e}^{-(\alpha+1) \ln \frac{k}{2}} = \frac{\alpha(1-\alpha)2^{\alpha}}{5k^{\alpha+1}}, \quad k \geqslant 3.$$

In addition,

$$|g_2^{(\alpha)}| = \frac{\alpha(1-\alpha)}{2} > \frac{\alpha(1-\alpha)2^{\alpha}}{5 \cdot 2^{\alpha+1}},$$

$$|g_1^{(\alpha)}| = \alpha > \alpha \cdot \frac{(1-\alpha)2^{\alpha}}{5 \cdot 1^{\alpha+1}}.$$

Hence,

$$|g_k^{(\alpha)}| > \frac{\alpha(1-\alpha)2^{\alpha}}{5k^{\alpha+1}}, \quad k \geqslant 1. \tag{1.22}$$

Combining (1.21) with (1.22) arrives at

$$\frac{\alpha(1-\alpha)2^{\alpha}}{5k^{\alpha+1}} < |g_k^{(\alpha)}| \leqslant \frac{\alpha 2^{\alpha+1}}{(k+1)^{\alpha+1}}, \quad k \geqslant 1. \tag{1.23}$$

(III) Summing up for k in (1.23) gives

$$\sum_{k=l}^{\infty} \frac{\alpha(1-\alpha)2^{\alpha}}{5k^{\alpha+1}} < \sum_{k=l}^{\infty} |g_k^{(\alpha)}| \leqslant \sum_{k=l}^{\infty} \frac{\alpha 2^{\alpha+1}}{(k+1)^{\alpha+1}}. \tag{1.24}$$

It follows from the function $1/x^{\alpha+1}$ being monotone decreasing when $x > 0$ that

$$\sum_{k=l}^{\infty} \frac{1}{(k+1)^{\alpha+1}} < \int_{l}^{\infty} \frac{1}{x^{\alpha+1}} \mathrm{d}x < \sum_{k=l}^{\infty} \frac{1}{k^{\alpha+1}},$$

namely,

$$\sum_{k=l}^{\infty} \frac{1}{(k+1)^{\alpha+1}} < \frac{1}{\alpha l^{\alpha}} < \sum_{k=l}^{\infty} \frac{1}{k^{\alpha+1}}.$$

From (1.24) and the inequality above, it is easy to get

$$\frac{1-\alpha}{5} \left(\frac{2}{l}\right)^{\alpha} < \sum_{k=l}^{\infty} |g_k^{(\alpha)}| < 2\left(\frac{2}{l}\right)^{\alpha}.$$

The proof ends. □

Lemma 1.4.4. *When* $1 < \alpha < 2$, *it holds*

$$\frac{\alpha(\alpha-1)(2-\alpha)(3-\alpha)}{180}\left(\frac{4}{k}\right)^{\alpha+1} < g_k^{(\alpha)} \leqslant \frac{\alpha(\alpha-1)}{2}\left(\frac{3}{k+1}\right)^{\alpha+1}, \quad k \geqslant 2;$$

$$\sum_{n=k}^{\infty} g_n^{(\alpha)} > \frac{(\alpha-1)(2-\alpha)(3-\alpha)}{45}\left(\frac{4}{k}\right)^{\alpha}, \quad k \geqslant 2.$$

Proof. (I) By (1.20) and Lemma 1.4.2 (I), we have

$$\begin{aligned} g_k^{(\alpha)} &= \left(1 - \frac{\alpha+1}{k}\right) g_{k-1}^{(\alpha)} \\ &< \mathrm{e}^{-\frac{\alpha+1}{k}} g_{k-1}^{(\alpha)} \\ &< \mathrm{e}^{-\frac{\alpha+1}{k}} \mathrm{e}^{-\frac{\alpha+1}{k-1}} g_{k-2}^{(\alpha)} \\ &< \cdots < \mathrm{e}^{-\frac{\alpha+1}{k}} \mathrm{e}^{-\frac{\alpha+1}{k-1}} \cdots \mathrm{e}^{-\frac{\alpha+1}{3}} g_2^{(\alpha)} \\ &= \frac{\alpha(\alpha-1)}{2} \mathrm{e}^{-(\alpha+1)\sum_{n=3}^{k}\frac{1}{n}}, \quad k \geqslant 3. \end{aligned}$$

Noticing

$$\sum_{n=3}^{k} \frac{1}{n} \geqslant \sum_{n=3}^{k} \int_{n}^{n+1} \frac{1}{x}\mathrm{d}x = \int_{3}^{k+1} \frac{1}{x}\mathrm{d}x = \ln\left(\frac{k+1}{3}\right),$$

it follows

$$g_k^{(\alpha)} < \frac{\alpha(\alpha-1)}{2}\left(\frac{3}{k+1}\right)^{\alpha+1}, \quad k \geqslant 3.$$

It is noted that when $k = 2$, it becomes

$$g_2^{(\alpha)} = \frac{\alpha(\alpha-1)}{2}\left(\frac{3}{2+1}\right)^{\alpha+1}.$$

(II) By (1.20) and Lemma 1.4.2 (II), we have

$$\begin{aligned} g_k^{(\alpha)} &= \left(1 - \frac{\alpha+1}{k}\right) g_{k-1}^{(\alpha)} \\ &> \mathrm{e}^{-\frac{\alpha+1}{k} - (\frac{\alpha+1}{k})^2} g_{k-1}^{(\alpha)} \\ &> \mathrm{e}^{-\frac{\alpha+1}{k} - (\frac{\alpha+1}{k})^2} \mathrm{e}^{-\frac{\alpha+1}{k-1} - (\frac{\alpha+1}{k-1})^2} g_{k-2}^{(\alpha)} \\ &> \cdots > \mathrm{e}^{-\frac{\alpha+1}{k} - (\frac{\alpha+1}{k})^2} \mathrm{e}^{-\frac{\alpha+1}{k-1} - (\frac{\alpha+1}{k-1})^2} \cdots \mathrm{e}^{-\frac{\alpha+1}{5} - (\frac{\alpha+1}{5})^2} g_4^{(\alpha)} \\ &= \frac{\alpha(\alpha-1)(2-\alpha)(3-\alpha)}{24} \mathrm{e}^{-(\alpha+1)\sum_{n=5}^{k}\frac{1}{n}} \mathrm{e}^{-(\alpha+1)^2\sum_{n=5}^{k}\frac{1}{n^2}}, \quad k \geqslant 5. \end{aligned} \tag{1.25}$$

Noticing, when $k \geqslant 5$, it holds

$$e^{-(\alpha+1)\sum_{n=5}^{k}\frac{1}{n}} \geqslant \left(\frac{4}{k}\right)^{\alpha+1} \tag{1.26}$$

due to

$$\sum_{n=5}^{k}\frac{1}{n} \leqslant \sum_{n=5}^{k}\int_{n-1}^{n}\frac{1}{x}\mathrm{d}x = \int_{4}^{k}\frac{1}{x}\mathrm{d}x = \ln\frac{k}{4}.$$

Similarly, when $k \geqslant 5$, it holds that

$$e^{-(\alpha+1)^2\sum_{n=5}^{k}\frac{1}{n^2}} \geqslant e^{-9(\frac{\pi^2}{6}-\frac{205}{144})} > \frac{2}{15} \tag{1.27}$$

because of

$$\sum_{n=5}^{k}\frac{1}{n^2} \leqslant \sum_{n=5}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}\right) = \frac{\pi^2}{6} - \frac{205}{144}.$$

The substitution of (1.26) and (1.27) into (1.25) gives

$$g_k^{(\alpha)} > \frac{\alpha(\alpha-1)(2-\alpha)(3-\alpha)}{180}\left(\frac{4}{k}\right)^{\alpha+1}, \quad k \geqslant 5.$$

It is not hard to verify that the inequality above is also true for $k = 4, 3, 2$.

(III) When $k \geqslant 2$, we have

$$\begin{aligned}
\sum_{n=k}^{\infty} g_n^{(\alpha)} &> \sum_{n=k}^{\infty}\frac{\alpha(\alpha-1)(2-\alpha)(3-\alpha)}{180}\left(\frac{4}{n}\right)^{\alpha+1} \\
&= \frac{\alpha(\alpha-1)(2-\alpha)(3-\alpha)}{45}4^{\alpha}\sum_{n=k}^{\infty}\left(\frac{1}{n}\right)^{\alpha+1} \\
&\geqslant \frac{\alpha(\alpha-1)(2-\alpha)(3-\alpha)}{45}4^{\alpha}\sum_{n=k}^{\infty}\int_{n}^{n+1}\frac{1}{x^{\alpha+1}}\mathrm{d}x \\
&= \frac{\alpha(\alpha-1)(2-\alpha)(3-\alpha)}{45}4^{\alpha}\int_{k}^{\infty}\frac{1}{x^{\alpha+1}}\mathrm{d}x \\
&= \frac{(\alpha-1)(2-\alpha)(3-\alpha)}{45}\left(\frac{4}{k}\right)^{\alpha}.
\end{aligned}$$

The proof ends. □

Define

$$\mathscr{C}^{n+\alpha}(\mathcal{R}) = \left\{f \mid f \in L^1(\mathcal{R}), \int_{-\infty}^{\infty}(1+|\omega|)^{n+\alpha}\big|F(\omega)\big|\mathrm{d}\omega < \infty\right\},$$

where $F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t$ is the Fourier transform of function $f(t)$.

Tuan and Gorenflo derived the asymptotic expansion of the standard Grünwald formula in [89]. As well, Tadjeran et al. established the asymptotic expansion of the shifted Grünwald formula in [86] (For the result of $n = 1$, see also [58]).

Theorem 1.4.1. *Suppose $f \in \mathscr{C}^{n+\alpha}(\mathcal{R})$, then it holds*

$$A_{h,p}^{\alpha} f(t) = {}_{-\infty}\mathbf{D}_t^{\alpha} f(t) + \sum_{l=1}^{n-1} c_l^{(\alpha,p)} {}_{-\infty}\mathbf{D}_t^{\alpha+l} f(t) h^l + O(h^n)$$

uniformly for $t \in \mathcal{R}$, where $\{c_l^{(\alpha,p)}\}$ is the coefficient of power series of function $W_{\alpha,p}(z) = (\frac{1-\mathrm{e}^{-z}}{z})^{\alpha}\mathrm{e}^{pz}$, namely,

$$W_{\alpha,p}(z) = \sum_{l=0}^{\infty} c_l^{(\alpha,p)} z^l = c_0^{(\alpha,p)} + c_1^{\alpha,p} z + c_2^{(\alpha,p)} z^2 + c_3^{(\alpha,p)} z^3 + O(|z|^4),$$

in particular,

$$c_0^{(\alpha,p)} = 1, \quad c_1^{(\alpha,p)} = p - \frac{\alpha}{2}, \quad c_2^{(\alpha,p)} = \frac{p^2}{2} - \frac{\alpha p}{2} + \frac{\alpha(3\alpha+1)}{24},$$
$$c_3^{(\alpha,p)} = \frac{p^3}{6} - \frac{\alpha p^2}{4} + \frac{\alpha(3\alpha+1)p}{24} - \frac{\alpha^2(\alpha+1)}{48}.$$

Proof. Let

$$\mathcal{F}[f(t);\omega] = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \equiv F(\omega).$$

According to

$$\mathcal{F}[f(t-h);\omega] = \mathrm{e}^{\mathrm{i}\omega h} F(\omega),$$

we have

$$\begin{aligned} &\mathcal{F}[A_{h,p}^{\alpha} f(t);\omega] \\ &= h^{-\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} \mathcal{F}[f(t-(k-p)h);\omega] \\ &= h^{-\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} \mathrm{e}^{\mathrm{i}\omega(k-p)h} F(\omega) \\ &= h^{-\alpha} \left[\sum_{k=0}^{\infty} g_k^{(\alpha)} \mathrm{e}^{\mathrm{i}k\omega h}\right] \mathrm{e}^{-\mathrm{i}p\omega h} F(\omega) \\ &= h^{-\alpha} (1-\mathrm{e}^{\mathrm{i}\omega h})^{\alpha} \mathrm{e}^{-\mathrm{i}p\omega h} F(\omega) \end{aligned}$$

$$= (-\mathrm{i}\omega)^\alpha\left(\frac{1-\mathrm{e}^{\mathrm{i}\omega h}}{-\mathrm{i}\omega h}\right)^\alpha \mathrm{e}^{-\mathrm{i}p\omega h}F(\omega)$$
$$= (-\mathrm{i}\omega)^\alpha W_{\alpha,p}(-\mathrm{i}\omega h)F(\omega). \tag{1.28}$$

Taking into account that the function $W_{\alpha,p}(z)$ is analytical in a neighborhood of the origin, there exists a positive constant R such that

$$W_{\alpha,p}(z) = \sum_{l=0}^{\infty} c_l^{(\alpha,p)} z^l, \quad \text{for all } |z| \leqslant R.$$

Next, we aim to show that there is a constant c_1 such that

$$\left|W_{\alpha,p}(-\mathrm{i}x) - \sum_{l=0}^{n-1} c_l^{(\alpha,p)}(-\mathrm{i}x)^l\right| \leqslant c_1|x|^n \tag{1.29}$$

uniformly holds for $x \in \mathcal{R}$.

When $|x| \leqslant R$,

$$\left|W_{\alpha,p}(-\mathrm{i}x) - \sum_{l=0}^{n-1} c_l^{(\alpha,p)}(-\mathrm{i}x)^l\right| = \left|\sum_{l=n}^{\infty} c_l^{(\alpha,p)}(-\mathrm{i}x)^l\right|$$
$$\leqslant |x|^n \sum_{l=n}^{\infty}\left|c_l^{(\alpha,p)}\right| |x|^{l-n} \leqslant c_2|x|^n,$$

where $c_2 = R^{-n}\sum_{l=n}^{\infty}|c_l^{(\alpha,p)}|R^l < \infty$.

When $|x| > R$, on the one hand,

$$\left|W_{\alpha,p}(-\mathrm{i}x)\right| = \left|\left(\frac{1-\mathrm{e}^{\mathrm{i}x}}{-\mathrm{i}x}\right)^\alpha \mathrm{e}^{-\mathrm{i}px}\right| \leqslant \frac{2^\alpha}{R^\alpha} \leqslant c_3|x|^n,$$

where $c_3 = \frac{2^\alpha}{R^{\alpha+n}} < \infty$; On the other hand,

$$\left|\sum_{l=0}^{n-1} c_l^{(\alpha,p)}(-\mathrm{i}x)^l\right| \leqslant |x|^n \sum_{l=0}^{n-1}\left|c_l^{(\alpha,p)}\right| \cdot |x|^{l-n} \leqslant c_4|x|^n,$$

where $c_4 = \sum_{l=0}^{n-1}|c_l^{(\alpha,p)}|R^{l-n} < \infty$.

Let $c_1 = \max\{c_2, c_3 + c_4\}$. It is apparent that (1.29) is uniformly true for $x \in \mathcal{R}$.

It follows from (1.28) that

$$\mathcal{F}[A_{h,p}^\alpha f(t);\omega] = \sum_{l=0}^{n-1} c_l^{(\alpha,p)}(-\mathrm{i}\omega)^{\alpha+l}h^l F(\omega) + \Phi(\omega,h)$$
$$= \sum_{l=0}^{n-1} c_l^{(\alpha,p)}\mathcal{F}[{}_{-\infty}\mathbf{D}_t^{\alpha+l}f(t);\omega]h^l + \Phi(\omega,h), \tag{1.30}$$

where

$$\Phi(\omega, h) = (-\mathrm{i}\omega)^{\alpha}\left[W_{\alpha,p}(-\mathrm{i}\omega h) - \sum_{l=0}^{n-1} c_l^{(\alpha,p)}(-\mathrm{i}\omega h)^l\right]F(\omega).$$

Taking the inverse Fourier transform on both hand sides of (1.30) produces

$$A_{h,p}^{\alpha}f(t) - \sum_{l=0}^{n-1} c_l^{(\alpha,p)}\,{}_{-\infty}\mathbf{D}_t^{\alpha+l}f(t)h^l = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega, h)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}\omega.$$

Combining with (1.29) and noticing $f \in \mathscr{C}^{n+\alpha}(\mathcal{R})$, we can get

$$\begin{aligned}
&\left|A_{h,p}^{\alpha}f(t) - \sum_{l=0}^{n-1} c_l^{(\alpha,p)}\,{}_{-\infty}\mathbf{D}_t^{\alpha+l}f(t)h^l\right| \\
&\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\Phi(\omega, h)\right|\mathrm{d}\omega \\
&\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} c_1|\omega|^{\alpha}|\omega h|^{n}\left|F(\omega)\right|\mathrm{d}\omega \\
&\leqslant \frac{c_1}{2\pi}h^n\int_{-\infty}^{\infty}(1 + |\omega|)^{n+\alpha}\left|F(\omega)\right|\mathrm{d}\omega \\
&\leqslant ch^n.
\end{aligned}$$

The proof ends. □

In what follows, some common approximations will be stated in detail.

The first-order approximation [88]

It is easy to see from Theorem 1.4.1 that

Theorem 1.4.2. *Suppose $f \in \mathscr{C}^{1+\alpha}(\mathcal{R})$, then it holds*

$$A_{h,p}^{\alpha}f(t) = {}_{-\infty}\mathbf{D}_t^{\alpha}f(t) + O(h)$$

uniformly for $t \in \mathcal{R}$.

The second-order approximation [88]

Theorem 1.4.3. *Suppose $f \in \mathscr{C}^{2+\alpha}(\mathcal{R})$ and $p \neq q$, then it holds*

$$\lambda_1 A_{h,p}^{\alpha}f(t) + \lambda_2 A_{h,q}^{\alpha}f(t) = {}_{-\infty}\mathbf{D}_t^{\alpha}f(t) + O(h^2) \tag{1.31}$$

uniformly for $t \in \mathcal{R}$, where

$$\lambda_1 = \frac{\alpha - 2q}{2(p-q)}, \quad \lambda_2 = \frac{2p - \alpha}{2(p-q)}.$$

Proof. It follows from Theorem 1.4.1 that

$$\begin{aligned}&\lambda_1 A_{h,p}^{\alpha} f(t) + \lambda_2 A_{h,q}^{\alpha} f(t)\\ =\ & (\lambda_1 + \lambda_2)\, {}_{-\infty}\mathbf{D}_t^{\alpha} f(t) + \big(\lambda_1 c_1^{(\alpha,p)} + \lambda_2 c_1^{(\alpha,q)}\big)\, {}_{-\infty}\mathbf{D}_t^{\alpha+1} f(t) h + O(h^2)\end{aligned}$$

uniformly holds for $t \in \mathcal{R}$. Let

$$\begin{cases}\lambda_1 + \lambda_2 = 1,\\ \lambda_1 c_1^{(\alpha,p)} + \lambda_2 c_1^{(\alpha,q)} = 0,\end{cases}$$

which implies

$$\lambda_1 = \frac{\alpha - 2q}{2(p-q)}, \quad \lambda_2 = \frac{2p - \alpha}{2(p-q)}$$

in view of

$$c_1^{(\alpha,p)} = p - \frac{\alpha}{2}, \quad c_1^{(\alpha,q)} = q - \frac{\alpha}{2}.$$

The proof ends. □

We call the left-hand side of (1.31) the **weighted and shifted G-L (WSGL) formula**.

Some common applications of Theorem 1.4.3 can be concluded as follows.

Corollary 1.4.1. *When $\alpha \in (0,1)$, taking $(p,q) = (0,-1)$, then $\lambda_1 = 1 + \frac{\alpha}{2}$, $\lambda_2 = -\frac{\alpha}{2}$.This result is often used to solve the* ***time-fractional*** *PDEs* [96]*. At this point,*

$$\begin{aligned}&\lambda_1 A_{h,0}^{\alpha} f(t) + \lambda_2 A_{h,-1}^{\alpha} f(t)\\ &= \left(1 + \frac{\alpha}{2}\right) h^{-\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} f(t - kh) + \left(-\frac{\alpha}{2}\right) h^{-\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} f(t - (k+1)h)\\ &= h^{-\alpha} \sum_{k=0}^{\infty} w_k^{(\alpha)} f(t - kh)\\ &= {}_{-\infty}\mathbf{D}_t^{\alpha} f(t) + O(h^2)\end{aligned} \tag{1.32}$$

uniformly holds for $t \in \mathcal{R}$, where

$$\begin{cases} w_0^{(\alpha)} = \left(1 + \frac{\alpha}{2}\right) g_0^{(\alpha)} = 1 + \frac{\alpha}{2}, & (1.33)\\ w_k^{(\alpha)} = \left(1 + \frac{\alpha}{2}\right) g_k^{(\alpha)} - \frac{\alpha}{2} g_{k-1}^{(\alpha)} & \\ \quad\ = \left[\left(1 + \frac{\alpha}{2}\right)\left(1 - \frac{\alpha+1}{k}\right) - \frac{\alpha}{2}\right] g_{k-1}^{(\alpha)}, \quad k \geqslant 1. & (1.34)\end{cases}$$

It is easy to verify that

$$\begin{cases} w_0^{(\alpha)} = 1 + \frac{\alpha}{2} > 0, \quad w_1^{(\alpha)} = -\frac{3\alpha + \alpha^2}{2} < 0, \quad w_2^{(\alpha)} = \frac{\alpha(\alpha^2 + 3\alpha - 2)}{4}, \\ w_1^{(\alpha)} \leqslant w_3^{(\alpha)} \leqslant w_4^{(\alpha)} \leqslant \cdots \leqslant 0, \quad w_0^{(\alpha)} + w_2^{(\alpha)} > 0, \\ \sum_{k=0}^{\infty} w_k^{(\alpha)} = 0, \quad \sum_{k=0}^{m} w_k^{(\alpha)} > 0, \quad m \geqslant 2. \end{cases}$$

Corollary 1.4.2. *When $\alpha \in (1,2)$, taking $(p,q) = (1,0)$, then $\lambda_1 = \frac{\alpha}{2}$, $\lambda_2 = 1 - \frac{\alpha}{2}$. This result is often used to solve the **space-fractional** PDEs [88]. At this point,*

$$\begin{aligned} &\lambda_1 A_{h,1}^{\alpha} f(t) + \lambda_2 A_{h,0}^{\alpha} f(t) \\ &= \frac{\alpha}{2} h^{-\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} f(t - (k-1)h) + \left(1 - \frac{\alpha}{2}\right) h^{-\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} f(t - kh) \\ &= h^{-\alpha} \sum_{k=0}^{\infty} \widetilde{w}_k^{(\alpha)} f(t - (k-1)h) \\ &= {}_{-\infty}\mathbf{D}_t^{\alpha} f(t) + O(h^2) \end{aligned}$$

uniformly holds for $t \in \mathcal{R}$, where

$$\widetilde{w}_0^{(\alpha)} = \frac{\alpha}{2} g_0^{(\alpha)}, \quad \widetilde{w}_k^{(\alpha)} = \frac{\alpha}{2} g_k^{(\alpha)} + \left(1 - \frac{\alpha}{2}\right) g_{k-1}^{(\alpha)}, \quad k \geqslant 1. \tag{1.35}$$

It is easy to show that

$$\begin{cases} \widetilde{w}_0^{(\alpha)} = \frac{\alpha}{2} > 0, \quad \widetilde{w}_1^{(\alpha)} = \frac{2 - \alpha - \alpha^2}{2} < 0, \quad \widetilde{w}_2^{(\alpha)} = \frac{\alpha(\alpha^2 + \alpha - 4)}{4}, \\ 1 \geqslant \widetilde{w}_0^{(\alpha)} \geqslant \widetilde{w}_3^{(\alpha)} \geqslant \widetilde{w}_4^{(\alpha)} \geqslant \cdots \geqslant 0, \quad \widetilde{w}_0^{(\alpha)} + \widetilde{w}_2^{(\alpha)} > 0, \\ \sum_{k=0}^{\infty} \widetilde{w}_k^{(\alpha)} = 0, \quad \sum_{k=0}^{m} \widetilde{w}_k^{(\alpha)} < 0, \quad m \geqslant 2. \end{cases} \tag{1.36}$$

The third-order approximation [117]

Theorem 1.4.4. *Suppose $f \in \mathscr{C}^{3+\alpha}(\mathcal{R})$ and the constants p, q and r are distinct, then it holds*

$$\lambda_1 A_{h,p}^{\alpha} f(t) + \lambda_2 A_{h,q}^{\alpha} f(t) + \lambda_3 A_{h,r}^{\alpha} f(t) = {}_{-\infty}\mathbf{D}_t^{\alpha} f(t) + O(h^3) \tag{1.37}$$

uniformly for $t \in \mathcal{R}$, where

$$\lambda_1 = \frac{12qr - (6q + 6r + 1)\alpha + 3\alpha^2}{12(qr - pq - pr + p^2)},$$

$$\lambda_2 = \frac{12pr - (6p + 6r + 1)\alpha + 3\alpha^2}{12(pr - pq - qr + q^2)},$$
$$\lambda_3 = \frac{12pq - (6p + 6q + 1)\alpha + 3\alpha^2}{12(pq - pr - qr + r^2)}.$$

Proof. It follows from Theorem 1.4.1 that

$$\begin{aligned}
&\lambda_1 A^{\alpha}_{h,p} f(t) + \lambda_2 A^{\alpha}_{h,q} f(t) + \lambda_3 A^{\alpha}_{h,r} f(t) \\
&= (\lambda_1 + \lambda_2 + \lambda_3)\,{}_{-\infty}\mathbf{D}^{\alpha}_t f(t) + \big(\lambda_1 c_1^{(\alpha,p)} + \lambda_2 c_1^{(\alpha,q)} + \lambda_3 c_1^{(\alpha,r)}\big)\,{}_{-\infty}\mathbf{D}^{\alpha+1}_t f(t) h \\
&\quad + \big(\lambda_1 c_2^{(\alpha,p)} + \lambda_2 c_2^{(\alpha,q)} + \lambda_3 c_2^{(\alpha,r)}\big)\,{}_{-\infty}\mathbf{D}^{\alpha+2}_t f(t) h^2 + O(h^3)
\end{aligned}$$

uniformly holds for $t \in \mathcal{R}$. Let

$$\begin{cases}
\lambda_1 + \lambda_2 + \lambda_3 = 1, \\
\lambda_1 c_1^{(\alpha,p)} + \lambda_2 c_1^{(\alpha,q)} + \lambda_3 c_1^{(\alpha,r)} = 0, \\
\lambda_1 c_2^{(\alpha,p)} + \lambda_2 c_2^{(\alpha,q)} + \lambda_3 c_2^{(\alpha,r)} = 0,
\end{cases}$$

which implies

$$\lambda_1 = \frac{12qr - (6q + 6r + 1)\alpha + 3\alpha^2}{12(qr - pq - pr + p^2)},$$
$$\lambda_2 = \frac{12pr - (6p + 6r + 1)\alpha + 3\alpha^2}{12(pr - pq - qr + q^2)},$$
$$\lambda_3 = \frac{12pq - (6p + 6q + 1)\alpha + 3\alpha^2}{12(pq - pr - qr + r^2)}.$$

The proof ends. □

We also call the left-hand side of (1.37) the **weighted and shifted G-L (WSGL) formula**.

When $\alpha \in (0, 1)$, taking $(p, q, r) = (0, -1, -2)$, then

$$\lambda_1 = \frac{24 + 17\alpha + 3\alpha^2}{24}, \quad \lambda_2 = -\frac{11\alpha + 3\alpha^2}{12}, \quad \lambda_3 = \frac{5\alpha + 3\alpha^2}{24}.$$

This result is often used to solve the **time**-fractional differential equations[40].

When $\alpha \in (1, 2)$, taking $(p, q, r) = (1, 0, -1)$, then

$$\lambda_1 = \frac{5\alpha + 3\alpha^2}{24}, \quad \lambda_2 = \frac{12 + \alpha - 3\alpha^2}{12}, \quad \lambda_3 = \frac{-7\alpha + 3\alpha^2}{24}.$$

This result is often used to solve the **space**-fractional differential equations[88].

The fourth-order approximation [35]

Theorem 1.4.5. *Suppose* $f \in \mathscr{C}^{4+\alpha}(\mathcal{R})$ *and denote*

$$\delta_t^\alpha f(t) = \lambda_1 A_{h,1}^\alpha f(t) + \lambda_0 A_{h,0}^\alpha f(t) + \lambda_{-1} A_{h,-1}^\alpha f(t), \tag{1.38}$$

then it holds

$$\delta_t^\alpha f(t) = {}_{-\infty}\mathbf{D}_t^\alpha f(t) + c_2^\alpha\, {}_{-\infty}\mathbf{D}_t^{\alpha+2} f(t) h^2 + O(h^4) \tag{1.39}$$

uniformly for $t \in \mathcal{R}$, *where*

$$\lambda_1 = \frac{\alpha^2 + 3\alpha + 2}{12}, \quad \lambda_0 = \frac{4-\alpha^2}{6}, \quad \lambda_{-1} = \frac{\alpha^2 - 3\alpha + 2}{12} \tag{1.40}$$

and

$$c_2^\alpha = \lambda_1 c_2^{(\alpha,1)} + \lambda_0 c_2^{(\alpha,0)} + \lambda_{-1} c_2^{(\alpha,-1)} = \frac{-\alpha^2 + \alpha + 4}{24}. \tag{1.41}$$

Moreover,

$$\begin{aligned} \delta_t^\alpha f(t) = {} & c_2^\alpha\, {}_{-\infty}\mathbf{D}_t^\alpha f(t-h) + (1 - 2c_2^\alpha)\, {}_{-\infty}\mathbf{D}_t^\alpha f(t) \\ & + c_2^\alpha\, {}_{-\infty}\mathbf{D}_t^\alpha f(t+h) + O(h^4) \end{aligned} \tag{1.42}$$

uniformly holds for $t \in \mathcal{R}$.

Proof. It follows from Theorem 1.4.1 that

$$\begin{aligned} A_{h,p}^\alpha f(t) = {} & {}_{-\infty}\mathbf{D}_t^\alpha f(t) + c_1^{(\alpha,p)}\, {}_{-\infty}\mathbf{D}_t^{\alpha+1} f(t) h \\ & + c_2^{(\alpha,p)}\, {}_{-\infty}\mathbf{D}_t^{\alpha+2} f(t) h^2 + c_3^{(\alpha,p)}\, {}_{-\infty}\mathbf{D}_t^{\alpha+3} f(t) h^3 + O(h^4). \end{aligned}$$

Taking $p = 1, 0$ and -1 in the equality above, respectively, and then taking the weighted sum of the results arrive at

$$\begin{aligned} \delta_t^\alpha f(t) = {} & (\lambda_1 + \lambda_0 + \lambda_{-1})\, {}_{-\infty}\mathbf{D}_t^\alpha f(t) \\ & + (c_1^{(\alpha,1)} \lambda_1 + c_1^{(\alpha,0)} \lambda_0 + c_1^{(\alpha,-1)} \lambda_{-1})\, {}_{-\infty}\mathbf{D}_t^{\alpha+1} f(t) h + c_2^\alpha\, {}_{-\infty}\mathbf{D}_t^{\alpha+2} f(t) h^2 \\ & + (c_3^{(\alpha,1)} \lambda_1 + c_3^{(\alpha,0)} \lambda_0 + c_3^{(\alpha,-1)} \lambda_{-1})\, {}_{-\infty}\mathbf{D}_t^{\alpha+3} f(t) h^3 + O(h^4). \end{aligned}$$

Let

$$\begin{cases} \lambda_1 + \lambda_0 + \lambda_{-1} = 1, \\ c_1^{(\alpha,1)} \lambda_1 + c_1^{(\alpha,0)} \lambda_0 + c_1^{(\alpha,-1)} \lambda_{-1} = 0, \\ c_3^{(\alpha,1)} \lambda_1 + c_3^{(\alpha,0)} \lambda_0 + c_3^{(\alpha,-1)} \lambda_{-1} = 0, \end{cases}$$

which can be solved to give (1.40). Furthermore, (1.41) and (1.39) are followed.

On the other hand, noticing that the R-L operator ${}_{-\infty}\mathbf{D}_t^{\alpha+2}$ can be written as ${}_{-\infty}\mathbf{D}_t^{\alpha+2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}({}_{-\infty}\mathbf{D}_t^{\alpha})$, it follows

$$\delta_t^{\alpha} f(t) = \left(\mathcal{I} + c_2^{\alpha} h^2 \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) {}_{-\infty}\mathbf{D}_t^{\alpha} f(t) + O(h^4), \tag{1.43}$$

where $\mathcal{I}$ is the identity operator.

In view of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} v(t) = \frac{1}{h^2}\left[v(t+h) - 2v(t) + v(t-h)\right] + O(h^2),$$

we have

$$\begin{aligned}
&\left(\mathcal{I} + c_2^{\alpha} h^2 \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) {}_{-\infty}\mathbf{D}_t^{\alpha} f(t) \\
&= {}_{-\infty}\mathbf{D}_t^{\alpha} f(t) + c_2^{\alpha} h^2 \Bigg[\frac{1}{h^2}\big({}_{-\infty}\mathbf{D}_t^{\alpha} f(t-h) - 2\,{}_{-\infty}\mathbf{D}_t^{\alpha} f(t) \\
&\quad + {}_{-\infty}\mathbf{D}_t^{\alpha} f(t+h)\big) + O(h^2)\Bigg] \\
&= c_2^{\alpha}\ {}_{-\infty}\mathbf{D}_t^{\alpha} f(t-h) + \left(1 - 2c_2^{\alpha}\right)\ {}_{-\infty}\mathbf{D}_t^{\alpha} f(t) \\
&\quad + c_2^{\alpha}\ {}_{-\infty}\mathbf{D}_t^{\alpha} f(t+h) + O(h^4).
\end{aligned} \tag{1.44}$$

Then (1.42) is followed from (1.43) and (1.44). The proof ends. □

Substituting (1.40) into (1.38) leads to

$$\begin{aligned}
\delta_t^{\alpha} f(t) &= \frac{\alpha^2 + 3\alpha + 2}{12} A_{h,1}^{\alpha} f(t) + \frac{4 - \alpha^2}{6} A_{h,0}^{\alpha} f(t) + \frac{\alpha^2 - 3\alpha + 2}{12} A_{h,-1}^{\alpha} f(t) \\
&= \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} \hat{w}_k^{(\alpha)} f(t - (k-1)h),
\end{aligned}$$

where

$$\begin{cases}
\hat{w}_0^{(\alpha)} = \dfrac{\alpha^2 + 3\alpha + 2}{12} g_0^{(\alpha)}, \quad \hat{w}_1^{(\alpha)} = \dfrac{\alpha^2 + 3\alpha + 2}{12} g_1^{(\alpha)} + \dfrac{4 - \alpha^2}{6} g_0^{(\alpha)}, \\
\hat{w}_k^{(\alpha)} = \dfrac{\alpha^2 + 3\alpha + 2}{12} g_k^{(\alpha)} + \dfrac{4 - \alpha^2}{6} g_{k-1}^{(\alpha)} + \dfrac{\alpha^2 - 3\alpha + 2}{12} g_{k-2}^{(\alpha)}, \quad k \geqslant 2.
\end{cases} \tag{1.45}$$

It is easy to verify when $\alpha \in [1, 2]$ that

$$\begin{cases}
\hat{w}_0^{(\alpha)} > 0, \quad \hat{w}_1^{(\alpha)} \leqslant 0, \quad \hat{w}_k^{(\alpha)} \geqslant 0, \quad k \geqslant 3, \\
\displaystyle\sum_{k=0}^{\infty} \hat{w}_k^{(\alpha)} = 0, \quad \sum_{k=0}^{m} \hat{w}_k^{(\alpha)} \leqslant 0, \quad m \geqslant 2, \\
\hat{w}_0^{(\alpha)} + \hat{w}_2^{(\alpha)} \geqslant 0.
\end{cases} \tag{1.46}$$

It is worth to mention that all results in this section are the G-L approximation of left R-L fractional derivative and the corresponding numerical differentiation formulae are called the left G-L formulae; similarly, the results for the right R-L fractional derivative can be obtained and the corresponding numerical differentiation formulae are called the right G-L formulae.

1.5 Central difference quotient approximations of Riesz fractional derivatives

The Riesz fractional derivative

$$\frac{\partial^\alpha f(x)}{\partial |x|^\alpha} = -\Psi_\alpha \big({}_{-\infty}\mathbf{D}_x^\alpha f(x) + {}_x\mathbf{D}_{+\infty}^\alpha f(x) \big), \quad \Psi_\alpha = \frac{1}{2\cos(\frac{\alpha\pi}{2})}$$

is a weighted sum of the left R-L fractional derivative ${}_{-\infty}\mathbf{D}_x^\alpha f(x)$ and the right R-L fractional derivative ${}_x\mathbf{D}_{+\infty}^\alpha f(x)$.

Denote $x_i = ih$, $i = 0, \pm 1, \pm 2, \ldots$. When $\alpha \in [1, 2]$, by the left G-L formula and the right one, it follows from Theorem 1.4.1 that

$$\frac{\partial^\alpha f(x)}{\partial |x|^\alpha}\bigg|_{x=x_i} = -\frac{\Psi_\alpha}{h^\alpha} \left[\sum_{k=0}^{\infty} g_k^{(\alpha)} f(x_{i-k+1}) + \sum_{k=0}^{\infty} g_k^{(\alpha)} f(x_{i+k-1}) \right] + O(h). \tag{1.47}$$

Applying the left weighted and shifted G-L formula and the right weighted and shifted G-L formula, we have

$$\frac{\partial^\alpha f(x)}{\partial |x|^\alpha}\bigg|_{x=x_i} = -\frac{\Psi_\alpha}{h^\alpha} \left[\sum_{k=0}^{\infty} \widetilde{w}_k^{(\alpha)} f(x_{i-k+1}) + \sum_{k=0}^{\infty} \widetilde{w}_k^{(\alpha)} f(x_{i+k-1}) \right] + O(h^2),$$

where $\{\widetilde{w}_k^{(\alpha)}\}$ is defined by (1.35).

Ortigueira[62] introduced a fractional central difference operator as follows:

$$\Delta_h^\alpha f(x) \equiv \sum_{k=-\infty}^{\infty} \hat{g}_k^{(\alpha)} f(x - kh),$$

where

$$\hat{g}_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\alpha/2 - k + 1)\Gamma(\alpha/2 + k + 1)}. \tag{1.48}$$

As that pointed out in [62], the coefficient $\{\hat{g}_k^{(\alpha)}\}$ satisfies

$$\left| 2\sin\left(\frac{x}{2}\right)\right|^\alpha = \sum_{k=-\infty}^{\infty} \hat{g}_k^{(\alpha)} e^{ikx}, \quad x \in \mathcal{R}.$$

When $\alpha > -1$, the following recursive relations are true for $\{\hat{g}_k^{(\alpha)}\}$:

$$\hat{g}_0^{(\alpha)} = \frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha/2+1)}, \quad \hat{g}_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{\alpha/2+k}\right)\hat{g}_{k-1}^{(\alpha)}, \quad k \geqslant 1; \tag{1.49}$$

$$g_{-k}^{(\alpha)} = g_k^{(\alpha)}, \quad k \geqslant 1. \tag{1.50}$$

The following lemma has been proved by Çelik and Duman in [4].

Lemma 1.5.1. [4] *Suppose $f \in C^5(\mathcal{R})$ and its all derivatives of order up to 5 belong to $L^1(\mathcal{R})$. Then it holds*

$$-\frac{\Delta_h^\alpha f(x)}{h^\alpha} = \frac{\partial^\alpha f(x)}{\partial |x|^\alpha} + O(h^2).$$

It is easy to see from the lemma above that

$$\lim_{h\to 0}\left[-\frac{\Delta_h^\alpha f(x)}{h^\alpha}\right] = \frac{\partial^\alpha f(x)}{\partial |x|^\alpha}.$$

Moreover, we have the following conclusion.

Theorem 1.5.1. *Suppose $f \in \mathscr{C}^{2n+\alpha}(\mathcal{R})$, then it holds*

$$-\frac{\Delta_h^\alpha f(x)}{h^\alpha} = \frac{\partial^\alpha f(x)}{\partial |x|^\alpha} + \sum_{l=1}^{n-1}(-1)^l \hat{c}_l^\alpha \frac{\partial^{2l+\alpha} f(x)}{\partial |x|^{2l+\alpha}}\left(\frac{h}{2}\right)^{2l} + O(h^{2n})$$

uniformly for $x \in \mathcal{R}$, where $\{\hat{c}_l^\alpha\}$ is the coefficient of power series of function $\left|\frac{\sin z}{z}\right|^\alpha$, that is,

$$\left|\frac{\sin z}{z}\right|^\alpha = \hat{c}_0^\alpha + \hat{c}_1^\alpha z^2 + \hat{c}_2^\alpha z^4 + \hat{c}_3^\alpha z^6 + \cdots,$$

in particular,

$$\hat{c}_0^\alpha = 1, \quad \hat{c}_1^\alpha = -\frac{\alpha}{6}, \quad \hat{c}_2^\alpha = \frac{(5\alpha-2)\alpha}{360}, \quad \hat{c}_3^\alpha = -\frac{(35\alpha^2-42\alpha+16)\alpha}{45360}.$$

Proof. Making the Fourier transform of function $[-\frac{\Delta_h^\alpha f(x)}{h^\alpha}]$ gives

$$\begin{aligned}
&\mathcal{F}\left\{-\frac{\Delta_h^\alpha f(x)}{h^\alpha};\omega\right\} \\
&= -\frac{1}{h^\alpha}\sum_{k=-\infty}^{\infty} \hat{g}_k^{(\alpha)} \mathcal{F}\{f(x-kh);\omega\} \\
&= -\frac{1}{h^\alpha}\sum_{k=-\infty}^{\infty} \hat{g}_k^{(\alpha)} \mathrm{e}^{\mathrm{i}\omega kh} F(\omega)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{h^{\alpha}}\left|2\sin\left(\frac{\omega h}{2}\right)\right|^{\alpha}F(\omega)\\
&= -|\omega|^{\alpha}\cdot\left|\frac{\sin\frac{\omega h}{2}}{\frac{\omega h}{2}}\right|^{\alpha}F(\omega),
\end{aligned}
\tag{1.51}
$$

where $F(\omega)$ is the Fourier transform of $f(t)$. By the Fourier transform, we have

$$
\begin{aligned}
\mathcal{F}\left\{\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}};\omega\right\} &= -\Psi_{\alpha}[(\mathrm{i}\omega)^{\alpha}+(-\mathrm{i}\omega)^{\alpha}]F(\omega) = -|\omega|^{\alpha}F(\omega),\\
\mathcal{F}\left\{\frac{\partial^{2l+\alpha}f(x)}{\partial|x|^{2l+\alpha}};\omega\right\} &= (\mathrm{i}\omega)^{2l}\mathcal{F}\left\{\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}};\omega\right\} = (-1)^{l+1}\omega^{2l}|\omega|^{\alpha}F(\omega).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathcal{F}\left\{\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}}+\sum_{l=1}^{n-1}(-1)^{l}\hat{c}_{l}^{\alpha}\frac{\partial^{2l+\alpha}f(x)}{\partial|x|^{2l+\alpha}}\left(\frac{h}{2}\right)^{2l};\omega\right\}\\
&= \mathcal{F}\left\{\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}};\omega\right\}+\sum_{l=1}^{n-1}(-1)^{l}\hat{c}_{l}^{\alpha}\left(\frac{h}{2}\right)^{2l}\mathcal{F}\left\{\frac{\partial^{2l+\alpha}f(x)}{\partial|x|^{2l+\alpha}};\omega\right\} \qquad (1.52)\\
&= -|\omega|^{\alpha}F(\omega)-\sum_{l=1}^{n-1}\hat{c}_{l}^{\alpha}\left(\frac{h}{2}\right)^{2l}\omega^{2l}|\omega|^{\alpha}F(\omega)\\
&= -|\omega|^{\alpha}\left[1+\sum_{l=1}^{n-1}\hat{c}_{l}^{\alpha}\left(\frac{\omega h}{2}\right)^{2l}\right]F(\omega). \qquad (1.53)
\end{aligned}
$$

It is easy to know that for any $z \in \mathcal{R}$, there is a constant c such that

$$
\left|\left|\frac{\sin z}{z}\right|^{\alpha}-\sum_{l=0}^{n-1}\hat{c}_{l}^{\alpha}z^{2l}\right| \leqslant c\,z^{2n}.
$$

Subtracting (1.53) from (1.51) leads to

$$
\begin{aligned}
&\mathcal{F}\left\{-\frac{\Delta_{h}^{\alpha}f(x)}{h^{\alpha}};\omega\right\}-\mathcal{F}\left\{\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}}+\sum_{l=1}^{n-1}(-1)^{l}\hat{c}_{l}^{\alpha}\frac{\partial^{2l+\alpha}f(x)}{\partial|x|^{2l+\alpha}}\left(\frac{h}{2}\right)^{2l};\omega\right\}\\
&= -|\omega|^{\alpha}\left\{\left|\frac{\sin\frac{\omega h}{2}}{\frac{\omega h}{2}}\right|^{\alpha}-\left[1+\sum_{l=1}^{n-1}\hat{c}_{l}^{\alpha}\left(\frac{\omega h}{2}\right)^{2l}\right]\right\}F(\omega) \equiv \Phi(\omega,h).
\end{aligned}
$$

Taking the inverse Fourier transforms on both hand sides of the equality above gives

$$
\begin{aligned}
&\left|-\frac{\Delta_{h}^{\alpha}f(x)}{h^{\alpha}}-\left[\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}}+\sum_{l=1}^{n-1}(-1)^{l}\hat{c}_{l}^{\alpha}\frac{\partial^{2l+\alpha}f(x)}{\partial|x|^{2l+\alpha}}\left(\frac{h}{2}\right)^{2l}\right]\right|\\
&= \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega,h)\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{d}\omega\right|
\end{aligned}
$$

$$\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\Phi(\omega,h)\right|\mathrm{d}\omega$$

$$\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}|\omega|^{\alpha}\cdot\left|\left|\frac{\sin\frac{\omega h}{2}}{\frac{\omega h}{2}}\right|^{\alpha}-\left[1+\sum_{l=1}^{n-1}\hat{c}_l^{\alpha}\left(\frac{\omega h}{2}\right)^{2l}\right]\right|\cdot\left|F(\omega)\right|\mathrm{d}\omega$$

$$\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}|\omega|^{\alpha}c\left(\frac{\omega h}{2}\right)^{2n}\cdot\left|F(\omega)\right|\mathrm{d}\omega$$

$$= \frac{c}{2\pi}\left(\frac{h}{2}\right)^{2n}\int_{-\infty}^{\infty}|\omega|^{2n+\alpha}\left|F(\omega)\right|\mathrm{d}\omega = \hat{c}h^{2n},$$

where $\hat{c} = \frac{c}{2^{2n+1}\pi}\int_{-\infty}^{\infty}|\omega|^{2n+\alpha}|F(\omega)|\mathrm{d}\omega$, independent of h and x. The proof ends. □

When $n = 2$, the theorem says that

$$\begin{aligned}
-\frac{\Delta_h^{\alpha}f(x)}{h^{\alpha}} &= \frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}} - \hat{c}_1^{\alpha}\frac{\partial^{2+\alpha}f(x)}{\partial|x|^{2+\alpha}}\left(\frac{h}{2}\right)^2 + O(h^4)\\
&= \frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}} + \frac{\alpha}{24}h^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}}\right) + O(h^4)\\
&= \frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}} + \frac{\alpha}{24}\left[\frac{\partial^{\alpha}f(x-h)}{\partial|x|^{\alpha}} - 2\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}} + \frac{\partial^{\alpha}f(x+h)}{\partial|x|^{\alpha}}\right] + O(h^4)\\
&= \frac{\alpha}{24}\frac{\partial^{\alpha}f(x-h)}{\partial|x|^{\alpha}} + \left(1-\frac{\alpha}{12}\right)\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}} + \frac{\alpha}{24}\frac{\partial^{\alpha}f(x+h)}{\partial|x|^{\alpha}} + O(h^4).
\end{aligned}$$

Hence the following theorem is true.

Theorem 1.5.2. [116] *Suppose $f \in \mathscr{C}^{4+\alpha}(\mathcal{R})$, then it holds*

$$-\frac{\Delta_h^{\alpha}f(x)}{h^{\alpha}} = \frac{\alpha}{24}\frac{\partial^{\alpha}f(x-h)}{\partial|x|^{\alpha}} + \left(1-\frac{\alpha}{12}\right)\frac{\partial^{\alpha}f(x)}{\partial|x|^{\alpha}} + \frac{\alpha}{24}\frac{\partial^{\alpha}f(x+h)}{\partial|x|^{\alpha}} + O(h^4) \tag{1.54}$$

uniformly for $x \in \mathcal{R}$.

Lemma 1.5.2. *When $\alpha \in [1,2]$, the coefficient $\{\hat{g}_k^{(\alpha)}\}$ satisfies*

$$\hat{g}_0^{(\alpha)} = \frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha/2+1)} \geqslant 0,\quad \hat{g}_{-k}^{(\alpha)} = \hat{g}_k^{(\alpha)} \leqslant 0,\quad k = 1,2,\ldots,$$

$$\sum_{k=-\infty}^{\infty}\hat{g}_k^{(\alpha)} = 0,\quad -\sum_{\substack{k=-M+i\\k\neq 0}}^{i}\hat{g}_k^{(\alpha)} \leqslant \hat{g}_0^{(\alpha)},\quad 1\leqslant i\leqslant M-1.$$

Moreover, the next lemma can be proved.

Lemma 1.5.3. *When* $1 < \alpha \leqslant 2$, *it holds*

(I) $|\hat{g}_k^{(\alpha)}| \leqslant \frac{\alpha}{2+\alpha}\frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha/2+1)}(\frac{\alpha/2+2}{\alpha/2+k+1})^{\alpha+1}, \quad |k| \geqslant 1$;

(II) $|\hat{g}_k^{(\alpha)}| \geqslant \frac{r_\alpha}{(k+1)^{\alpha+1}}, \quad |k| \geqslant 1$;

(III) $\displaystyle\sum_{|k|=l}^{\infty} |\hat{g}_k^{(\alpha)}| \geqslant \frac{c_*^{(\alpha)}}{(l+1)^\alpha}, \quad l \geqslant 1,$

where

$$\begin{aligned} r_\alpha &= \mathrm{e}^{-2}\frac{(4-\alpha)(2-\alpha)\alpha}{(6+\alpha)(4+\alpha)(2+\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha/2+1)}\left(3+\frac{\alpha}{2}\right)^{\alpha+1}, \\ c_*^{(\alpha)} &= \frac{2}{\alpha}r_\alpha. \end{aligned} \tag{1.55}$$

Proof. (I) From (1.49) and Lemma 1.4.2(I), we have

$$\begin{aligned} |\hat{g}_k^{(\alpha)}| &= \left(1-\frac{\alpha+1}{\alpha/2+k}\right)|\hat{g}_{k-1}^{(\alpha)}| \\ &\leqslant \mathrm{e}^{-\frac{\alpha+1}{\alpha/2+k}}|\hat{g}_{k-1}^{(\alpha)}| \\ &\leqslant \cdots \leqslant \mathrm{e}^{-\sum_{m=2}^{k}\frac{\alpha+1}{\alpha/2+m}}|\hat{g}_1^{(\alpha)}|, \quad k \geqslant 2. \end{aligned}$$

Noticing

$$\sum_{m=2}^{k}\frac{1}{\alpha/2+m} \geqslant \sum_{m=2}^{k}\int_m^{m+1}\frac{1}{\alpha/2+x}\mathrm{d}x = \int_2^{k+1}\frac{1}{\alpha/2+x}\mathrm{d}x = \ln\frac{\alpha/2+k+1}{\alpha/2+2},$$

it follows

$$\begin{aligned} |\hat{g}_k^{(\alpha)}| &\leqslant \mathrm{e}^{-(\alpha+1)\ln\frac{\alpha/2+k+1}{\alpha/2+2}}|\hat{g}_1^{(\alpha)}| \\ &= \frac{\alpha}{2+\alpha}\frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha/2+1)}\left(\frac{\alpha/2+2}{\alpha/2+k+1}\right)^{\alpha+1}, \quad k \geqslant 2. \end{aligned}$$

Obviously, it is also true for $k = 1$. Hence combining with (1.50), the conclusion is true for all k ($|k| \geqslant 1$).

(II) From (1.49) and Lemma 1.4.2(II), we have

$$\begin{aligned} |\hat{g}_k^{(\alpha)}| &= \left(1-\frac{\alpha+1}{\alpha/2+k}\right)|\hat{g}_{k-1}^{(\alpha)}| \\ &\geqslant \mathrm{e}^{-\frac{\alpha+1}{\alpha/2+k}-(\frac{\alpha+1}{\alpha/2+k})^2}|\hat{g}_{k-1}^{(\alpha)}| \\ &\geqslant \mathrm{e}^{-\frac{\alpha+1}{\alpha/2+k}-(\frac{\alpha+1}{\alpha/2+k})^2}\mathrm{e}^{-\frac{\alpha+1}{\alpha/2+k-1}-(\frac{\alpha+1}{\alpha/2+k-1})^2}|\hat{g}_{k-2}^{(\alpha)}| \\ &\geqslant \cdots \geqslant \mathrm{e}^{-(\alpha+1)\sum_{m=4}^{k}\frac{1}{\alpha/2+m}}\mathrm{e}^{-(\alpha+1)^2\sum_{m=4}^{k}(\frac{1}{\alpha/2+m})^2}|\hat{g}_3^{(\alpha)}|, \quad k \geqslant 4. \end{aligned} \tag{1.56}$$

Noticing

$$\sum_{m=4}^{\infty}\left(\frac{1}{\alpha/2+m}\right)^2 \leqslant \sum_{m=4}^{\infty}\frac{1}{(\alpha/2+m+1/2)(\alpha/2+m-1/2)}$$

$$= \sum_{m=4}^{\infty}\left(\frac{1}{\alpha/2+m-1/2} - \frac{1}{\alpha/2+m+1/2}\right) = \frac{2}{\alpha+7},$$

it follows

$$e^{-(\alpha+1)^2 \sum_{m=4}^{k}(\frac{1}{\alpha/2+m})^2} \geqslant e^{-2(\alpha+1)^2/(\alpha+7)} \geqslant e^{-2}.$$

The combination with (1.56) gives

$$|\hat{g}_k^{(\alpha)}| \geqslant e^{-2}|\hat{g}_3^{(\alpha)}|e^{-(\alpha+1)\sum_{m=4}^{k}\frac{1}{\alpha/2+m}}, \quad k \geqslant 4.$$

Noticing that the function $\frac{1}{\alpha/2+x}$ is monotone decreasing for $x > 0$, we have

$$\sum_{m=4}^{k}\frac{1}{\alpha/2+m} < \sum_{m=4}^{k}\int_{m-1}^{m}\frac{1}{\alpha/2+x}dx = \int_{3}^{k}\frac{1}{\alpha/2+x}dx = \ln\left(\frac{k+\alpha/2}{3+\alpha/2}\right).$$

Hence,

$$\begin{aligned}
|\hat{g}_k^{(\alpha)}| &\geqslant e^{-2}|\hat{g}_3^{(\alpha)}|e^{-(\alpha+1)\ln(\frac{k+\alpha/2}{3+\alpha/2})} \\
&= e^{-2}|\hat{g}_3^{(\alpha)}|\left(\frac{3+\alpha/2}{k+\alpha/2}\right)^{\alpha+1} \\
&= \frac{r_\alpha}{(k+\frac{\alpha}{2})^{\alpha+1}} \\
&\geqslant \frac{r_\alpha}{(k+1)^{\alpha+1}}, \quad k \geqslant 4.
\end{aligned}$$

It can be verified that the inequality above is also fulfilled for $k = 3, 2, 1$. Combining with (1.50), it follows

$$|\hat{g}_k^{(\alpha)}| \geqslant \frac{r_\alpha}{(k+1)^{\alpha+1}}, \quad |k| \geqslant 1.$$

(III) Summing up for k in the inequality above leads to

$$\sum_{k=l}^{\infty}|\hat{g}_k^{(\alpha)}| \geqslant r_\alpha \sum_{k=l}^{\infty}\frac{1}{(k+1)^{\alpha+1}} = r_\alpha \sum_{k=l+1}^{\infty}\frac{1}{k^{\alpha+1}}, \quad l \geqslant 1.$$

Noticing that the function $1/x^{\alpha+1}$ is monotone decreasing for $x > 0$, we have

$$\sum_{k=l+1}^{\infty}\frac{1}{k^{\alpha+1}} \geqslant \sum_{k=l+1}^{\infty}\int_{k}^{k+1}\frac{1}{x^{\alpha+1}}dx = \int_{l+1}^{\infty}\frac{1}{x^{\alpha+1}}dx = \frac{1}{\alpha(l+1)^\alpha}.$$

Hence,

$$\sum_{k=l}^{\infty}|\hat{g}_k^{(\alpha)}| \geqslant \frac{r_\alpha}{\alpha(l+1)^\alpha}, \quad l \geqslant 1$$

and

$$\sum_{|k|=l}^{\infty} |\hat{g}_k^{(\alpha)}| \geqslant \frac{2r_\alpha}{\alpha(l+1)^\alpha} = \frac{c_*^{(\alpha)}}{(l+1)^\alpha}, \quad l \geqslant 1.$$

The proof ends. □

1.6 Interpolation approximations of Caputo fractional derivatives

1.6.1 L1 approximation

For the Caputo fractional derivative of order α $(0 < \alpha < 1)$,

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} \, \mathrm{d}s,$$

one of the popular ways to discretize it is the so-called L1 approximation, which approximates the function $f(s)$ on each small interval by a linear interpolation polynomial. We will state it here in detail.

Take a positive integer N. Denote $\tau = \frac{T}{N}$, $t_k = k\tau$, $0 \leqslant k \leqslant N$ and

$$a_l^{(\alpha)} = (l+1)^{1-\alpha} - l^{1-\alpha}, \quad l \geqslant 0. \tag{1.57}$$

Considering the Caputo fractional derivative at $t = t_n$ gives

$$\begin{aligned} {}_0^C D_t^\alpha f(t)|_{t=t_n} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{f'(t)}{(t_n - t)^\alpha} \, \mathrm{d}t \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{f'(t)}{(t_n - t)^\alpha} \, \mathrm{d}t. \end{aligned} \tag{1.58}$$

On the small interval $[t_{k-1}, t_k]$, the linear interpolation polynomial for $f(t)$ is

$$L_{1,k}(t) = \frac{t_k - t}{\tau} f(t_{k-1}) + \frac{t - t_{k-1}}{\tau} f(t_k)$$

and the interpolation remainder is

$$f(t) - L_{1,k}(t) = \frac{1}{2} f''(\xi_k)(t - t_{k-1})(t - t_k), \quad t \in [t_{k-1}, t_k], \tag{1.59}$$

where $\xi_k = \xi_k(t) \in (t_{k-1}, t_k)$. Approximating the function $f(t)$ in (1.58) using $L_{1,k}(t)$ leads to

$${}_0^C D_t^\alpha f(t)|_{t=t_n} \approx \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{L'_{1,k}(t)}{(t_n - t)^\alpha} \, \mathrm{d}t$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \frac{f(t_k)-f(t_{k-1})}{\tau} \cdot \int_{t_{k-1}}^{t_k} \frac{1}{(t_n-t)^{\alpha}}\,\mathrm{d}t \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \frac{f(t_k)-f(t_{k-1})}{\tau} \cdot \frac{1}{1-\alpha}\left[(t_n-t_{k-1})^{1-\alpha}-(t_n-t_k)^{1-\alpha}\right] \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [f(t_k)-f(t_{k-1})] \cdot \left[(n-k+1)^{1-\alpha}-(n-k)^{1-\alpha}\right] \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} a_{n-k}^{(\alpha)} [f(t_k)-f(t_{k-1})] \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[a_0^{(\alpha)} f(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) f(t_k) - a_{n-1}^{(\alpha)} f(t_0) \right].
\end{aligned}
$$

Now we have derived the numerical differentiation formula to calculate ${}_0^C D_t^{\alpha} f(t)|_{t=t_n}$ as follows:

$$
D_t^{\alpha} f(t_n) \equiv \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[a_0^{(\alpha)} f(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) f(t_k) - a_{n-1}^{(\alpha)} f(t_0) \right]. \tag{1.60}
$$

It is usually called the **L1 approximation** or **L1 formula**.

The coefficient $\{a_l^{(\alpha)} \mid l \geqslant 0\}$ has the following properties.

Lemma 1.6.1. *Suppose* $\alpha \in (0,1)$ *and* $\{a_l^{(\alpha)} \mid l = 0,1,2,\dots,\}$ *is defined by* (1.57), *then it holds:*

(I) $1 = a_0^{(\alpha)} > a_1^{(\alpha)} > a_2^{(\alpha)} > \cdots > a_l^{(\alpha)} > 0; a_l^{(\alpha)} \to 0$, *when* $l \to \infty$;

(II) $(1-\alpha)l^{-\alpha} < a_{l-1}^{(\alpha)} < (1-\alpha)(l-1)^{-\alpha}, \quad l \geqslant 1.$

In [31, 47, 49, 50, 84], different techniques have been used to prove that L1 formula has the accuracy of order $2-\alpha$. Now we will discuss the approximation error

$$
R(f(t_n)) = {}_0^C D_t^{\alpha} f(t)|_{t=t_n} - D_t^{\alpha} f(t_n)
$$

using the technique in [31]. The following theorem is true.

Theorem 1.6.1. *Suppose* $f \in C^2[t_0, t_n]$, *then it holds*

$$
|R(f(t_n))| \leqslant \frac{1}{2\Gamma(1-\alpha)} \left[\frac{1}{4} + \frac{\alpha}{(1-\alpha)(2-\alpha)} \right] \max_{t_0 \leqslant t \leqslant t_n} |f''(t)|\, \tau^{2-\alpha}.
$$

Proof. From the definition of $R(f(t_n))$, we have

$$
R(f(t_n)) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{f'(t)}{(t_n-t)^{\alpha}}\,\mathrm{d}t
$$

$$- \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{f(t_k) - f(t_{k-1})}{\tau} \cdot \frac{1}{(t_n - t)^\alpha} \, \mathrm{d}t$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} [f(t) - L_{1,k}(t)]' \frac{1}{(t_n - t)^\alpha} \, \mathrm{d}t.$$

Noticing (1.59), the application of integration by parts arrives at

$$\begin{aligned} R(f(t_n)) &= -\frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} [f(t) - L_{1,k}(t)] \mathrm{d}\Big(\frac{1}{(t_n - t)^\alpha}\Big) \\ &= -\frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} [f(t) - L_{1,k}(t)] \alpha (t_n - t)^{-\alpha-1} \mathrm{d}t \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{1}{2} f''(\xi_k)(t - t_{k-1})(t_k - t) \alpha (t_n - t)^{-\alpha-1} \mathrm{d}t. \end{aligned}$$

Hence,

$$\big|R(f(t_n))\big| \leqslant \frac{1}{2\Gamma(1-\alpha)} \max_{t_0 \leqslant t \leqslant t_n} \big|f''(t)\big| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t - t_{k-1})(t_k - t) \alpha (t_n - t)^{-\alpha-1} \mathrm{d}t. \tag{1.61}$$

Some direct calculations produce

$$\begin{aligned} &\sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} (t - t_{k-1})(t_k - t) \alpha (t_n - t)^{-\alpha-1} \mathrm{d}t \\ \leqslant\; &\frac{\tau^2}{4} \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \alpha (t_n - t)^{-\alpha-1} \mathrm{d}t \\ =\; &\frac{\tau^2}{4} \int_{t_0}^{t_{n-1}} \alpha (t_n - t)^{-\alpha-1} \mathrm{d}t \\ =\; &\frac{\tau^2}{4} \big(\tau^{-\alpha} - t_n^{-\alpha}\big) \leqslant \frac{1}{4} \tau^{2-\alpha} \end{aligned} \tag{1.62}$$

and

$$\int_{t_{n-1}}^{t_n} (t - t_{n-1})(t_n - t) \alpha (t_n - t)^{-\alpha-1} \mathrm{d}t$$

$$
\begin{aligned}
&= \alpha \int_{t_{n-1}}^{t_n} (t - t_{n-1})(t_n - t)^{-\alpha} \mathrm{d}t \\
&= \alpha \int_0^{\tau} (\tau - \xi)\xi^{-\alpha} \mathrm{d}\xi \\
&= \frac{\alpha}{(1-\alpha)(2-\alpha)} \tau^{2-\alpha}.
\end{aligned}
\tag{1.63}
$$

Substituting (1.62) and (1.63) into (1.61), we have

$$
|R(f(t_n))| \leqslant \frac{1}{2\Gamma(1-\alpha)} \left[\frac{1}{4} + \frac{\alpha}{(1-\alpha)(2-\alpha)} \right] \max_{t_0 \leqslant t \leqslant t_n} |f''(t)| \tau^{2-\alpha}.
$$

The proof ends. □

Now we consider the numerical approximation of the Caputo fractional derivative of order γ $(1 < \gamma < 2)$,

$$
{}_0^C D_t^{\gamma} f(t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{f''(s)}{(t-s)^{\gamma-1}} \mathrm{d}s.
$$

Let

$$
g(t) = f'(t), \quad \alpha = \gamma - 1,
$$

then

$$
{}_0^C D_t^{\gamma} f(t) = \frac{1}{\Gamma(1-(\gamma-1))} \int_0^t \frac{g'(s)}{(t-s)^{\gamma-1}} \mathrm{d}s = {}_0^C D_t^{\gamma-1} g(t) = {}_0^C D_t^{\alpha} g(t),
$$

that is, the γ-th order derivative of function $f(t)$ is exactly the $(\gamma-1)$-th order derivative of function $g(t)$.

By Theorem 1.6.1, we have

$$
\begin{aligned}
{}_0^C D_t^{\alpha} g(t)|_{t=t_n} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \Bigg[& a_0^{(\alpha)} g(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) g(t_k) \\
& - a_{n-1}^{(\alpha)} g(t_0) \Bigg] + R(g(t_n)),
\end{aligned}
$$

where

$$
\begin{aligned}
|R(g(t_n))| &\leqslant \frac{1}{2\Gamma(1-\alpha)} \left[\frac{1}{4} + \frac{\alpha}{(1-\alpha)(2-\alpha)} \right] \cdot \max_{t_0 \leqslant t \leqslant t_n} |g''(t)| \, \tau^{2-\alpha} \\
&= \frac{1}{2\Gamma(2-\gamma)} \left[\frac{1}{4} + \frac{\gamma-1}{(2-\gamma)(3-\gamma)} \right] \cdot \max_{t_0 \leqslant t \leqslant t_n} |f'''(t)| \, \tau^{3-\gamma}.
\end{aligned}
$$

Denote

$$b_l^{(\gamma)} = a_l^{(\alpha)} = (l+1)^{1-\alpha} - l^{1-\alpha} = (l+1)^{2-\gamma} - l^{2-\gamma}, \quad l = 0, 1, 2, \ldots. \tag{1.64}$$

Then

$$\begin{aligned} {}_0^C D_t^\gamma f(t)|_{t=t_n} = \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)} \Big[b_0^{(\gamma)} g(t_n) - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)}) g(t_k) - b_{n-1}^{(\gamma)} g(t_0) \Big] \\ + R(g(t_n)). \end{aligned} \tag{1.65}$$

Similarly, we have

$$\begin{aligned} {}_0^C D_t^\gamma f(t)|_{t=t_{n-1}} = \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)} \Big[b_0^{(\gamma)} g(t_{n-1}) - \sum_{k=1}^{n-2} (b_{n-k-2}^{(\gamma)} - b_{n-k-1}^{(\gamma)}) g(t_k) - b_{n-2}^{(\gamma)} g(t_0) \Big] \\ + R(g(t_{n-1})). \end{aligned} \tag{1.66}$$

Adding (1.65) and (1.66) and taking an average arrive at

$$\begin{aligned} &\frac{1}{2}\big[{}_0^C D_t^\gamma f(t)|_{t=t_n} + {}_0^C D_t^\gamma f(t)|_{t=t_{n-1}}\big] \\ &= \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)} \Big[b_0^{(\gamma)} \frac{g(t_n)+g(t_{n-1})}{2} - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)}) \frac{g(t_k)+g(t_{k-1})}{2} \\ &\quad - b_{n-1}^{(\gamma)} g(t_0) \Big] + \frac{1}{2}\big[R(g(t_n)) + R(g(t_{n-1}))\big]. \end{aligned} \tag{1.67}$$

Noticing

$$\begin{aligned} \frac{g(t_k)+g(t_{k-1})}{2} &= \frac{f'(t_k)+f'(t_{k-1})}{2} \\ &= \frac{f(t_k)-f(t_{k-1})}{\tau} + \frac{\tau^2}{12} f'''(\eta_k), \quad \eta_k \in (t_{k-1}, t_k) \end{aligned}$$

and denoting

$$\delta_t f^{k-\frac{1}{2}} = \frac{f(t_k)-f(t_{k-1})}{\tau},$$

it follows immediately from (1.67) that

$$\begin{aligned} &\frac{1}{2}\big[{}_0^C D_t^\gamma f(t)|_{t=t_n} + {}_0^C D_t^\gamma f(t)|_{t=t_{n-1}}\big] \\ &= \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)} \Big[b_0^{(\gamma)} \delta_t f^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)}) \delta_t f^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)} f'(t_0) \Big] \\ &\quad + \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)} \Big[b_0^{(\gamma)} \frac{\tau^2}{12} f'''(\eta_n) - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)}) \frac{\tau^2}{12} f'''(\eta_k) \Big] \end{aligned}$$

$$+\frac{1}{2}\big[R(g(t_n))+R(g(t_{n-1}))\big].$$

Let

$$\hat{R}^{n-\frac{1}{2}}=\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\frac{\tau^2}{12}f'''(\eta_n)-\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\frac{\tau^2}{12}f'''(\eta_k)\right]$$
$$+\frac{1}{2}\big[R(g(t_n))+R(g(t_{n-1}))\big],$$

then we have

$$\big|\hat{R}^{n-\frac{1}{2}}\big|\leqslant\left\{\frac{1}{6\Gamma(3-\gamma)}+\frac{1}{2\Gamma(2-\gamma)}\left[\frac{1}{4}+\frac{\gamma-1}{(2-\gamma)(3-\gamma)}\right]\right\}\max_{t_0\leqslant t\leqslant t_n}\big|f'''(t)\big|\,\tau^{3-\gamma}. \tag{1.68}$$

Therefore, the following theorem is obtained.

Theorem 1.6.2. *Suppose $f\in C^3[t_0,t_n]$, then it holds*

$$\frac{1}{2}\big[{}_0^CD_t^\gamma f(t)|_{t=t_n}+{}_0^CD_t^\gamma f(t)|_{t=t_{n-1}}\big]$$
$$=\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\delta_t f^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\delta_t f^{k-\frac{1}{2}}-b_{n-1}^{(\gamma)}f'(t_0)\right]+\hat{R}^{n-\frac{1}{2}}, \tag{1.69}$$

where $\hat{R}^{n-\frac{1}{2}}$ satisfies (1.68).

Remark 1.6.1. Denote

$$\nabla_t f^k=\frac{f(t_k)-f(t_{k-1})}{\tau}.$$

If we directly use

$$g(t_k)=f'(t_k)=\nabla_t f^k+\tau\int_0^1 f''(t_k-\theta\tau)(1-\theta)\mathrm{d}\theta$$

in (1.65), we can obtain

$${}_0^CD_t^\gamma f(t)|_{t=t_n}=\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\nabla_t f^n-\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\nabla_t f^k-b_{n-1}^{(\gamma)}f'(t_0)\right]$$
$$+r_n+R(g(t_n))$$
$$=\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[\sum_{k=2}^{n}b_{n-k}^{(\gamma)}(\nabla_t f^k-\nabla_t f^{k-1})+b_{n-1}^{(\gamma)}(\nabla_t f^1-f'(t_0))\right]$$
$$+r_n+R(g(t_n))$$
$$=\frac{1}{\Gamma(2-\gamma)}\left[\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\frac{\nabla_t f^k-\nabla_t f^{k-1}}{\tau}\cdot\frac{\mathrm{d}t}{(t_n-t)^{\gamma-1}}\right.$$

$$+\int_{t_0}^{t_1}\frac{\nabla_t f^1-f'(t_0)}{\tau}\cdot\frac{\mathrm{d}t}{(t_n-t)^{\gamma-1}}\Bigg]+r_n+R(g(t_n)),\tag{1.70}$$

where

$$\begin{aligned}
r_n &= \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\Bigg[b_0^{(\gamma)}\tau\int_0^1 f''(t_n-\theta\tau)(1-\theta)\mathrm{d}\theta\\
&\quad-\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\tau\int_0^1 f''(t_k-\theta\tau)(1-\theta)\mathrm{d}\theta\Bigg]\\
&=\frac{\tau^{2-\gamma}}{\Gamma(3-\gamma)}\Bigg[\sum_{k=2}^{n}b_{n-k}^{(\gamma)}\Bigg(\int_0^1 f''(t_k-\theta\tau)(1-\theta)\mathrm{d}\theta\\
&\quad-\int_0^1 f''(t_{k-1}-\theta\tau)(1-\theta)\mathrm{d}\theta\Bigg)+b_{n-1}^{(\gamma)}\int_0^1 f''(t_1-\theta\tau)(1-\theta)\mathrm{d}\theta\Bigg].
\end{aligned}$$

It is easy to know

$$\begin{aligned}
|r_n| &\leqslant \frac{\tau^{2-\gamma}}{\Gamma(3-\gamma)}\Bigg[\sum_{k=2}^{n}b_{n-k}^{(\gamma)}\cdot\frac{\tau}{2}\max_{t_0\leqslant t\leqslant t_n}|f'''(t)|+b_{n-1}^{(\gamma)}\cdot\frac{1}{2}\max_{t_0\leqslant t\leqslant t_1}|f''(t)|\Bigg]\\
&=\frac{1}{\Gamma(2-\gamma)}\Bigg[\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\frac{\mathrm{d}t}{(t_n-t)^{\gamma-1}}\cdot\frac{\tau}{2}\max_{t_0\leqslant t\leqslant t_n}|f'''(t)|\\
&\quad+\int_{t_0}^{t_1}\frac{\mathrm{d}t}{(t_n-t)^{\gamma-1}}\cdot\frac{1}{2}\max_{t_0\leqslant t\leqslant t_1}|f''(t)|\Bigg].
\end{aligned}$$

Noticing

$$\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\frac{\mathrm{d}t}{(t_n-t)^{\gamma-1}}=\int_{t_1}^{t_n}\frac{\mathrm{d}t}{(t_n-t)^{\gamma-1}}=\frac{(t_n-t_1)^{2-\gamma}}{2-\gamma}\leqslant\frac{t_n^{2-\gamma}}{2-\gamma},$$

if $n=1$,

$$\int_{t_0}^{t_1}\frac{\mathrm{d}t}{(t_n-t)^{\gamma-1}}=\frac{\tau^{2-\gamma}}{2-\gamma}$$

and if $n\geqslant 2$,

$$\int_{t_0}^{t_1}\frac{\mathrm{d}t}{(t_n-t)^{\gamma-1}}\leqslant\frac{\tau}{(t_n-t_1)^{\gamma-1}}\leqslant\frac{\tau}{(t_n/2)^{\gamma-1}}=2^{\gamma-1}\frac{\tau}{t_n^{\gamma-1}},$$

we have

$$|r_n| \leqslant \frac{1}{\Gamma(2-\gamma)}\left[\frac{t_n^{2-\gamma}}{2-\gamma}\cdot\frac{\tau}{2}\max_{t_0\leqslant t\leqslant t_n}\left|f'''(t)\right| + \frac{1}{2-\gamma}\cdot\frac{\tau}{t_n^{\gamma-1}}\cdot\frac{1}{2}\max_{t_0\leqslant t\leqslant t_1}\left|f''(t)\right|\right]$$
$$= \frac{1}{2\Gamma(3-\gamma)}\left[t_n^{2-\gamma}\cdot\tau\max_{t_0\leqslant t\leqslant t_n}\left|f'''(t)\right| + \frac{\tau}{t_n^{\gamma-1}}\max_{t_0\leqslant t\leqslant t_1}\left|f''(t)\right|\right].$$

Denote

$$\mathbb{D}_t^{\gamma}f(t_n) = \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\nabla_t f^n - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\nabla_t f^k - b_{n-1}^{(\gamma)}f'(t_0)\right].$$

Noticing

$$R(g(t_n)) = O(\tau^{3-\gamma})\max_{t_0\leqslant t\leqslant t_n}\left|f'''(t)\right|,$$

we obtain

$${}_0^C D_t^{\gamma}f(t)|_{t=t_n} - \mathbb{D}_t^{\gamma}f(t_n) = O(\tau)\max_{t_0\leqslant t\leqslant t_n}\left|f'''(t)\right| + O\left(\frac{\tau}{t_n^{\gamma-1}}\right)\max_{t_0\leqslant t\leqslant t_1}\left|f''(t)\right|.$$

If $f''(0) = 0$, we have

$${}_0^C D_t^{\gamma}f(t)|_{t=t_n} - \mathbb{D}_t^{\gamma}f(t_n) = O(\tau)\max_{t_0\leqslant t\leqslant t_n}\left|f'''(t)\right|.$$

It follows from (1.70) that

$$\begin{aligned}
&{}_0^C D_t^{\gamma}f(t)|_{t=t_n} - \mathbb{D}_t^{\gamma}f(t_n)\\
&= \frac{1}{\Gamma(2-\gamma)}\left\{\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\left[f''(t) - \frac{\nabla_t f^k - \nabla_t f^{k-1}}{\tau}\right]\cdot\frac{\mathrm{d}t}{(t_n-t)^{\gamma-1}}\right.\\
&\quad\left. + \int_{t_0}^{t_1}\left[f''(t) - \frac{\nabla_t f^1 - f'(t_0)}{\tau}\right]\cdot\frac{\mathrm{d}t}{(t_n-t)^{\gamma-1}}\right\}.
\end{aligned}$$

Notice the fact that

$$\begin{aligned}
f''(t) - \frac{\nabla_t f^k - \nabla_t f^{k-1}}{\tau} &= O(\tau), \quad t\in(t_{k-1},t_k), 2\leqslant k\leqslant n,\\
f''(t) - \frac{2[\nabla_t f^1 - f'(t_0)]}{\tau} &= O(\tau), \quad t\in(t_0,t_1).
\end{aligned}$$

If $f''(0) \neq 0$, we can modify $\mathbb{D}_t^{\gamma}f(t_n)$ as

$$\widetilde{\mathbb{D}}_t^{\gamma}f(t_n) = \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\nabla_t f^n - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\nabla_t f^k\right.$$

$$-b_{n-1}^{(\gamma)} f'(t_0) + b_{n-1}^{(\gamma)} (\nabla_t f^1 - f'(t_0)) \Big] .$$

Then we have

$$\begin{aligned}
&{}_0^C D_t^\gamma f(t)|_{t=t_n} - \widetilde{\mathbb{D}}_t^\gamma f(t_n) \\
&= \frac{1}{\Gamma(2-\gamma)} \left\{ \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} \left[f''(t) - \frac{\nabla_t f^k - \nabla_t f^{k-1}}{\tau} \right] \cdot \frac{\mathrm{d}t}{(t_n - t)^{\gamma-1}} \right. \\
&\quad \left. + \int_{t_0}^{t_1} \left[f''(t) - \frac{2(\nabla_t f^1 - f'(t_0))}{\tau} \right] \cdot \frac{\mathrm{d}t}{(t_n - t)^{\gamma-1}} \right\} \\
&= O(\tau) \max_{t_0 \leqslant t \leqslant t_n} |f'''(t)|.
\end{aligned}$$

This is why we consider the approximation of

$$\frac{1}{2} [{}_0^C D_t^\gamma f(t)|_{t=t_n} + {}_0^C D_t^\gamma f(t)|_{t=t_{n-1}}]$$

instead of directly considering the approximation of ${}_0^C D_t^\gamma f(t)|_{t=t_n}$.

1.6.2 L1-2 approximation

Gao et al. gave an L1-2 interpolation approximation formula for the fractional derivative of order $\alpha\,(0 < \alpha < 1)$ in [31]. Make a linear interpolation polynomial $L_{1,1}(t)$ for $f(t)$ on $[t_0, t_1]$ by using two points $(t_0, f(t_0))$, $(t_1, f(t_1))$; make a quadratic interpolation polynomial $L_{2,k}(t)$ for $f(t)$ on $[t_k, t_{k+1}]$ by using three points $(t_{k-1}, f(t_{k-1}))$, $(t_k, f(t_k))$ and $(t_{k+1}, f(t_{k+1}))$. Differentiating $L_{1,1}(t)$ and $L_{2,k}(t)$ with respect to t once produces

$$L'_{1,1}(t) = \delta_t f^{\frac{1}{2}}, \quad L'_{2,k}(t) = \frac{t_{k+\frac{1}{2}} - t}{\tau} \delta_t f^{k-\frac{1}{2}} + \frac{t - t_{k-\frac{1}{2}}}{\tau} \delta_t f^{k+\frac{1}{2}},$$

where $t_{k-\frac{1}{2}} = (t_k + t_{k-1})/2$, $\delta_t f^{k-\frac{1}{2}} = (f(t_k) - f(t_{k-1}))/\tau$. Using the above two equalities, we can get

$$\begin{aligned}
{}_0^C D_t^\alpha f(t)|_{t=t_n} &= \frac{1}{\Gamma(1-\alpha)} \left[\int_{t_0}^{t_1} f'(t)(t_n - t)^{-\alpha} \mathrm{d}t + \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} f'(t)(t_n - t)^{-\alpha} \mathrm{d}t \right] \\
&\approx \frac{1}{\Gamma(1-\alpha)} \left[\int_{t_0}^{t_1} L'_{1,1}(t)(t_n - t)^{-\alpha} \mathrm{d}t + \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} L'_{2,k}(t)(t_n - t)^{-\alpha} \mathrm{d}t \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\int_{t_0}^{t_1} (\delta_t f^{\frac{1}{2}})(t_n - t)^{-\alpha} \mathrm{d}t \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \left(\frac{t_{k+\frac{1}{2}} - t}{\tau} \delta_t f^{k-\frac{1}{2}} + \frac{t - t_{k-\frac{1}{2}}}{\tau} \delta_t f^{k+\frac{1}{2}} \right) (t_n - t)^{-\alpha} \mathrm{d}t \Bigg] \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\int_{t_0}^{t_1} (\delta_t f^{\frac{1}{2}})(t_n - t)^{-\alpha} \mathrm{d}t + \sum_{k=1}^{n-1} \delta_t f^{k-\frac{1}{2}} \int_{t_k}^{t_{k+1}} \frac{t_{k+\frac{1}{2}} - t}{\tau} (t_n - t)^{-\alpha} \mathrm{d}t \right. \\
&\quad \left. + \sum_{k=2}^{n} \delta_t f^{k-\frac{1}{2}} \int_{t_{k-1}}^{t_k} \frac{t - t_{k-\frac{3}{2}}}{\tau} (t_n - t)^{-\alpha} \mathrm{d}t \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \left\{ \left[\int_{t_0}^{t_1} (t_n - t)^{-\alpha} \mathrm{d}t + \int_{t_1}^{t_2} \frac{t_{\frac{3}{2}} - t}{\tau} (t_n - t)^{-\alpha} \mathrm{d}t \right] \delta_t f^{\frac{1}{2}} \right. \\
&\quad + \sum_{k=2}^{n-1} \left[\int_{t_k}^{t_{k+1}} \frac{t_{k+\frac{1}{2}} - t}{\tau} (t_n - t)^{-\alpha} \mathrm{d}t + \int_{t_{k-1}}^{t_k} \frac{t - t_{k-\frac{3}{2}}}{\tau} (t_n - t)^{-\alpha} \mathrm{d}t \right] \delta_t f^{k-\frac{1}{2}} \\
&\quad \left. + \left[\int_{t_{n-1}}^{t_n} \frac{t - t_{n-\frac{3}{2}}}{\tau} (t_n - t)^{-\alpha} \mathrm{d}t \right] \delta_t f^{n-\frac{1}{2}} \right\} \\
&\equiv \widehat{\mathbb{D}}_t^{\alpha} f(t_n).
\end{aligned}
$$

The integrals in the above equality can be computed by substitution of the integral variable.

When $n = 1$,

$$
\begin{aligned}
\widehat{\mathbb{D}}_t^{\alpha} f(t_n) &= \frac{1}{\Gamma(1-\alpha)} \left[\int_{t_0}^{t_1} (t_1 - t)^{-\alpha} \mathrm{d}t \right] \delta_t f^{\frac{1}{2}} \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} [f(t_1) - f(t_0)] \\
&\equiv \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \tilde{c}_0^{(1,\alpha)} [f(t_1) - f(t_0)],
\end{aligned}
\tag{1.71}
$$

where

$$
\tilde{c}_0^{(1,\alpha)} = 1.
$$

When $n \geqslant 2$,

$$
\widehat{\mathbb{D}}_t^{\alpha} f(t_n) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \tilde{c}_k^{(n,\alpha)} [f(t_{n-k}) - f(t_{n-k-1})],
\tag{1.72}
$$

where

$$
\tilde{c}_0^{(n,\alpha)} = \frac{1}{2} + \frac{1}{2-\alpha},
\tag{1.73}
$$

$$\tilde{c}_k^{(n,\alpha)} = \frac{(k+1)^{1-\alpha} - 2k^{1-\alpha} + (k-1)^{1-\alpha}}{2} + \frac{(k+1)^{2-\alpha} - 2k^{2-\alpha} + (k-1)^{2-\alpha}}{2-\alpha}, \quad 1 \leqslant k \leqslant n-2, \tag{1.74}$$

$$\tilde{c}_{n-1}^{(n,\alpha)} = \frac{(n-2)^{1-\alpha} - (n-1)^{1-\alpha} + 2n^{1-\alpha}}{2} - \frac{(n-1)^{2-\alpha} - (n-2)^{2-\alpha}}{2-\alpha}. \tag{1.75}$$

The following results are given in [31].

Theorem 1.6.3. *Suppose* $\alpha \in (0,1)$, $f \in C^3[t_0, t_n]$, $\widehat{\mathbb{D}}_t^\alpha f(t_n)$ *is defined by* (1.71) *and* (1.72). *Then we have the following error estimates:*

$$\left|{}_0^C D_t^\alpha f(t)|_{t=t_1} - \widehat{\mathbb{D}}_t^\alpha f(t_1)\right| \leqslant \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_0 \leqslant t \leqslant t_1} |f''(t)| \tau^{2-\alpha},$$

$$\left|{}_0^C D_t^\alpha f(t)|_{t=t_n} - \widehat{\mathbb{D}}_t^\alpha f(t_n)\right| \leqslant \frac{1}{\Gamma(1-\alpha)} \left\{ \frac{\alpha}{12} \max_{t_0 \leqslant t \leqslant t_1} |f''(t)| (t_n - t_1)^{-\alpha-1} \tau^3 + \left[\frac{1}{12} + \frac{\alpha}{3(1-\alpha)(2-\alpha)}\left(\frac{1}{2} + \frac{1}{3-\alpha}\right)\right] \max_{t_0 \leqslant t \leqslant t_n} |f'''(t)| \tau^{3-\alpha} \right\}, \quad n \geqslant 2.$$

If $f''(0) = 0$, it is easy to see that

$${}_0^C D_t^\alpha f(t)|_{t=t_n} - \widehat{\mathbb{D}}_t^\alpha f(t_n) = O(\tau^{3-\alpha}), \quad 1 \leqslant n \leqslant N.$$

The following lemma provides the properties of the coefficient $\{\tilde{c}_k^{(n,\alpha)} \mid 0 \leqslant k \leqslant n-1\}$.

Lemma 1.6.2. *The coefficient* $\{\tilde{c}_k^{(n,\alpha)} \mid 0 \leqslant k \leqslant n-1\}$ *defined by* (1.73)–(1.75) *satisfies the following properties:*

(I) *When* $n = 2$,

$$\tilde{c}_0^{(2,\alpha)} = \frac{1}{2} + \frac{1}{2-\alpha}, \quad \tilde{c}_1^{(2,\alpha)} = 2^{1-\alpha} - \left(\frac{1}{2} + \frac{1}{2-\alpha}\right);$$

(II) *When* $n \geqslant 3$,

$$\tilde{c}_0^{(n,\alpha)} = \frac{1}{2} + \frac{1}{2-\alpha},$$
$$\tilde{c}_0^{(n,\alpha)} > \tilde{c}_2^{(n,\alpha)} > \tilde{c}_3^{(n,\alpha)} > \cdots > \tilde{c}_{n-1}^{(n,\alpha)},$$
$$\tilde{c}_1^{(n,\alpha)} = 2^{-\alpha} - 1 + \frac{2^{2-\alpha} - 2}{2-\alpha}, \quad \tilde{c}_0^{(n,\alpha)} > |\tilde{c}_1^{(n,\alpha)}|.$$

Let α^* *be the unique root of equation*

$$6 - \alpha = (4-\alpha)2^\alpha$$

on the interval $[0,1]$ ($\alpha^* \approx 0.68029$). *Then when* $\alpha \in (0, \alpha^*)$, $\tilde{c}_1^{(n,\alpha)} > 0$; *When* $\alpha \in (\alpha^*, 1)$, $\tilde{c}_1^{(n,\alpha)} < 0$.

1.6.3 L2-1_σ approximation

Just as what was stated in last two subsections, for the α-th order $(0 < \alpha < 1)$ Caputo derivative, the classical L1 formula can attain the $(2-\alpha)$-th order convergence uniformly. Similarly, based on the piecewise interpolation approximation, the L1-2 approximation was established in [31]. On this basis, Alikhanov[1] discovered some superconvergent interpolation points and built L2-1_σ formula, which can reach the $(3-\alpha)$-th order convergence uniformly. Let us get into it in detail.

Suppose $0 < \alpha < 1$. Denote

$$\sigma = 1 - \frac{\alpha}{2}, \quad t_{n+\sigma} = (n+\sigma)\tau, \quad t_{n-\frac{1}{2}} = \frac{1}{2}(t_n + t_{n-1}),$$
$$f^n = f(t_n), \quad \delta_t f^{n-\frac{1}{2}} = \frac{1}{\tau}(f^n - f^{n-1}), \quad \delta_t^2 f^n = \frac{1}{\tau}(\delta_t f^{n+\frac{1}{2}} - \delta_t f^{n-\frac{1}{2}}).$$

Reformulate the fractional derivative as

$$ {}_0^C D_t^\alpha f(t)|_{t=t_{n-1+\sigma}} = \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} \frac{f'(\xi)}{(t_{n-1+\sigma}-\xi)^\alpha}\mathrm{d}\xi + \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{f'(\xi)}{(t_{n-1+\sigma}-\xi)^\alpha}\mathrm{d}\xi\right]. \tag{1.76}$$

On the small interval $[t_{k-1}, t_k]$, the quadratic interpolation polynomial of function $f(t)$ using three points $(t_{k-1}, f(t_{k-1}))$, $(t_k, f(t_k))$ and $(t_{k+1}, f(t_{k+1}))$ reads

$$L_{2,k}(t) = \sum_{j=-1}^{1} f(t_{k+j}) \prod_{l=-1, l\neq j}^{1} \frac{t - t_{k+l}}{t_{k+j} - t_{k+l}}, \quad k = 1, 2, \ldots, n-1,$$

the remainder for which is

$$f(t) - L_{2,k}(t) = \frac{f'''(\xi_k)}{6}(t - t_{k-1})(t - t_k)(t - t_{k+1}), \quad \xi_k \in (t_{k-1}, t_{k+1}).$$

Differentiating $L_{2,k}(t)$ with respect to t once gives

$$L'_{2,k}(t) = \frac{t_{k+\frac{1}{2}} - t}{\tau}\,\delta_t f^{k-\frac{1}{2}} + \frac{t - t_{k-\frac{1}{2}}}{\tau}\,\delta_t f^{k+\frac{1}{2}}.$$

On the interval $[t_{n-1}, t_{n-1+\sigma}]$, the linear interpolation polynomial of function $f(t)$ using two points $(t_{n-1}, f(t_{n-1}))$ and $(t_n, f(t_n))$ reads

$$L_{1,n}(t) = f(t_{n-1})\frac{t - t_n}{t_{n-1} - t_n} + f(t_n)\frac{t - t_{n-1}}{t_n - t_{n-1}}.$$

In addition, we have

$$L'_{1,n}(t) = \delta_t f^{n-\frac{1}{2}}.$$

Approximating the function $f(t)$ on the right-hand side of (1.76) using $L_{2,k}(t)$ ($1 \leqslant k \leqslant n-1$) and $L_{1,n}(t)$, respectively, leads to

$$
\begin{aligned}
{}_0^C D_t^\alpha f(t)|_{t=t_{n-1+\sigma}} &\approx \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{L_{2,k}'(t)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t + \int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{L_{1,n}'(t)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t\right] \\
&= \frac{1}{\Gamma(1-\alpha)}\left\{\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\left[\frac{t_{k+\frac{1}{2}}-t}{\tau}\delta_t f^{k-\frac{1}{2}} + \frac{t-t_{k-\frac{1}{2}}}{\tau}\delta_t f^{k+\frac{1}{2}}\right](t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t\right. \\
&\quad \left. + \int_{t_{n-1}}^{t_{n-1+\sigma}}(\delta_t f^{n-\frac{1}{2}})(t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t\right\} \\
&= \frac{1}{\Gamma(1-\alpha)}\left\{\left[\int_{t_0}^{t_1}\frac{t_{\frac{3}{2}}-t}{\tau}(t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t\right]\delta_t f^{\frac{1}{2}}\right. \\
&\quad + \sum_{k=2}^{n-1}\left[\int_{t_{k-1}}^{t_k}\frac{t_{k+\frac{1}{2}}-t}{\tau}(t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t\right. \\
&\quad \left. + \int_{t_{k-2}}^{t_{k-1}}\frac{t-t_{k-\frac{3}{2}}}{\tau}(t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t\right]\delta_t f^{k-\frac{1}{2}} \\
&\quad \left. + \left[\int_{t_{n-2}}^{t_{n-1}}\frac{t-t_{n-\frac{3}{2}}}{\tau}(t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t + \int_{t_{n-1}}^{t_{n-1+\sigma}}(t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t\right]\delta_t f^{n-\frac{1}{2}}\right\} \\
&= \frac{\tau^{1-\alpha}}{\Gamma(1-\alpha)}\left\{\left[\int_0^1\left(\frac{3}{2}-\theta\right)(n-1+\sigma-\theta)^{-\alpha}\mathrm{d}\theta\right]\delta_t f^{\frac{1}{2}}\right. \\
&\quad + \sum_{k=2}^{n-1}\left[\int_0^1\left(\frac{3}{2}-\theta\right)(n-k+\sigma-\theta)^{-\alpha}\mathrm{d}\theta\right. \\
&\quad \left. + \int_0^1\left(\theta-\frac{1}{2}\right)(n-k+1+\sigma-\theta)^{-\alpha}\mathrm{d}\theta\right]\delta_t f^{k-\frac{1}{2}} \\
&\quad \left. + \left[\int_0^1\left(\theta-\frac{1}{2}\right)(1+\sigma-\theta)^{-\alpha}\mathrm{d}\theta + \int_0^\sigma(\sigma-\theta)^{-\alpha}\mathrm{d}\theta\right]\delta_t f^{n-\frac{1}{2}}\right\} \\
&= \frac{\tau^{1-\alpha}}{\Gamma(1-\alpha)}\left\{\left[\int_0^1\left(\frac{3}{2}-\theta\right)(n-1+\sigma-\theta)^{-\alpha}\mathrm{d}\theta\right]\delta_t f^{\frac{1}{2}}\right. \\
&\quad + \sum_{k=2}^{n-1}\left[\int_0^1\left(\frac{3}{2}-\theta\right)(n-k+\sigma-\theta)^{-\alpha}\mathrm{d}\theta\right.
\end{aligned}
$$

$$+\int_0^{\frac{1}{2}}\theta\left(\alpha\int_{-\theta}^{\theta}\left(n-k+\frac{1}{2}+\sigma+\xi\right)^{-\alpha-1}\mathrm{d}\xi\right)\mathrm{d}\theta\Bigg]\delta_t f^{k-\frac{1}{2}}$$

$$+\left[\int_0^{\frac{1}{2}}\theta\left(\alpha\int_{-\theta}^{\theta}\left(\frac{1}{2}+\sigma+\xi\right)^{-\alpha-1}\mathrm{d}\xi\right)\mathrm{d}\theta+\frac{\sigma^{1-\alpha}}{1-\alpha}\right]\delta_t f^{n-\frac{1}{2}}\Bigg\}$$

$$\equiv \Delta_t^\alpha f(t_{n-1+\sigma}).$$

Denote

$$c_0^{(1,\alpha)}=\sigma^{1-\alpha}. \tag{1.77}$$

If $n \geqslant 2$, denote

$$\begin{cases} c_0^{(n,\alpha)} = (1-\alpha)\displaystyle\int_0^{\frac{1}{2}}\theta\left(\alpha\int_{-\theta}^{\theta}\left(\frac{1}{2}+\sigma+\xi\right)^{-\alpha-1}\mathrm{d}\xi\right)\mathrm{d}\theta+\sigma^{1-\alpha}, & (1.78)\\[2ex] c_k^{(n,\alpha)} = (1-\alpha)\left[\displaystyle\int_0^1\left(\frac{3}{2}-\theta\right)(k+\sigma-\theta)^{-\alpha}\mathrm{d}\theta\right. \\[2ex] \qquad\left.+\displaystyle\int_0^{\frac{1}{2}}\theta\left(\alpha\int_{-\theta}^{\theta}\left(k+\frac{1}{2}+\sigma+\xi\right)^{-\alpha-1}\mathrm{d}\xi\right)\mathrm{d}\theta\right],\quad 1\leqslant k\leqslant n-2, & (1.79)\\[2ex] c_{n-1}^{(n,\alpha)} = (1-\alpha)\displaystyle\int_0^1\left(\frac{3}{2}-\theta\right)(n-1+\sigma-\theta)^{-\alpha}\mathrm{d}\theta. & (1.80)\end{cases}$$

Then we have

$$\begin{aligned}&\Delta_t^\alpha f(t_{n-1+\sigma})\\ &=\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}c_{n-k}^{(n,\alpha)}\delta_t f^{k-\frac{1}{2}}\\ &=\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}c_{k}^{(n,\alpha)}\delta_t f^{n-k-\frac{1}{2}}\\ &=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}c_{k}^{(n,\alpha)}[f(t_{n-k})-f(t_{n-k-1})],\quad 1\leqslant n\leqslant N.\end{aligned} \tag{1.81}$$

We call (1.81) the **L2-1$_\sigma$ formula** or **L2-1$_\sigma$ approximation**.

Computing yields, if $n \geqslant 2$,

$$\begin{cases} c_0^{(n,\alpha)} = \dfrac{(1+\sigma)^{2-\alpha}-\sigma^{2-\alpha}}{2-\alpha} - \dfrac{(1+\sigma)^{1-\alpha}-\sigma^{1-\alpha}}{2}, \\ c_k^{(n,\alpha)} = \dfrac{1}{2-\alpha}\left[(k+1+\sigma)^{2-\alpha}-2(k+\sigma)^{2-\alpha}+(k-1+\sigma)^{2-\alpha}\right] \\ \qquad -\dfrac{1}{2}\left[(k+1+\sigma)^{1-\alpha}-2(k+\sigma)^{1-\alpha}+(k-1+\sigma)^{1-\alpha}\right], \\ \qquad 1 \leqslant k \leqslant n-2, \\ c_{n-1}^{(n,\alpha)} = \dfrac{1}{2}\left[3(n-1+\sigma)^{1-\alpha}-(n-2+\sigma)^{1-\alpha}\right] \\ \qquad -\dfrac{1}{2-\alpha}\left[(n-1+\sigma)^{2-\alpha}-(n-2+\sigma)^{2-\alpha}\right]. \end{cases}$$

The computational cost is $O(N^2)$ when using L2-1_σ formula (1.81) to compute $\Delta_t^\alpha f(t_{n-1+\sigma})$ $(1 \leqslant n \leqslant N)$.

Theorem 1.6.4. [1] *Suppose $f \in C^3[t_0, t_n]$, then it holds*

$$\left|{}_0^C D_t^\alpha f(t)|_{t=t_{n-1+\sigma}} - \Delta_t^\alpha f(t_{n-1+\sigma})\right| \leqslant \frac{(4\sigma-1)\sigma^{-\alpha}}{12\Gamma(2-\alpha)} \max_{t_0 \leqslant t \leqslant t_n} |f'''(t)| \tau^{3-\alpha},$$

where $\sigma = 1 - \frac{\alpha}{2}$, $0 < \alpha < 1$.

Proof. Denote

$$R^n = {}_0^C D_t^\alpha f(t)|_{t=t_{n-1+\sigma}} - \Delta_t^\alpha f(t_{n-1+\sigma}),$$

$$R_1^n = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \frac{f'(t) - L'_{2,k}(t)}{(t_{n-1+\sigma}-t)^\alpha} \mathrm{d}t,$$

$$R_2^n = \frac{1}{\Gamma(1-\alpha)} \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{f'(t) - \delta_t f^{n-\frac{1}{2}}}{(t_{n-1+\sigma}-t)^\alpha} \mathrm{d}t,$$

then

$$R^n = R_1^n + R_2^n. \tag{1.82}$$

Now the two terms R_1^n and R_2^n will be estimated, respectively.

It is easy to see that

$$R_1^n = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n-1} \left[\frac{f(t) - L_{2,k}(t)}{(t_{n-1+\sigma}-t)^\alpha}\bigg|_{t=t_{k-1}}^{t_k} - \int_{t_{k-1}}^{t_k} \alpha[f(t) - L_{2,k}(t)](t_{n-1+\sigma}-t)^{-\alpha-1} \mathrm{d}t \right]$$

$$= \frac{-\alpha}{\Gamma(1-\alpha)} \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \frac{f'''(\xi_k)}{6}(t-t_{k-1})(t-t_k) \cdot (t-t_{k+1})(t_{n-1+\sigma}-t)^{-\alpha-1}\mathrm{d}t,$$

hence,

$$\begin{aligned}|R_1^n| &\leqslant \frac{\alpha}{6\Gamma(1-\alpha)} \max_{t_0\leqslant t\leqslant t_n} |f'''(t)| \cdot \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} (t-t_{k-1})(t_k-t)(t_{k+1}-t)(t_{n-1+\sigma}-t)^{-\alpha-1}\mathrm{d}t \\ &\leqslant \frac{\alpha\tau^3}{12\Gamma(1-\alpha)} \max_{t_0\leqslant t\leqslant t_n} |f'''(t)| \int_{t_0}^{t_{n-1}} (t_{n-1+\sigma}-t)^{-\alpha-1}\mathrm{d}t \\ &\leqslant \frac{\sigma^{-\alpha}}{12\Gamma(1-\alpha)} \max_{t_0\leqslant t\leqslant t_n} |f'''(t)|\tau^{3-\alpha}. \end{aligned} \tag{1.83}$$

Due to that

$$f'(t) = f'(t_{n-\frac{1}{2}}) + (t-t_{n-\frac{1}{2}})f''(t_{n-\frac{1}{2}}) + \frac{1}{2}(t-t_{n-\frac{1}{2}})^2 f'''(\eta_n), \quad t, \eta_n \in (t_{n-1}, t_{n-1+\sigma}),$$

it follows

$$\begin{aligned}R_2^n = &\frac{1}{\Gamma(1-\alpha)} \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{f'(t_{n-\frac{1}{2}}) - \delta_t f^{n-\frac{1}{2}}}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t \\ &+ \frac{1}{\Gamma(1-\alpha)} \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{(t-t_{n-\frac{1}{2}})f''(t_{n-\frac{1}{2}})}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t \\ &+ \frac{1}{\Gamma(1-\alpha)} \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{\frac{1}{2}(t-t_{n-\frac{1}{2}})^2 f'''(\eta_n)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t.\end{aligned}$$

Computing the three integrals in the equality above gives

$$\begin{aligned}&\left|\frac{1}{\Gamma(1-\alpha)} \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{f'(t_{n-\frac{1}{2}}) - \delta_t f^{n-\frac{1}{2}}}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t\right| \\ &\leqslant \frac{1}{\Gamma(1-\alpha)} \left|f'(t_{n-\frac{1}{2}}) - \delta_t f^{n-\frac{1}{2}}\right| \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{1}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t\end{aligned}$$

$$\leqslant \frac{1}{\Gamma(1-\alpha)} \cdot \frac{\tau^2}{24} \max_{t_{n-1}\leqslant t\leqslant t_n} |f'''(t)| \cdot \frac{1}{1-\alpha}\sigma^{1-\alpha}\tau^{1-\alpha};$$

$$\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)} \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{(t-t_{n-\frac{1}{2}})f''(t_{n-\frac{1}{2}})}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t \\
&= \frac{1}{\Gamma(1-\alpha)} f''(t_{n-\frac{1}{2}}) \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{t-t_{n-\frac{1}{2}}}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t \\
&= \frac{1}{\Gamma(1-\alpha)} f''(t_{n-\frac{1}{2}}) \cdot \frac{\sigma^{1-\alpha}}{(1-\alpha)(2-\alpha)}\left[\sigma - \left(1-\frac{\alpha}{2}\right)\right]\tau^{2-\alpha} = 0
\end{aligned} \tag{1.84}$$

and

$$\begin{aligned}
&\left|\frac{1}{\Gamma(1-\alpha)} \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{\frac{1}{2}(t-t_{n-\frac{1}{2}})^2 f'''(\eta_n)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t\right| \\
&\leqslant \frac{\tau^2}{8\Gamma(1-\alpha)} \max_{t_{n-1}\leqslant t\leqslant t_n} |f'''(t)| \cdot \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{1}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t \\
&= \frac{\tau^2}{8\Gamma(1-\alpha)} \max_{t_{n-1}\leqslant t\leqslant t_n} |f'''(t)| \cdot \frac{1}{1-\alpha}\sigma^{1-\alpha}\tau^{1-\alpha}.
\end{aligned}$$

Therefore,

$$|R_2^n| \leqslant \frac{\sigma^{1-\alpha}}{6\Gamma(2-\alpha)} \max_{t_{n-1}\leqslant t\leqslant t_n} |f'''(t)|\tau^{3-\alpha}. \tag{1.85}$$

The substitution of (1.83) and (1.85) into (1.82) leads to

$$|R^n| \leqslant \frac{(4\sigma-1)\sigma^{-\alpha}}{12\Gamma(2-\alpha)} \max_{t_0\leqslant t\leqslant t_n} |f'''(t)|\tau^{3-\alpha}.$$

The proof ends. □

Remark 1.6.2. The parameter σ is chosen as $1-\frac{\alpha}{2}$ such that the integral in (1.84)

$$\int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{t-t_{n-\frac{1}{2}}}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t$$

happens to be 0. If $\sigma \neq 1-\frac{\alpha}{2}$, then

$${}_0^C D_t^\alpha f(t)|_{t=t_{n-1+\sigma}} - \Delta_t^\alpha f(t_{n-1+\sigma}) = O(\tau^{2-\alpha}).$$

Lemma 1.6.3. [1] *Suppose* $\alpha \in (0,1)$, $\sigma = 1 - \frac{\alpha}{2}$, $c_k^{(n,\alpha)} (0 \leqslant k \leqslant n-1, n \geqslant 1)$ *is defined in* (1.77)–(1.80), *then it holds*

$$c_0^{(n,\alpha)} > c_1^{(n,\alpha)} > c_2^{(n,\alpha)} > \cdots > c_{n-2}^{(n,\alpha)} > c_{n-1}^{(n,\alpha)} > (1-\alpha)n^{-\alpha}, \tag{1.86}$$

$$(2\sigma - 1)c_0^{(n,\alpha)} - \sigma c_1^{(n,\alpha)} > 0. \tag{1.87}$$

Proof. It is easy to know from (1.79)–(1.80) that

$$c_1^{(n,\alpha)} > c_2^{(n,\alpha)} > \cdots > c_{n-2}^{(n,\alpha)} > c_{n-1}^{(n,\alpha)} > (1-\alpha)n^{-\alpha}.$$

If $n = 1$,

$$c_0^{(n,\alpha)} = \sigma^{1-\alpha} = \left(1 - \frac{\alpha}{2}\right)^{1-\alpha} > 1 - \frac{\alpha}{2} > 1 - \alpha,$$

hence, (1.86) is true.

If $n \geqslant 2$ and (1.87) is valid, then we have

$$c_0^{(n,\alpha)} - c_1^{(n,\alpha)} = \frac{1}{\sigma}\left[(2\sigma-1)c_0^{(n,\alpha)} - \sigma c_1^{(n,\alpha)}\right] + \frac{\alpha}{2\sigma} c_0^{(n,\alpha)} > 0.$$

Consequently, we only need to prove (1.87).

(I) If $n = 2$, we have

$$\begin{aligned}
&(2\sigma - 1)c_0^{(n,\alpha)} - \sigma c_1^{(n,\alpha)} \\
&= (2\sigma - 1)\left[\frac{(1+\sigma)^{2-\alpha} - \sigma^{2-\alpha}}{2-\alpha} - \frac{(1+\sigma)^{1-\alpha} - \sigma^{1-\alpha}}{2}\right] \\
&\quad - \sigma\left[\frac{1}{2}\left(3(1+\sigma)^{1-\alpha} - \sigma^{1-\alpha}\right) - \frac{1}{2-\alpha}\left((1+\sigma)^{2-\alpha} - \sigma^{2-\alpha}\right)\right] \\
&= \frac{1}{2}\left(3 - 2\sigma - \frac{1}{\sigma}\right)(1+\sigma)^{1-\alpha} \\
&= \frac{1}{2\sigma}(2\sigma - 1)(1-\sigma)(1+\sigma)^{1-\alpha} \\
&> 0.
\end{aligned}$$

(II) If $n \geqslant 3$, we have

$$\begin{aligned}
&(2\sigma - 1)c_0^{(n,\alpha)} - \sigma c_1^{(n,\alpha)} \\
&= (2\sigma - 1)\left[\frac{(1+\sigma)^{2-\alpha} - \sigma^{2-\alpha}}{2-\alpha} - \frac{(1+\sigma)^{1-\alpha} - \sigma^{1-\alpha}}{2}\right] \\
&\quad - \sigma\left\{\frac{1}{2-\alpha}\left[(2+\sigma)^{2-\alpha} - 2(1+\sigma)^{2-\alpha} + \sigma^{2-\alpha}\right]\right. \\
&\quad \left. - \frac{1}{2}\left[(2+\sigma)^{1-\alpha} - 2(1+\sigma)^{1-\alpha} + \sigma^{1-\alpha}\right]\right\}
\end{aligned}$$

$$= \frac{4\sigma - 1}{2\sigma}(1+\sigma)^{1-\alpha} - (2+\sigma)^{1-\alpha}.$$

Noticing

$$\begin{aligned}(2+\sigma)^{1-\alpha} &= (1+\sigma)^{1-\alpha}\left(1+\frac{1}{1+\sigma}\right)^{1-\alpha} \\ &\leqslant (1+\sigma)^{1-\alpha}\left(1+\frac{1-\alpha}{1+\sigma}\right) \\ &= (1+\sigma)^{1-\alpha}\frac{3\sigma}{1+\sigma},\end{aligned}$$

we obtain

$$\begin{aligned}&(2\sigma-1)c_0^{(n,\alpha)} - \sigma c_1^{(n,\alpha)} \\ \geqslant\ & \frac{4\sigma-1}{2\sigma}(1+\sigma)^{1-\alpha} - (1+\sigma)^{1-\alpha}\frac{3\sigma}{1+\sigma} \\ =\ & \frac{(2\sigma-1)(1-\sigma)}{2\sigma(1+\sigma)}(1+\sigma)^{1-\alpha} > 0.\end{aligned}$$

This completes the proof. □

1.6.4 L2-1_σ approximation of multi-term fractional derivatives

In [19], Gao et al. considered L2-1_σ interpolation approximation of multiterm Caputo fractional derivatives

$$\mathbf{D}_t f(t) = \sum_{r=0}^{m} \lambda_r {}_0^C D_t^{\alpha_r} f(t), \tag{1.88}$$

where λ_r, $r = 0, 1, \ldots, m$ are positive constants, $0 \leqslant \alpha_m < \alpha_{m-1} < \cdots < \alpha_0 \leqslant 1$, and at least one $\alpha_r \in (0,1)$, ${}_0^C D_t^{\alpha} f(t)$ is defined by

$${}_0^C D_t^{\alpha} f(t) = \begin{cases} f(t) - f(0), & \alpha = 0, \\ \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(s)(t-s)^{-\alpha} \mathrm{d}s, & \alpha \in (0,1), \\ f'(t), & \alpha = 1. \end{cases}$$

Denote

$$t_n = n\tau, \quad n = 0, 1, 2, \ldots,$$

$$a = \min_{0\leqslant r\leqslant m}\left\{1 - \frac{\alpha_r}{2}\right\}, \quad b = \max_{0\leqslant r\leqslant m}\left\{1 - \frac{\alpha_r}{2}\right\}.$$

It is easy to know that

$$a = 1 - \frac{\alpha_0}{2} \geqslant \frac{1}{2}, \quad b = 1 - \frac{\alpha_m}{2} \leqslant 1.$$

Define

$$F(\sigma) = \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(3-\alpha_r)} \sigma^{1-\alpha_r}\left[\sigma - \left(1 - \frac{\alpha_r}{2}\right)\right]\tau^{2-\alpha_r}, \quad \sigma > 0.$$

Two useful lemmas will be stated below.

Lemma 1.6.4. *The equation $F(\sigma) = 0$ has a unique positive root $\sigma^* \in [a, b]$.*

Proof. When $m = 0$, $F(\sigma) = \frac{\lambda_0}{\Gamma(3-\alpha_0)} \sigma^{1-\alpha_0}[\sigma - (1 - \frac{\alpha_0}{2})]\tau^{2-\alpha_0}$. It is easy to know that the equation $F(\sigma) = 0$ has a unique root $\sigma^* = 1 - \frac{\alpha_0}{2}$.

Now, we suppose $m \geqslant 1$. When $0 \leqslant \sigma \leqslant a$, $F(\sigma) \leqslant 0$. When $\sigma \geqslant b$, $F(\sigma) \geqslant 0$. When $\sigma \in [a, b]$,

$$F'(\sigma) = \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(2-\alpha_r)} \sigma^{-\alpha_r}\left[\sigma - \frac{1}{2}(1-\alpha_r)\right]\tau^{2-\alpha_r} > 0.$$

Thus, the equation $F(\sigma) = 0$ has a unique root $\sigma^* \in [a, b]$. The proof ends. □

Lemma 1.6.5. *Suppose $m \geqslant 1$. The* Newton *iteration sequence $\{\sigma_k\}_{k=0}^{\infty}$, generated by*

$$\begin{cases} \sigma_{k+1} = \sigma_k - \frac{F(\sigma_k)}{F'(\sigma_k)}, & k = 0, 1, 2, \ldots, \\ \sigma_0 = b \end{cases} \tag{1.89}$$

is monotone decreasing and convergent to σ^*.

Proof. In view of the proof for Lemma 1.6.4, we have $F(a) < 0$, $F(b) > 0$, and when $\sigma \in [a, b]$, $F'(\sigma) > 0$. In addition, when $\sigma \in [a, b]$,

$$F''(\sigma) = \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \sigma^{-\alpha_r-1}\left(\sigma + \frac{1}{2}\alpha_r\right)\tau^{2-\alpha_r} > 0.$$

Noticing

$$F(\sigma_0)F''(\sigma_0) > 0,$$

the Newton iteration sequence $\{\sigma_k\}_{k=0}^{\infty}$ generated by (1.89) is monotone decreasing and convergent to σ^* [82]. The proof ends. □

For simplicity, we denote $\sigma = \sigma^*$ in this subsection, which implies this $\sigma \in [\frac{1}{2}, 1]$ satisfying $F(\sigma) = 0$.

In addition, denote $t_{n-1+\sigma} = (n - 1 + \sigma)\tau$.

When $n = 1$,

$${}_0^C D_t^{\alpha} f(t)|_{t=t_{n-1+\sigma}} = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_\sigma} f'(t)(t_\sigma - t)^{-\alpha} \mathrm{d}t.$$

When $n \geqslant 2$,

$$
{}_0^C D_t^\alpha f(t)|_{t=t_{n-1+\sigma}} = \frac{1}{\Gamma(1-\alpha)}\Bigg[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} f'(t)(t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t + \int_{t_{n-1}}^{t_{n-1+\sigma}} f'(t)(t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t\Bigg].
$$

The following theorem will give a numerical approximation formula of (1.88) at the point $t = t_{n-1+\sigma}$ and reveal its numerical accuracy.

Theorem 1.6.5. *Suppose $f \in C^3[t_0, t_n]$. Let*

$$
\mathcal{D}_t f(t_{n-1+\sigma}) \equiv \sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Bigg[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L'_{2,k}(t)(t_{n-1+\sigma}-t)^{-\alpha_r}\mathrm{d}t + \int_{t_{n-1}}^{t_{n-1+\sigma}} L'_{1,n}(t)(t_{n-1+\sigma}-t)^{-\alpha_r}\mathrm{d}t\Bigg].
$$

Then we have the following error estimate:

$$
\begin{aligned}
&\left|\mathbf{D}_t f(t_{n-1+\sigma}) - \mathcal{D}_t f(t_{n-1+\sigma})\right| \\
&\leqslant \sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(2-\alpha_r)}\cdot\left(\frac{1-\alpha_r}{12}+\frac{\sigma}{6}\right)\sigma^{-\alpha_r}\tau^{3-\alpha_r}\cdot\max_{t_0\leqslant t\leqslant t_n}|f'''(t)| = O(\tau^{3-\alpha_0}).
\end{aligned}
$$

Proof. It is easy to know that

$$
\begin{aligned}
&\mathbf{D}_t f(t_{n-1+\sigma}) - \mathcal{D}_t f(t_{n-1+\sigma}) \\
&= \sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Bigg[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}(f'(t)-L'_{2,k}(t))(t_{n-1+\sigma}-t)^{-\alpha_r}\mathrm{d}t \\
&\quad + \int_{t_{n-1}}^{t_{n-1+\sigma}}(f'(t)-L'_{1,n}(t))(t_{n-1+\sigma}-t)^{-\alpha_r}\mathrm{d}t\Bigg] \\
&= \sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}(f'(t)-L'_{2,k}(t))(t_{n-1+\sigma}-t)^{-\alpha_r}\mathrm{d}t \\
&\quad + \sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\int_{t_{n-1}}^{t_{n-1+\sigma}}(f'(t)-L'_{1,n}(t))(t_{n-1+\sigma}-t)^{-\alpha_r}\mathrm{d}t.
\end{aligned}
\tag{1.90}
$$

Denote

$$
M = \max_{t_0\leqslant t\leqslant t_n}|f'''(t)|.
$$

Since

$$\begin{aligned}A_r &\equiv \sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}(f'(t)-L'_{2,k}(t))(t_{n-1+\sigma}-t)^{-\alpha_r}\mathrm{d}t\\ &=\sum_{k=1}^{n-1}\bigg[(f(t)-L_{2,k}(t))(t_{n-1+\sigma}-t)^{-\alpha_r}|_{t=t_{k-1}}^{t_k}\\ &\quad-\int_{t_{k-1}}^{t_k}(f(t)-L_{2,k}(t))\alpha_r(t_{n-1+\sigma}-t)^{-\alpha_r-1}\mathrm{d}t\bigg]\\ &=-\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}(f(t)-L_{2,k}(t))\alpha_r(t_{n-1+\sigma}-t)^{-\alpha_r-1}\mathrm{d}t\end{aligned}$$

and

$$\max_{t_{k-1}\leqslant t\leqslant t_k}\left|f(t)-L_{2,k}(t)\right|\leqslant\frac{1}{12}M\tau^3,$$

we have

$$\begin{aligned}|A_r| &\leqslant \sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\left|f(t)-L_{2,k}(t)\right|\alpha_r(t_{n-1+\sigma}-t)^{-\alpha_r-1}\mathrm{d}t\\ &\leqslant\frac{1}{12}M\tau^3\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\alpha_r(t_{n-1+\sigma}-t)^{-\alpha_r-1}\mathrm{d}t\\ &=\frac{1}{12}M\tau^3\int_{t_0}^{t_{n-1}}\alpha_r(t_{n-1+\sigma}-t)^{-\alpha_r-1}\mathrm{d}t\\ &=\frac{1}{12}M\tau^3\big[(t_{n-1+\sigma}-t_{n-1})^{-\alpha_r}-(t_{n-1+\sigma}-t_0)^{-\alpha_r}\big]\\ &\leqslant\frac{1}{12}M\tau^3\cdot(\sigma\tau)^{-\alpha_r}\\ &=\frac{1}{12}M\sigma^{-\alpha_r}\tau^{3-\alpha_r}.\end{aligned}\tag{1.91}$$

For the second term of (1.90), according to

$$\begin{aligned}f'(t)-L'_{1,n}(t) &= f'(t)-\delta_t f^{n-\frac{1}{2}}\\ &=\big[f'(t)-f'(t_{n-\frac{1}{2}})\big]+\big[f'(t_{n-\frac{1}{2}})-\delta_t f^{n-\frac{1}{2}}\big]\\ &=\Big[f''(t_{n-\frac{1}{2}})(t-t_{n-\frac{1}{2}})+\frac{1}{2}f'''(\eta_n)(t-t_{n-\frac{1}{2}})^2\Big]-\frac{1}{24}\tau^2f'''(\tilde{\eta}_n),\\ &\qquad t\in[t_{n-1},t_n],\quad \eta_n,\ \tilde{\eta}_n\in(t_{n-1},t_n),\end{aligned}$$

we have

$$\begin{aligned}
B &\equiv \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \int_{t_{n-1}}^{t_{n-1+\sigma}} [f'(t) - L'_{1,n}(t)](t_{n-1+\sigma} - t)^{-\alpha_r} \mathrm{d}t \\
&= \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \int_{t_{n-1}}^{t_{n-1+\sigma}} \Big[f''(t_{n-\frac{1}{2}})(t - t_{n-\frac{1}{2}}) + \frac{1}{2} f'''(\eta_n)(t - t_{n-\frac{1}{2}})^2 \\
&\quad - \frac{1}{24}\tau^2 f'''(\tilde{\eta}_n)\Big](t_{n-1+\sigma} - t)^{-\alpha_r} \mathrm{d}t \\
&= f''(t_{n-\frac{1}{2}}) \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \int_{t_{n-1}}^{t_{n-1+\sigma}} (t - t_{n-\frac{1}{2}})(t_{n-1+\sigma} - t)^{-\alpha_r} \mathrm{d}t \\
&\quad + \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \int_{t_{n-1}}^{t_{n-1+\sigma}} \Big[\frac{1}{2} f'''(\eta_n)(t - t_{n-\frac{1}{2}})^2 \\
&\quad - \frac{1}{24}\tau^2 f'''(\tilde{\eta}_n)\Big](t_{n-1+\sigma} - t)^{-\alpha_r} \mathrm{d}t.
\end{aligned}$$

Noticing $F(\sigma) = 0$, we know that

$$\begin{aligned}
&\sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \int_{t_{n-1}}^{t_{n-1+\sigma}} (t - t_{n-\frac{1}{2}})(t_{n-1+\sigma} - t)^{-\alpha_r} \mathrm{d}t \\
&= \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \int_{0}^{\sigma\tau} \Big[\Big(\sigma - \frac{1}{2}\Big)\tau - \xi\Big]\xi^{-\alpha_r} \mathrm{d}\xi \\
&= \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \Big[\Big(\sigma - \frac{1}{2}\Big)\tau \frac{(\sigma\tau)^{1-\alpha_r}}{1-\alpha_r} - \frac{(\sigma\tau)^{2-\alpha_r}}{2-\alpha_r}\Big] \\
&= \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(3-\alpha_r)} \sigma^{1-\alpha_r} \Big[\sigma - \Big(1 - \frac{\alpha_r}{2}\Big)\Big]\tau^{2-\alpha_r} \\
&= F(\sigma) \\
&= 0.
\end{aligned}$$

Therefore,

$$B = \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \int_{t_{n-1}}^{t_{n-1+\sigma}} \Big[\frac{1}{2} f'''(\eta_n)(t - t_{n-\frac{1}{2}})^2 - \frac{1}{24}\tau^2 f'''(\tilde{\eta}_n)\Big](t_{n-1+\sigma} - t)^{-\alpha_r} \mathrm{d}t.$$

Furthermore, we have

$$|B| \leqslant \frac{1}{6} M \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \cdot \frac{\sigma^{1-\alpha_r}}{1-\alpha_r} \tau^{3-\alpha_r}. \tag{1.92}$$

Substituting (1.91) and (1.92) into (1.90), we have

$$\begin{aligned}&\left|\mathbf{D}_t f(t_{n-1+\sigma}) - \mathcal{D}_t f(t_{n-1+\sigma})\right| \\ &\leqslant M \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \cdot \left(\frac{1}{12} + \frac{1}{6} \cdot \frac{\sigma}{1-\alpha_r}\right) \sigma^{-\alpha_r} \tau^{3-\alpha_r}.\end{aligned}$$

This completes the proof. □

With application of (1.81), we have

$$\begin{aligned}\mathcal{D}_t f(t_{n-1+\sigma}) &\equiv \sum_{r=0}^{m} \lambda_r \cdot \frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)} \sum_{k=0}^{n-1} c_k^{(n,\alpha_r)} [f(t_{n-k}) - f(t_{n-k-1})] \\ &= \sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} [f(t_{n-k}) - f(t_{n-k-1})],\end{aligned}$$

where

$$\hat{c}_k^{(n,\alpha)} = \sum_{r=0}^{m} \lambda_r \cdot \frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)} c_k^{(n,\alpha_r)}, \quad 0 \leqslant k \leqslant n-1, \tag{1.93}$$

and $\{c_k^{(n,\alpha_r)}\}$ is defined by (1.77)–(1.80).

The following two lemmas give the properties of coefficient $\{\hat{c}_k^{(n,\alpha)}\}$.

Lemma 1.6.6. *Given any nonnegative integer m, and positive constants $\lambda_0, \lambda_1, \ldots, \lambda_m$, for any $\alpha_r \in [0,1]$ $(0 \leqslant r \leqslant m)$, where at least one $\alpha_r \in (0,1)$, it holds the following inequalities:*

$$\hat{c}_1^{(n,\alpha)} > \hat{c}_2^{(n,\alpha)} > \cdots > \hat{c}_{n-2}^{(n,\alpha)} > \hat{c}_{n-1}^{(n,\alpha)} > \sum_{r=0}^{m} \lambda_r \frac{\tau^{-\alpha_r}}{\Gamma(1-\alpha_r)} n^{-\alpha_r}. \tag{1.94}$$

Proof. When $m = 0$, the conclusion has been obtained in Lemma 1.6.3. Now, we suppose $m \geqslant 1$. For any $\alpha_r \in (0,1)$, we have (see Lemma 1.6.3)

$$c_1^{(n,\alpha_r)} > c_2^{(n,\alpha_r)} > \cdots > c_{n-2}^{(n,\alpha_r)} > c_{n-1}^{(n,\alpha_r)} > (1-\alpha_r) n^{-\alpha_r}. \tag{1.95}$$

In particular, if $\alpha_r = 0$, we have

$$c_1^{(n,\alpha_r)} = c_2^{(n,\alpha_r)} = \cdots = c_{n-1}^{(n,\alpha_r)} = 1 = (1-0) n^{-0};$$

If $\alpha_r = 1$, we have

$$c_1^{(n,\alpha_r)} = c_2^{(n,\alpha_r)} = \cdots = c_{n-1}^{(n,\alpha_r)} = 0 = (1-1) n^{-1}. \tag{1.96}$$

Combining (1.95) with (1.96) and noticing that at least one $\alpha_r \in (0,1)$, the conclusion (1.94) holds. The proof ends. □

Lemma 1.6.7. *Given any nonnegative integer m, positive constants $\lambda_0, \lambda_1, \ldots, \lambda_m$, for any $\alpha_r \in [0,1]$ $(0 \leqslant r \leqslant m)$, where at least one $\alpha_r \in (0,1)$, then there exists a positive constant τ_0, such that, when $\tau \leqslant \tau_0$, it holds*

$$(2\sigma - 1)\hat{c}_0^{(n,\alpha)} - \sigma\hat{c}_1^{(n,\alpha)} > 0, \tag{1.97}$$

which implies

$$\hat{c}_0^{(n,\alpha)} > \hat{c}_1^{(n,\alpha)}.$$

Proof. If $m = 0$, the conclusion can be found in Lemma 1.6.3. Now we suppose $m \geqslant 1$, hence $\sigma \in (\frac{1}{2}, 1)$.

(I) When $n \geqslant 3$, for every $\alpha_r \in (0,1)$, we have (see Lemma 1.6.3)

$$\begin{aligned}
&(2\sigma - 1)c_0^{(n,\alpha_r)} - \sigma c_1^{(n,\alpha_r)} \\
&= (2\sigma - 1)\left[\frac{(1+\sigma)^{2-\alpha_r} - \sigma^{2-\alpha_r}}{2-\alpha_r} - \frac{(1+\sigma)^{1-\alpha_r} - \sigma^{1-\alpha_r}}{2}\right] \\
&\quad - \sigma\left\{\frac{1}{2-\alpha_r}\left[(2+\sigma)^{2-\alpha_r} - 2(1+\sigma)^{2-\alpha_r} + \sigma^{2-\alpha_r}\right]\right. \\
&\quad \left. - \frac{1}{2}\left[(2+\sigma)^{1-\alpha_r} - 2(1+\sigma)^{1-\alpha_r} + \sigma^{1-\alpha_r}\right]\right\} \\
&= -\frac{s_r(2+\sigma) - \sigma}{2}(2+\sigma)^{1-\alpha_r} + \frac{(4\sigma^2 + 3\sigma - 1)s_r - 4\sigma^2 + \sigma}{2\sigma}(1+\sigma)^{1-\alpha_r} \\
&\quad - \frac{3\sigma - 1}{2}(s_r - 1)\sigma^{1-\alpha_r},
\end{aligned}$$

where $s_r = \frac{\sigma}{1-\alpha_r/2}$, $r = 0, 1, \ldots, m$.

Noticing

$$\begin{aligned}
(2+\sigma)^{1-\alpha_r} &= (1+\sigma)^{1-\alpha_r}\left(1 + \frac{1}{1+\sigma}\right)^{1-\alpha_r} \\
&\leqslant (1+\sigma)^{1-\alpha_r}\left(1 + \frac{1-\alpha_r}{1+\sigma}\right) \\
&= (1+\sigma)^{1-\alpha_r}\frac{\sigma s_r + 2\sigma}{s_r(1+\sigma)},
\end{aligned}$$

we have

$$\begin{aligned}
&(2\sigma - 1)c_0^{(n,\alpha_r)} - \sigma c_1^{(n,\alpha_r)} \\
&\geqslant \frac{1}{2}\left[\left(3\sigma^2 + 5\sigma + 2 - \frac{1}{\sigma}\right)s_r + \frac{2\sigma^2}{s_r} - 5\sigma^2 - 7\sigma + 1\right](1+\sigma)^{-\alpha_r} \\
&\quad - \frac{3\sigma - 1}{2}(s_r - 1)\sigma^{1-\alpha_r}.
\end{aligned} \tag{1.98}$$

For $\sigma \in (1/2, 1)$, consider the function

$$f_\sigma(t) = \left(3\sigma^2 + 5\sigma + 2 - \frac{1}{\sigma}\right)t + \frac{2\sigma^2}{t} - 5\sigma^2 - 7\sigma + 1.$$

When $t \geqslant 1$,

$$\begin{aligned} f'_\sigma(t) &= \left(3\sigma^2 + 5\sigma + 2 - \frac{1}{\sigma}\right) - \frac{2\sigma^2}{t^2} \\ &\geqslant \left(3\sigma^2 + 5\sigma + 2 - \frac{1}{\sigma}\right) - 2\sigma^2 \\ &\geqslant \sigma^2 + 5\sigma > 0. \end{aligned}$$

Due to $s_0 > 1$, we have

$$f_\sigma(s_0) > f_\sigma(1) = \frac{(2\sigma - 1)(1 - \sigma)}{\sigma} > 0.$$

Noticing

$$\sum_{r=0}^{m} \lambda_r \frac{\tau^{-\alpha_r}}{\Gamma(2 - \alpha_r)} \sigma^{1-\alpha_r}(s_r - 1) = \frac{2}{\tau^2} F(\sigma) = 0,$$

with the help of (1.98), we have

$$\begin{aligned} &(2\sigma - 1)\hat{c}_0^{(n,\alpha)} - \sigma\hat{c}_1^{(n,\alpha)} \\ &= \sum_{r=0}^{m} \lambda_r \frac{\tau^{-\alpha_r}}{\Gamma(2 - \alpha_r)} \left[(2\sigma - 1)c_0^{(n,\alpha_r)} - \sigma c_1^{(n,\alpha_r)}\right] \\ &\geqslant \frac{1}{2} \sum_{r=0}^{m} \lambda_r \frac{\tau^{-\alpha_r}}{\Gamma(2 - \alpha_r)} f_\sigma(s_r)(1 + \sigma)^{-\alpha_r} \\ &\quad - \frac{3\sigma - 1}{2} \sum_{r=0}^{m} \lambda_r \frac{\tau^{-\alpha_r}}{\Gamma(2 - \alpha_r)} \sigma^{1-\alpha_r}(s_r - 1) \\ &= \frac{1}{2} \sum_{r=0}^{m} \lambda_r \frac{\tau^{-\alpha_r}}{\Gamma(2 - \alpha_r)} f_\sigma(s_r)(1 + \sigma)^{-\alpha_r} \\ &= \frac{1}{2}\lambda_0 \frac{\tau^{-\alpha_0}}{\Gamma(2 - \alpha_0)} f_\sigma(s_0)(1 + \sigma)^{-\alpha_0} \\ &\quad + \frac{1}{2} \sum_{r=1}^{m} \lambda_r \frac{\tau^{-\alpha_r}}{\Gamma(2 - \alpha_r)} f_\sigma(s_r)(1 + \sigma)^{-\alpha_r} \\ &> \frac{1}{2}\lambda_0 \frac{\tau^{-\alpha_0}}{\Gamma(2 - \alpha_0)} (1 + \sigma)^{-\alpha_0} \Bigg[f_\sigma(1) \\ &\quad + \sum_{r=1}^{m} \lambda_r \frac{\tau^{\alpha_0-\alpha_r}\Gamma(2 - \alpha_0)}{\lambda_0 \Gamma(2 - \alpha_r)} f_\sigma(s_r)(1 + \sigma)^{\alpha_0-\alpha_r} \Bigg] \end{aligned}$$

$$= \frac{1}{2}\lambda_0 \frac{\tau^{-\alpha_0}}{\Gamma(2-\alpha_0)}(1+\sigma)^{-\alpha_0}[f_\sigma(1) + O(\tau^{\alpha_0-\alpha_1})]. \tag{1.99}$$

Since $\alpha_0 - \alpha_1 > 0$, there exists a positive constant τ_0 such that $f_\sigma(1) + O(\tau^{\alpha_0-\alpha_1}) \geqslant 0$ when $\tau \leqslant \tau_0$. Therefore, the inequality (1.97) holds.

(II) When $n = 2$, we have

$$\begin{aligned}
&(2\sigma-1)c_0^{(2,\alpha_r)} - \sigma c_1^{(2,\alpha_r)} \\
&= (2\sigma-1)\left[\frac{(1+\sigma)^{2-\alpha_r} - \sigma^{2-\alpha_r}}{2-\alpha_r} - \frac{(1+\sigma)^{1-\alpha_r} - \sigma^{1-\alpha_r}}{2}\right] \\
&\quad - \sigma\left\{\frac{1}{2}[3(1+\sigma)^{1-\alpha_r} - \sigma^{1-\alpha_r}] - \frac{1}{2-\alpha_r}[(1+\sigma)^{2-\alpha_r} - \sigma^{2-\alpha_r}]\right\} \\
&= \frac{1}{2}\left[\frac{(3\sigma-1)(1+\sigma)}{\sigma}s_r + 1 - 5\sigma\right](1+\sigma)^{1-\alpha_r} + \frac{1}{2}(3\sigma-1)(1-s_r)\sigma^{1-\alpha_r}.
\end{aligned}$$

For $\sigma \in (\frac{1}{2}, 1)$, consider the function

$$g_\sigma(t) = \frac{(3\sigma-1)(1+\sigma)}{\sigma}t + 1 - 5\sigma.$$

It is easy to know that

$$g_\sigma(s_0) > g_\sigma(1) = \frac{(3\sigma-1)(1+\sigma)}{\sigma} + 1 - 5\sigma = 3 - 2\sigma - \frac{1}{\sigma} > 0, \quad \sigma \in \left(\frac{1}{2}, 1\right).$$

Similar to the proof of (1.99), there exists a positive constant τ_0, such that the inequality (1.97) holds when $\tau \leqslant \tau_0$.

This completes the proof. □

According to Theorem 1.6.5, we know that the accuracy is no less than second order when using $\mathcal{D}_t f(t_{n-1+\sigma})$ to approximate the value of the sum (1.88) of the multi-term Caputo derivatives at $t = t_{n-1+\sigma}$. In the aftermentioned chapters, we will develop some high order accurate difference schemes for solving the multi-term time-fractional differential equations based on the approximation formula in Theorem 1.6.5.

1.6.5 H2N2 approximation

In [56] and [61], the authors presented L2 method and L2C method (a variant of L2 method) to approximate the R-L fractional derivative. In [46], the authors applied L2 method and L2C method to treat the approximation of Caputo derivative

$${}_0^C D_t^\gamma f(t) = \frac{1}{\Gamma(2-\gamma)}\int_0^t f''(s)(t-s)^{1-\gamma}\mathrm{d}s, \quad \gamma \in (1,2).$$

L2 method:

$$
\begin{aligned}
{}_0^C D_t^\gamma f(t)|_{t=t_n} &= \frac{1}{\Gamma(2-\gamma)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f''(s)(t_n - s)^{1-\gamma} \mathrm{d}s \\
&\approx \frac{1}{\Gamma(2-\gamma)} \sum_{k=1}^{n} \frac{f(t_{k-1}) - 2f(t_k) + f(t_{k+1})}{\tau^2} \cdot \int_{t_{k-1}}^{t_k} (t_n - s)^{1-\gamma} \mathrm{d}s.
\end{aligned}
$$

L2C method:

$$
\begin{aligned}
{}_0^C D_t^\gamma f(t)|_{t=t_n} &= \frac{1}{\Gamma(2-\gamma)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f''(s)(t_n - s)^{1-\gamma} \mathrm{d}s \\
&\approx \frac{1}{\Gamma(2-\gamma)} \sum_{k=1}^{n} \frac{f(t_{k-2}) - f(t_{k-1}) - f(t_k) + f(t_{k+1})}{2\tau^2} \cdot \int_{t_{k-1}}^{t_k} (t_n - s)^{1-\gamma} \mathrm{d}s.
\end{aligned}
$$

In this subsection, we introduce the H2N2 interpolation approximation method[71]. Consider the numerical evaluation of

$$
{}_0^C D_t^\gamma f(t)|_{t=t_{n-\frac{1}{2}}} = \frac{1}{\Gamma(2-\gamma)} \int_{t_0}^{t_{n-\frac{1}{2}}} f''(t)(t_{n-\frac{1}{2}} - t)^{1-\gamma} \mathrm{d}t, \quad 1 \leqslant n \leqslant N,
$$

which can be written as

$$
{}_0^C D_t^\gamma f(t)|_{t=t_{n-\frac{1}{2}}} = \frac{1}{\Gamma(2-\gamma)} \left[\int_{t_0}^{t_{\frac{1}{2}}} f''(t)(t_{n-\frac{1}{2}} - t)^{1-\gamma} \mathrm{d}t + \sum_{k=1}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} f''(t)(t_{n-\frac{1}{2}} - t)^{1-\gamma} \mathrm{d}t \right].
$$

Using the data $(t_0, f(t_0))$, $(t_0, f'(t_0))$ and $(t_1, f(t_1))$, the quadratic Hermite interpolation polynomial of $f(t)$ reads

$$
H_{2,0}(t) = f(t_0) + f'(t_0)(t - t_0) + \frac{1}{\tau}(\delta_t f^{\frac{1}{2}} - f'(t_0))(t - t_0)^2.
$$

It is easy to know that

$$
H''_{2,0}(t) = \frac{2}{\tau}(\delta_t f^{\frac{1}{2}} - f'(t_0)) \tag{1.100}
$$

and there exists a $\zeta_0 \in (t_0, t_1)$ satisfying

$$
\frac{2}{\tau}(\delta_t f^{\frac{1}{2}} - f'(t_0)) = f''(\zeta_0). \tag{1.101}
$$

Using three points $(t_{k-1}, f(t_{k-1}))$, $(t_k, f(t_k))$ and $(t_{k+1}, f(t_{k+1}))$, the quadratic Newton interpolation polynomial of $f(t)$ reads

$$N_{2,k}(t) = f(t_{k-1}) + \big(\delta_t f^{k-\frac{1}{2}}\big)(t - t_{k-1}) + \frac{1}{2}(\delta_t^2 f^k)(t - t_{k-1})(t - t_k),$$

where

$$\delta_t^2 f^k = \frac{1}{\tau}(\delta_t f^{k+\frac{1}{2}} - \delta_t f^{k-\frac{1}{2}}).$$

It is easy to know that

$$N''_{2,k}(t) = \delta_t^2 f^k \tag{1.102}$$

and there exists a $\zeta_k \in (t_{k-1}, t_{k+1})$ satisfying

$$\delta_t^2 f^k = f''(\zeta_k). \tag{1.103}$$

From (1.100) and (1.102), we have

$$\begin{aligned}
&{}_0^C D_t^\gamma f(t)|_{t=t_{n-\frac{1}{2}}} \\
&\approx \frac{1}{\Gamma(2-\gamma)}\left[\int_{t_0}^{t_{\frac{1}{2}}} H''_{2,0}(t)(t_{n-\frac{1}{2}} - t)^{1-\gamma}\mathrm{d}t + \sum_{k=1}^{n-1}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N''_{2,k}(t)(t_{n-\frac{1}{2}} - t)^{1-\gamma}\mathrm{d}t\right] \\
&= \frac{1}{\Gamma(2-\gamma)}\left[\int_{t_0}^{t_{\frac{1}{2}}} \frac{2}{\tau}(\delta_t f^{\frac{1}{2}} - f'(t_0))(t_{n-\frac{1}{2}} - t)^{1-\gamma}\mathrm{d}t\right. \\
&\quad \left. + \sum_{k=1}^{n-1}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (\delta_t^2 f^k)(t_{n-\frac{1}{2}} - t)^{1-\gamma}\mathrm{d}t\right] \\
&= \frac{1}{\Gamma(2-\gamma)}\left[\frac{2}{\tau}\int_{t_0}^{t_{\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\gamma}\mathrm{d}t \cdot (\delta_t f^{\frac{1}{2}} - f'(t_0))\right. \\
&\quad \left. + \sum_{k=1}^{n-1}\frac{1}{\tau}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\gamma}\mathrm{d}t \cdot (\delta_t f^{k+\frac{1}{2}} - \delta_t f^{k-\frac{1}{2}})\right] \\
&\equiv \hat{\mathcal{D}}^\gamma f(t_{n-\frac{1}{2}}).
\end{aligned}$$

Let

$$\hat{b}_k^{(n,\gamma)} = \begin{cases} \frac{\tau^{1-\gamma}}{2-\gamma}[(k+1)^{2-\gamma} - k^{2-\gamma}], & 0 \leqslant k \leqslant n-2, \\ \frac{2\tau^{1-\gamma}}{2-\gamma}[(n-\frac{1}{2})^{2-\gamma} - (n-1)^{2-\gamma}], & k = n-1. \end{cases}$$

Computing arrives at

$$\begin{aligned}
&\frac{1}{\tau}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}}(t_{n-\frac{1}{2}}-t)^{1-\gamma}\mathrm{d}t \\
&=\frac{1}{\tau}\cdot\frac{1}{2-\gamma}\left[(t_{n-\frac{1}{2}}-t_{k-\frac{1}{2}})^{2-\gamma}-(t_{n-\frac{1}{2}}-t_{k+\frac{1}{2}})^{2-\gamma}\right] \\
&=\frac{\tau^{1-\gamma}}{2-\gamma}\left[(n-k)^{2-\gamma}-(n-k-1)^{2-\gamma}\right] \\
&=\hat{b}_{n-k-1}^{(n,\gamma)},\quad 1\leqslant k\leqslant n-1
\end{aligned}\tag{1.104}$$

and

$$\begin{aligned}
&\frac{2}{\tau}\int_{t_0}^{t_{\frac{1}{2}}}(t_{n-\frac{1}{2}}-t)^{1-\gamma}\mathrm{d}t \\
&=\frac{2}{\tau}\cdot\frac{1}{2-\gamma}\left[(t_{n-\frac{1}{2}}-t_0)^{2-\gamma}-(t_{n-\frac{1}{2}}-t_{\frac{1}{2}})^{2-\gamma}\right] \\
&=\frac{2\tau^{1-\gamma}}{2-\gamma}\left[\left(n-\frac{1}{2}\right)^{2-\gamma}-(n-1)^{2-\gamma}\right] \\
&=\hat{b}_{n-1}^{(n,\gamma)}.
\end{aligned}\tag{1.105}$$

Thus we get the approximation formula of ${}_0^CD_t^{\gamma}f(t)|_{t=t_{n-\frac{1}{2}}}$ as follows:

$$\begin{aligned}
\hat{\mathcal{D}}_t^{\gamma}f(t_{n-\frac{1}{2}})&=\frac{1}{\Gamma(2-\gamma)}\left[\hat{b}_{n-1}^{(n,\gamma)}\cdot(\delta_t f^{\frac{1}{2}}-f'(t_0))+\sum_{k=1}^{n-1}\hat{b}_{n-k-1}^{(n,\gamma)}\cdot(\delta_t f^{k+\frac{1}{2}}-\delta_t f^{k-\frac{1}{2}})\right] \\
&=\frac{1}{\Gamma(2-\gamma)}\left[\hat{b}_0^{(n,\gamma)}\delta_t f^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(\hat{b}_{n-k-1}^{(n,\gamma)}-\hat{b}_{n-k}^{(n,\gamma)})\delta_t f^{k-\frac{1}{2}}-\hat{b}_{n-1}^{(n,\gamma)}f'(t_0)\right].
\end{aligned}\tag{1.106}$$

We call (1.106) the **H2N2 formula** or **H2N2 approximation**.

The coefficients in formula (1.106) satisfy

Lemma 1.6.8.

$$\hat{b}_0^{(n,\gamma)}>\hat{b}_1^{(n,\gamma)}>\hat{b}_2^{(n,\gamma)}>\cdots>\hat{b}_{n-1}^{(n,\gamma)}>0.$$

Proof. Using (1.104), we have

$$\hat{b}_{n-k-1}^{(n,\gamma)}=\frac{1}{\tau}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}}(t_{n-\frac{1}{2}}-t)^{1-\gamma}\mathrm{d}t=\tau^{1-\gamma}\int_0^1(n-k-\xi)^{1-\gamma}\mathrm{d}\xi,\quad 1\leqslant k\leqslant n-1,$$

that is,

$$\hat{b}_k^{(n,\gamma)} = \tau^{1-\gamma} \int_0^1 (k+1-\xi)^{1-\gamma} \mathrm{d}\xi, \quad 0 \leqslant k \leqslant n-2.$$

It is easy to know that

$$\hat{b}_0^{(n,\gamma)} > \hat{b}_1^{(n,\gamma)} > \hat{b}_2^{(n,\gamma)} > \cdots > \hat{b}_{n-2}^{(n,\gamma)}.$$

Particularly,

$$\hat{b}_{n-2}^{(n,\gamma)} = \tau^{1-\gamma} \int_0^1 (n-1-\xi)^{1-\gamma} \mathrm{d}\xi.$$

Using (1.105), we have

$$\hat{b}_{n-1}^{(n,\gamma)} = \frac{2}{\tau} \int_{t_0}^{t_{\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\gamma} \mathrm{d}t = \tau^{1-\gamma} \int_0^1 \left(n - \frac{1+\xi}{2}\right)^{1-\gamma} \mathrm{d}\xi.$$

Noticing that when $\xi \in (0,1)$,

$$(n-1-\xi)^{1-\gamma} > \left(n - \frac{1+\xi}{2}\right)^{1-\gamma},$$

hence,

$$\int_0^1 (n-1-\xi)^{1-\gamma} \mathrm{d}\xi > \int_0^1 \left(n - \frac{1+\xi}{2}\right)^{1-\gamma} \mathrm{d}\xi,$$

which implies that

$$\hat{b}_{n-2}^{(n,\gamma)} > \hat{b}_{n-1}^{(n,\gamma)}.$$

The proof is completed. □

Now we estimate the error.

Theorem 1.6.6. *Suppose $f \in C^3[t_0, t_n]$. Denote*

$$R_n = {}_0^C D_t^\gamma f(t)|_{t=t_{n-\frac{1}{2}}} - \hat{\mathcal{D}}_t^\gamma f(t_{n-\frac{1}{2}}).$$

Then we have

$$|R_n| \leqslant \left[\frac{1}{8\Gamma(2-\gamma)} + \frac{1}{12\Gamma(3-\gamma)} + \frac{\gamma-1}{2\Gamma(4-\gamma)}\right] \max_{t_0 \leqslant t \leqslant t_n} |f'''(t)| \tau^{3-\gamma}.$$

Proof. Let

$$g(t) = f'(t), \ \alpha = \gamma - 1.$$

We have

$$ {}_0^C D_t^\gamma f(t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{f''(s)}{(t-s)^{\gamma-1}} \mathrm{d}s = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g'(s)}{(t-s)^{\alpha}} \mathrm{d}s = {}_0^C D_t^\alpha g(t). \tag{1.107}$$

Make the following piecewise linear interpolations for $g(t)$ as follows:

$$L_{1,0}(t) = \frac{t - t_{\frac{1}{2}}}{t_0 - t_{\frac{1}{2}}} g(t_0) + \frac{t - t_0}{t_{\frac{1}{2}} - t_0} g(t_{\frac{1}{2}}), \quad t \in [t_0, t_{\frac{1}{2}}],$$

$$L_{1,k}(t) = \frac{t - t_{k+\frac{1}{2}}}{t_{k-\frac{1}{2}} - t_{k+\frac{1}{2}}} g(t_{k-\frac{1}{2}}) + \frac{t - t_{k-\frac{1}{2}}}{t_{k+\frac{1}{2}} - t_{k-\frac{1}{2}}} g(t_{k+\frac{1}{2}}), \quad t \in [t_{k-\frac{1}{2}}, t_{k+\frac{1}{2}}],$$
$$1 \leqslant k \leqslant n-1.$$

It is obvious that there exist $\xi_0 \in (t_0, t_{\frac{1}{2}})$, $\xi_k \in (t_{k-\frac{1}{2}}, t_{k+\frac{1}{2}})$, $1 \leqslant k \leqslant n-1$, such that

$$g(t) - L_{1,0}(t) = \frac{1}{2} g''(\xi_0)(t - t_0)(t - t_{\frac{1}{2}}), \quad t \in [t_0, t_{\frac{1}{2}}], \tag{1.108}$$

$$g(t) - L_{1,k}(t) = \frac{1}{2} g''(\xi_k)(t - t_{k-\frac{1}{2}})(t - t_{k+\frac{1}{2}}), \quad t \in [t_{k-\frac{1}{2}}, t_{k+\frac{1}{2}}], \quad 1 \leqslant k \leqslant n-1. \tag{1.109}$$

Noticing

$$\begin{aligned} {}_0^C D_t^\gamma f(t)|_{t=t_{n-\frac{1}{2}}} &= {}_0^C D_t^\alpha g(t)|_{t=t_{n-\frac{1}{2}}} \\ &= \frac{1}{\Gamma(1-\alpha)} \left[\int_{t_0}^{t_{\frac{1}{2}}} \frac{g'(t)}{(t_{n-\frac{1}{2}} - t)^\alpha} \mathrm{d}t + \sum_{k=1}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \frac{g'(t)}{(t_{n-\frac{1}{2}} - t)^\alpha} \mathrm{d}t \right], \end{aligned}$$

we can rewrite $\hat{\mathcal{D}}_t^\gamma f(t_{n-\frac{1}{2}})$ as

$$\begin{aligned} \hat{\mathcal{D}}_t^\gamma f(t_{n-\frac{1}{2}}) &= \frac{1}{\Gamma(2-\gamma)} \left[\frac{2}{\tau} \int_{t_0}^{t_{\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\gamma} \mathrm{d}t \cdot (\delta_t f^{\frac{1}{2}} - f'(t_0)) \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \frac{1}{\tau} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\gamma} \mathrm{d}t \cdot (\delta_t f^{k+\frac{1}{2}} - \delta_t f^{k-\frac{1}{2}}) \right] \\ &= \frac{1}{\Gamma(2-\gamma)} \left[\frac{2}{\tau} \int_{t_0}^{t_{\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\gamma} \mathrm{d}t \cdot (g(t_{\frac{1}{2}}) - g(t_0)) \right. \end{aligned}$$

$$+ \sum_{k=1}^{n-1} \frac{1}{\tau} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\gamma} \mathrm{d}t \cdot \big(g(t_{k+\frac{1}{2}}) - g(t_{k-\frac{1}{2}})\big)\Bigg]$$

$$- \frac{1}{\Gamma(2-\gamma)} \Bigg\{ \frac{2}{\tau} \int_{t_0}^{t_{\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\gamma} \mathrm{d}t \cdot \big(g(t_{\frac{1}{2}}) - \delta_t f^{\frac{1}{2}}\big)$$

$$+ \sum_{k=1}^{n-1} \frac{1}{\tau} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\gamma} \mathrm{d}t \cdot \big[\big(g(t_{k+\frac{1}{2}}) - \delta_t f^{k+\frac{1}{2}}\big)$$

$$- \big(g(t_{k-\frac{1}{2}}) - \delta_t f^{k-\frac{1}{2}}\big)\big]\Bigg\}$$

$$= \frac{1}{\Gamma(1-\alpha)} \Bigg[\int_{t_0}^{t_{\frac{1}{2}}} \frac{g(t_{\frac{1}{2}}) - g(t_0)}{\frac{\tau}{2}} (t_{n-\frac{1}{2}} - t)^{-\alpha} \mathrm{d}t$$

$$+ \sum_{k=1}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \frac{g(t_{k+\frac{1}{2}}) - g(t_{k-\frac{1}{2}})}{\tau} (t_{n-\frac{1}{2}} - t)^{-\alpha} \mathrm{d}t \Bigg]$$

$$- \frac{1}{\Gamma(2-\gamma)} \Bigg\{ \hat{b}_{n-1}^{(n,\gamma)} \big(g(t_{\frac{1}{2}}) - \delta_t f^{\frac{1}{2}}\big)$$

$$+ \sum_{k=1}^{n-1} \hat{b}_{n-k-1}^{(n,\gamma)} \big[\big(g(t_{k+\frac{1}{2}}) - \delta_t f^{k+\frac{1}{2}}\big) - \big(g(t_{k-\frac{1}{2}}) - \delta_t f^{k-\frac{1}{2}}\big)\big]\Bigg\}$$

$$= \frac{1}{\Gamma(1-\alpha)} \Bigg[\int_{t_0}^{t_{\frac{1}{2}}} L_{1,0}'(t)(t_{n-\frac{1}{2}} - t)^{-\alpha} \mathrm{d}t + \sum_{k=1}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} L_{1,k}'(t)(t_{n-\frac{1}{2}} - t)^{-\alpha} \mathrm{d}t \Bigg]$$

$$- \frac{1}{\Gamma(2-\gamma)} \Bigg[\hat{b}_0^{(n,\gamma)} \big(g(t_{n-\frac{1}{2}}) - \delta_t f^{n-\frac{1}{2}}\big)$$

$$- \sum_{k=1}^{n-1} \big(\hat{b}_{n-k-1}^{(n,\gamma)} - \hat{b}_{n-k}^{(n,\gamma)}\big)\big(g(t_{k-\frac{1}{2}}) - \delta_t f^{k-\frac{1}{2}}\big) \Bigg].$$

Thus,

$$R_n = \frac{1}{\Gamma(1-\alpha)} \Bigg[\int_{t_0}^{t_{\frac{1}{2}}} \frac{g'(t) - L_{1,0}'(t)}{(t_{n-\frac{1}{2}} - t)^{\alpha}} \mathrm{d}t + \sum_{k=1}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \frac{g'(t) - L_{1,k}'(t)}{(t_{n-\frac{1}{2}} - t)^{\alpha}} \mathrm{d}t \Bigg]$$

$$+ \frac{1}{\Gamma(2-\gamma)} \Bigg[\hat{b}_0^{(n,\gamma)} \big(g(t_{n-\frac{1}{2}}) - \delta_t f^{n-\frac{1}{2}}\big)$$

$$
\begin{aligned}
&\quad - \sum_{k=1}^{n-1} (\hat{b}_{n-k-1}^{(n,\gamma)} - \hat{b}_{n-k}^{(n,\gamma)})(g(t_{k-\frac{1}{2}}) - \delta_t f^{k-\frac{1}{2}})\Bigg] \\
&\equiv p_n + q_n,
\end{aligned}
\tag{1.110}
$$

where

$$
p_n = \frac{1}{\Gamma(1-\alpha)}\left[\int_{t_0}^{t_{\frac{1}{2}}} \frac{g'(t) - L'_{1,0}(t)}{(t_{n-\frac{1}{2}} - t)^{\alpha}}\mathrm{d}t + \sum_{k=1}^{n-1}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \frac{g'(t) - L'_{1,k}(t)}{(t_{n-\frac{1}{2}} - t)^{\alpha}}\mathrm{d}t\right],
$$

$$
\begin{aligned}
q_n = \frac{1}{\Gamma(2-\gamma)}\Bigg[&\hat{b}_0^{(n,\gamma)}(g(t_{n-\frac{1}{2}}) - \delta_t f^{n-\frac{1}{2}}) \\
&- \sum_{k=1}^{n-1} (\hat{b}_{n-k-1}^{(n,\gamma)} - \hat{b}_{n-k}^{(n,\gamma)})(g(t_{k-\frac{1}{2}}) - \delta_t f^{k-\frac{1}{2}})\Bigg].
\end{aligned}
$$

The estimate of p_n. Using the integration by parts and noticing (1.108)–(1.109), we obtain

$$
\begin{aligned}
p_n &= \frac{1}{\Gamma(1-\alpha)}\Bigg[\int_{t_0}^{t_{\frac{1}{2}}} (g(t) - L_{1,0}(t))(-\alpha)(t_{n-\frac{1}{2}} - t)^{-\alpha-1}\mathrm{d}t \\
&\quad + \sum_{k=1}^{n-1}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (g(t) - L_{1,k}(t))(-\alpha)(t_{n-\frac{1}{2}} - t)^{-\alpha-1}\mathrm{d}t\Bigg] \\
&= \frac{\alpha}{\Gamma(1-\alpha)}\Bigg[\int_{t_0}^{t_{\frac{1}{2}}} \frac{1}{2}g''(\xi_0)(t - t_0)(t_{\frac{1}{2}} - t)(t_{n-\frac{1}{2}} - t)^{-\alpha-1}\mathrm{d}t \\
&\quad + \sum_{k=1}^{n-1}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \frac{1}{2}g''(\xi_k)(t - t_{k-\frac{1}{2}})(t_{k+\frac{1}{2}} - t)(t_{n-\frac{1}{2}} - t)^{-\alpha-1}\mathrm{d}t\Bigg].
\end{aligned}
$$

When $n = 1$, we have

$$
\begin{aligned}
p_1 &= \frac{\alpha}{\Gamma(1-\alpha)}\int_{t_0}^{t_{\frac{1}{2}}} \frac{1}{2}g''(\xi_0)(t - t_0)(t_{\frac{1}{2}} - t)(t_{\frac{1}{2}} - t)^{-\alpha-1}\mathrm{d}t \\
&= \frac{\alpha}{\Gamma(1-\alpha)}\int_{t_0}^{t_{\frac{1}{2}}} \frac{1}{2}g''(\xi_0)(t - t_0)(t_{\frac{1}{2}} - t)^{-\alpha}\mathrm{d}t \\
&= \frac{\alpha}{\Gamma(1-\alpha)}\cdot\frac{1}{2}g''(\hat{\xi}_0)\int_{t_0}^{t_{\frac{1}{2}}} (t - t_0)(t_{\frac{1}{2}} - t)^{-\alpha}\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha}{\Gamma(1-\alpha)} \cdot \frac{1}{2} g''(\hat{\xi}_0)\left(\frac{\tau}{2}\right)^{2-\alpha} \int_0^1 \theta(1-\theta)^{-\alpha} \mathrm{d}\theta \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \cdot \frac{1}{2} g''(\hat{\xi}_0)\left(\frac{\tau}{2}\right)^{2-\alpha} \left(\frac{1}{1-\alpha} - \frac{1}{2-\alpha}\right) \\
&= \frac{\alpha}{\Gamma(3-\alpha)} \cdot \frac{1}{2} g''(\hat{\xi}_0)\left(\frac{\tau}{2}\right)^{2-\alpha}, \quad \hat{\xi}_0 \in (t_0, t_{\frac{1}{2}}).
\end{aligned}
$$

Therefore,

$$
|p_1| \leqslant \frac{\alpha}{2\Gamma(3-\alpha)} \cdot \max_{t_0 \leqslant t \leqslant t_{\frac{1}{2}}} |g''(t)| \left(\frac{\tau}{2}\right)^{2-\alpha}. \tag{1.111}
$$

When $n \geqslant 2$,

$$
\begin{aligned}
p_n = \frac{\alpha}{\Gamma(1-\alpha)} \Bigg[&\int_{t_0}^{t_{\frac{1}{2}}} \frac{1}{2} g''(\xi_0)(t-t_0)(t_{\frac{1}{2}}-t)(t_{n-\frac{1}{2}}-t)^{-\alpha-1} \mathrm{d}t \\
&+ \sum_{k=1}^{n-2} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \frac{1}{2} g''(\xi_k)(t-t_{k-\frac{1}{2}})(t_{k+\frac{1}{2}}-t)(t_{n-\frac{1}{2}}-t)^{-\alpha-1} \mathrm{d}t \\
&+ \int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} \frac{1}{2} g''(\xi_{n-1})(t-t_{n-\frac{3}{2}})(t_{n-\frac{1}{2}}-t)(t_{n-\frac{1}{2}}-t)^{-\alpha-1} \mathrm{d}t \Bigg].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|p_n| \leqslant \frac{\alpha}{\Gamma(1-\alpha)} \cdot \frac{1}{2} \max_{t_0 \leqslant t \leqslant t_{n-\frac{1}{2}}} |g''(t)| \Bigg[&\int_{t_0}^{t_{\frac{1}{2}}} (t-t_0)(t_{\frac{1}{2}}-t)(t_{n-\frac{1}{2}}-t)^{-\alpha-1} \mathrm{d}t \\
&+ \sum_{k=1}^{n-2} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (t-t_{k-\frac{1}{2}})(t_{k+\frac{1}{2}}-t)(t_{n-\frac{1}{2}}-t)^{-\alpha-1} \mathrm{d}t \\
&+ \int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} (t-t_{n-\frac{3}{2}})(t_{n-\frac{1}{2}}-t)^{-\alpha} \mathrm{d}t \Bigg] \\
\leqslant \frac{\alpha}{\Gamma(1-\alpha)} \cdot \frac{1}{2} \max_{t_0 \leqslant t \leqslant t_{n-\frac{1}{2}}} |g''(t)| \Bigg[&\frac{\tau^2}{16} \int_{t_0}^{t_{\frac{1}{2}}} (t_{n-\frac{1}{2}}-t)^{-\alpha-1} \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\tau^2}{4}\sum_{k=1}^{n-2}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}}(t_{n-\frac{1}{2}}-t)^{-\alpha-1}\mathrm{d}t+\int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}}(t-t_{n-\frac{3}{2}})(t_{n-\frac{1}{2}}-t)^{-\alpha}\mathrm{d}t\Bigg]\\
&\leqslant\frac{\alpha}{2\Gamma(1-\alpha)}\cdot\max_{t_0\leqslant t\leqslant t_{n-\frac{1}{2}}}|g''(t)|\\
&\quad\cdot\left[\frac{\tau^2}{4}\int_{t_0}^{t_{n-\frac{3}{2}}}(t_{n-\frac{1}{2}}-t)^{-\alpha-1}\mathrm{d}t+\frac{\tau^{2-\alpha}}{(1-\alpha)(2-\alpha)}\right]\\
&\leqslant\frac{\alpha}{2\Gamma(1-\alpha)}\cdot\max_{t_0\leqslant t\leqslant t_{n-\frac{1}{2}}}|g''(t)|\left[\frac{\tau^{2-\alpha}}{4\alpha}+\frac{\tau^{2-\alpha}}{(1-\alpha)(2-\alpha)}\right]\\
&=\left[\frac{1}{8\Gamma(1-\alpha)}+\frac{\alpha}{2\Gamma(3-\alpha)}\right]\max_{t_0\leqslant t\leqslant t_{n-\frac{1}{2}}}|g''(t)|\tau^{2-\alpha}. \qquad (1.112)
\end{aligned}
$$

The estimate of q_n. Noticing

$$
g(t_{k-\frac{1}{2}})-\delta_t f^{k-\frac{1}{2}}=f'(t_{k-\frac{1}{2}})-\delta_t f^{k-\frac{1}{2}}=-\frac{\tau^2}{24}f'''(\eta_k),\quad \eta_k\in(t_{k-1},t_k),
$$

we have

$$
q_n=\frac{1}{\Gamma(2-\gamma)}\left[\hat{b}_0^{(n,\gamma)}\left(-\frac{\tau^2}{24}f'''(\eta_n)\right)-\sum_{k=1}^{n-1}(\hat{b}_{n-k-1}^{(n,\gamma)}-\hat{b}_{n-k}^{(n,\gamma)})\left(-\frac{\tau^2}{24}f'''(\eta_k)\right)\right].
$$

Consequently,

$$
\begin{aligned}
|q_n|&\leqslant\frac{\tau^2}{24}\max_{t_0\leqslant t\leqslant t_n}|f'''(t)|\frac{1}{\Gamma(2-\gamma)}\left[\hat{b}_0^{(n,\gamma)}+\sum_{k=1}^{n-1}(\hat{b}_{n-k-1}^{(n,\gamma)}-\hat{b}_{n-k}^{(n,\gamma)})\right]\\
&\leqslant\frac{\tau^2}{24}\max_{t_0\leqslant t\leqslant t_n}|f'''(t)|\frac{1}{\Gamma(2-\gamma)}\cdot 2\hat{b}_0^{(n,\gamma)}\\
&=\frac{\tau^2}{12}\max_{t_0\leqslant t\leqslant t_n}|f'''(t)|\frac{1}{\Gamma(2-\gamma)}\frac{\tau^{1-\gamma}}{2-\gamma}\\
&=\frac{\tau^{3-\gamma}}{12\Gamma(3-\gamma)}\max_{t_0\leqslant t\leqslant t_n}|f'''(t)|. \qquad (1.113)
\end{aligned}
$$

Substituting (1.111)–(1.113) into (1.110), we get

$$
\begin{aligned}
&|R_n|\leqslant|p_n|+|q_n|\\
&\leqslant\left[\frac{1}{8\Gamma(1-\alpha)}+\frac{\alpha}{2\Gamma(3-\alpha)}\right]\max_{t_0\leqslant t\leqslant t_{n-\frac{1}{2}}}|g''(t)|\tau^{2-\alpha}+\frac{\tau^{3-\gamma}}{12\Gamma(3-\gamma)}\max_{t_0\leqslant t\leqslant t_n}|f'''(t)|\\
&\leqslant\left[\frac{1}{8\Gamma(2-\gamma)}+\frac{1}{12\Gamma(3-\gamma)}+\frac{\gamma-1}{2\Gamma(4-\gamma)}\right]\max_{t_0\leqslant t\leqslant t_n}|f'''(t)|\tau^{3-\gamma}.
\end{aligned}
$$

The proof ends. □

Remark 1.6.3. The formula (1.106) can also be obtained in the following way[46]. Let

$$g(t) = f'(t), \quad \alpha = \gamma - 1.$$

Then we have

$$\begin{aligned}
&{}_0^C D_t^{\gamma} f(t)|_{t=t_{n-\frac{1}{2}}} = {}_0^C D_t^{\alpha} g(t)|_{t=t_{n-\frac{1}{2}}} \\
&= \frac{1}{\Gamma(1-\alpha)}\left[\int_{t_0}^{t_{\frac{1}{2}}} g'(t)(t_{n-\frac{1}{2}} - t)^{-\alpha}\mathrm{d}t + \sum_{k=1}^{n-1}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} g'(t)(t_{n-\frac{1}{2}} - t)^{-\alpha}\mathrm{d}t\right] \\
&\approx \frac{1}{\Gamma(1-\alpha)}\left[\int_{t_0}^{t_{\frac{1}{2}}} L'_{1,0}(t)(t_{n-\frac{1}{2}} - t)^{-\alpha}\mathrm{d}t + \sum_{k=1}^{n-1}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} L'_{1,k}(t)(t_{n-\frac{1}{2}} - t)^{-\alpha}\mathrm{d}t\right] \\
&= \frac{1}{\Gamma(1-\alpha)}\left[\frac{2}{\tau}(g(t_{\frac{1}{2}}) - g(t_0))\cdot\int_{t_0}^{t_{\frac{1}{2}}}(t_{n-\frac{1}{2}} - t)^{-\alpha}\mathrm{d}t\right. \\
&\quad \left. + \sum_{k=1}^{n-1}\frac{1}{\tau}(g(t_{k+\frac{1}{2}}) - g(t_{k-\frac{1}{2}}))\cdot\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}}(t_{n-\frac{1}{2}} - t)^{-\alpha}\mathrm{d}t\right] \\
&= \frac{1}{\Gamma(2-\gamma)}\left[\hat{b}_{n-1}^{(n,\gamma)}\cdot(g(t_{\frac{1}{2}}) - f'(t_0)) + \sum_{k=1}^{n-1}\hat{b}_{n-k-1}^{(n,\gamma)}\cdot(g(t_{k+\frac{1}{2}}) - g(t_{k-\frac{1}{2}}))\right] \\
&= \frac{1}{\Gamma(2-\gamma)}\left[\hat{b}_0^{(n,\gamma)} g(t_{n-\frac{1}{2}}) - \sum_{k=1}^{n-1}(\hat{b}_{n-k-1}^{(n,\gamma)} - \hat{b}_{n-k}^{(n,\gamma)}) g(t_{k-\frac{1}{2}}) - \hat{b}_{n-1}^{(n,\gamma)} f'(t_0)\right] \\
&\approx \frac{1}{\Gamma(2-\gamma)}\left[\hat{b}_0^{(n,\gamma)}\delta_t f^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(\hat{b}_{n-k-1}^{(n,\gamma)} - \hat{b}_{n-k}^{(n,\gamma)})\delta_t f^{k-\frac{1}{2}} - \hat{b}_{n-1}^{(n,\gamma)} f'(t_0)\right],
\end{aligned}$$

in which two approximate equalities have been used.

1.7 Fast interpolation approximations of Caputo fractional derivatives

1.7.1 Fast L1 approximation

Jiang et al. gave the approximation formula of sum-of-exponentials (SOE) for the kernel function $t^{-\alpha}$ in Caputo derivative.

Lemma 1.7.1. [41] *For given $\alpha \in (0,1)$, $\epsilon > 0$, $\hat{\tau} > 0$ and $T > 0$, where $\hat{\tau} < T$, there exist a positive integer $N_{\exp}^{(\alpha)}$, positive points $s_l^{(\alpha)}$ and corresponding positive weights $\omega_l^{(\alpha)}$,*

($l = 1, 2, \ldots, N_{\exp}^{(\alpha)}$) *satisfying*

$$\left| t^{-\alpha} - \sum_{l=1}^{N_{\exp}^{(\alpha)}} \omega_l^{(\alpha)} e^{-s_l^{(\alpha)} t} \right| \leqslant \epsilon, \quad \forall\, t \in [\hat{\tau}, T].$$

In addition, the number of exponentials has the following estimate:

$$N_{\exp}^{(\alpha)} = O\Big(\Big(\log\frac{1}{\epsilon}\Big)\Big(\log\log\frac{1}{\epsilon} + \log\frac{T}{\hat{\tau}}\Big) + \Big(\log\frac{1}{\hat{\tau}}\Big)\Big(\log\log\frac{1}{\epsilon} + \log\frac{T}{\hat{\tau}}\Big)\Big).$$

It is worth to note that $N_{\exp}^{(\alpha)}, \omega_l^{(\alpha)}, s_l^{(\alpha)}$ not only depend on α but also depend on $\epsilon, \hat{\tau}$ and T.

Without confusion, $N_{\exp}^{(\alpha)}$, $s_l^{(\alpha)}$ and $\omega_l^{(\alpha)}$ will be briefly written as $N_{\exp}$, s_l and ω_l.

Table 1.1 lists the numbers of exponentials, $N_{\exp}$ needed to approximate $t^{-\alpha}$ ($t \in (\hat{\tau}, T)$) for different α, $\hat{\tau}$, ϵ, with $T = 1$. One can find that the number of exponentials is very limited and no more than 200 in general.

Table 1.1: $N_{\exp}$ needed to approximate $t^{-\alpha}$ ($t \in (\hat{\tau}, T)$) for different α, $\hat{\tau}$, ϵ with $T = 1$.

		$\hat{\tau}$				
α	ϵ	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
0.1	10^{-6}	31	37	42	48	53
	10^{-8}	40	47	55	62	69
	10^{-10}	48	57	66	75	84
	10^{-12}	57	68	78	89	100
	10^{-14}	66	78	90	102	115
0.5	10^{-6}	32	37	43	48	54
	10^{-8}	40	48	55	62	70
	10^{-10}	49	58	66	75	84
	10^{-12}	58	68	79	90	100
	10^{-14}	66	78	91	103	115
0.9	10^{-6}	32	38	43	49	55
	10^{-8}	41	48	56	63	70
	10^{-10}	49	58	67	76	85
	10^{-12}	58	69	80	90	101
	10^{-14}	66	79	91	104	116

In what follows, a fast algorithm for Caputo fractional derivatives will be proposed. According to Lemma 1.7.1, we have

$$ {}_{0}^{C}D_t^{\alpha} f(t)|_{t=t_n} = \frac{1}{\Gamma(1-\alpha)} \left[\sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} f'(t)(t_n - t)^{-\alpha} \mathrm{d}t + \int_{t_{n-1}}^{t_n} f'(t)(t_n - t)^{-\alpha} \mathrm{d}t \right]$$

$$\approx \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L'_{1,k}(t)\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t)}\mathrm{d}t + \int_{t_{n-1}}^{t_n} L'_{1,n}(t)(t_n-t)^{-\alpha}\mathrm{d}t\right] \quad (1.114)$$

$$\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{l=1}^{N_{\exp}}\omega_l\left(\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L'_{1,k}(t)e^{-s_l(t_n-t)}\mathrm{d}t\right)\right.\\
&\qquad\left. + \int_{t_{n-1}}^{t_n} L'_{1,n}(t)(t_n-t)^{-\alpha}\mathrm{d}t\right]\\
&= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{l=1}^{N_{\exp}}\omega_l F_l^n + \frac{\tau^{-\alpha}}{1-\alpha}\big(f(t_n)-f(t_{n-1})\big)\right]\\
&\equiv {}^{\mathcal{F}}D_t^{\alpha} f(t_n),
\end{aligned} \quad (1.115)$$

where

$$F_l^n = \sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L'_{1,k}(t)e^{-s_l(t_n-t)}\mathrm{d}t, \quad 1 \leqslant l \leqslant N_{\exp}, \quad 1 \leqslant n \leqslant N.$$

It is noted that F_l^n can be evaluated by the following recurrence relation:

$$\begin{aligned}
F_l^n &= \sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L'_{1,k}(t)e^{-s_l(t_n-t)}\mathrm{d}t\\
&= \sum_{k=1}^{n-2}\int_{t_{k-1}}^{t_k} L'_{1,k}(t)e^{-s_l(t_n-t)}\mathrm{d}t + \int_{t_{n-2}}^{t_{n-1}} L'_{1,n-1}(t)e^{-s_l(t_n-t)}\mathrm{d}t\\
&= e^{-s_l\tau}\sum_{k=1}^{n-2}\int_{t_{k-1}}^{t_k} L'_{1,k}(t)e^{-s_l(t_{n-1}-t)}\mathrm{d}t + \delta_t f^{n-\frac{3}{2}}\int_{t_{n-2}}^{t_{n-1}} e^{-s_l(t_n-t)}\mathrm{d}t\\
&= e^{-s_l\tau}F_l^{n-1} + \tau\,\delta_t f^{n-\frac{3}{2}}\int_0^1 e^{-s_l(1+\theta)\tau}\mathrm{d}\theta\\
&= e^{-s_l\tau}F_l^{n-1} + B_l[f(t_{n-1})-f(t_{n-2})], \quad 2 \leqslant n \leqslant N,
\end{aligned}$$

where

$$F_l^1 = 0, \quad B_l = \int_0^1 e^{-s_l(1+\theta)\tau}\mathrm{d}\theta, \quad 1 \leqslant l \leqslant N_{\exp}. \quad (1.116)$$

A fast algorithm for evaluating ${}_0^C D_t^\alpha f(t)|_{t=t_n}$ can be obtained from (1.115)–(1.116) as follows:

$$\begin{cases} {}^{\mathcal{F}}D_t^\alpha f(t_n) = \dfrac{1}{\Gamma(1-\alpha)}\left[\displaystyle\sum_{l=1}^{N_{\exp}} \omega_l F_l^n + \frac{\tau^{-\alpha}}{1-\alpha}(f(t_n)-f(t_{n-1}))\right], \quad n \geqslant 1, & (1.117)\\ F_l^1 = 0, \quad 1 \leqslant l \leqslant N_{\exp}, & (1.118)\\ F_l^n = e^{-s_l\tau}F_l^{n-1} + B_l[f(t_{n-1}) - f(t_{n-2})], \quad 1 \leqslant l \leqslant N_{\exp} \quad n \geqslant 2. & (1.119)\end{cases}$$

The computational complexity for evaluating ${}_0^C D_t^\alpha f(t)|_{t=t_n}$ $(1 \leqslant n \leqslant N)$ by L1 formula (1.60) is $O(N^2)$, and it is $O(NN_{\exp})$ by the formula (1.117)–(1.119). When N is large, $O(NN_{\exp}) << O(N^2)$. Therefore, we call (1.117)–(1.119) a fast algorithm based on L1 interpolation approximation, or, a **fast L1 approximation**.

A direct calculation for (1.114) yields

$$\begin{aligned} {}^{\mathcal{F}}D_t^\alpha f(t_n) &= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L_{1,k}'(t)\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t)}\mathrm{d}t + \int_{t_{n-1}}^{t_n} L_{1,n}'(t)(t_n-t)^{-\alpha}\mathrm{d}t\right]\\ &= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\delta_t f^{k-\frac{1}{2}}\int_{t_{k-1}}^{t_k}\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t)}\mathrm{d}t + \delta_t f^{n-\frac{1}{2}}\int_{t_{n-1}}^{t_n}(t_n-t)^{-\alpha}\mathrm{d}t\right]\\ &= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\tau\delta_t f^{k-\frac{1}{2}}\int_0^1\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t_k+\theta\tau)}\mathrm{d}\theta + \tau\delta_t f^{n-\frac{1}{2}}\frac{\tau^{-\alpha}}{1-\alpha}\right]. \end{aligned}$$

Denote

$$\begin{cases} \hat{a}_0^{(\alpha)} = \dfrac{\tau^{-\alpha}}{1-\alpha}, & (1.120)\\ \hat{a}_k^{(\alpha)} = \displaystyle\int_0^1\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_k+\theta\tau)}\mathrm{d}\theta, \quad k \geqslant 1. & (1.121)\end{cases}$$

Then we obtain an equivalent form for (1.117)–(1.119) as follows:

$$\begin{aligned} {}^{\mathcal{F}}D_t^\alpha f(t_n) &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\hat{a}_k^{(\alpha)}[f(t_{n-k}) - f(t_{n-k-1})]\\ &= \frac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)}f(t_n) - \sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)} - \hat{a}_k^{(\alpha)})f(t_{n-k}) - \hat{a}_{n-1}^{(\alpha)}f(t_0)\right]. \end{aligned}$$

It is easy to know that

$$\begin{cases} \hat{a}_0^{(\alpha)} = \dfrac{1}{\tau}\displaystyle\int_{t_{n-1}}^{t_n} (t_n - t)^{-\alpha}\mathrm{d}t, & (1.122) \\ \hat{a}_{n-k}^{(\alpha)} = \dfrac{1}{\tau}\displaystyle\int_{t_{k-1}}^{t_k} \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_n-t)}\mathrm{d}t, \quad 1 \leqslant k \leqslant n-1. & (1.123) \end{cases}$$

The coefficient $\{\hat{a}_k^{(\alpha)} \mid 0 \leqslant k \leqslant n-1\}$ satisfies the following lemma.

Lemma 1.7.2. *The coefficient* $\{\hat{a}_k^{(\alpha)} \mid 0 \leqslant k \leqslant n-1\}$ *defined by* (1.120)–(1.121) *satisfies*

$$\hat{a}_1^{(\alpha)} > \hat{a}_2^{(\alpha)} > \cdots > \hat{a}_{n-1}^{(\alpha)} > 0. \tag{1.124}$$

If $\epsilon \leqslant \frac{2-2^{1-\alpha}}{1-\alpha}\tau^{-\alpha}$, *it holds*

$$\hat{a}_0^{(\alpha)} \geqslant \hat{a}_1^{(\alpha)}.$$

In addition,

$$\hat{a}_0^{(\alpha)} = \frac{\tau^{-\alpha}}{1-\alpha},$$
$$t_{k+1}^{-\alpha} - \epsilon < \hat{a}_k^{(\alpha)} < t_k^{-\alpha} + \epsilon, \quad k \geqslant 1. \tag{1.125}$$

Proof. According to (1.121), we have

$$\hat{a}_k^{(\alpha)} = \sum_{l=1}^{N_{\exp}} \omega_l \int_0^1 e^{-s_l(k\tau+\tau\theta)}\mathrm{d}\theta, \quad 1 \leqslant k \leqslant n-1,$$

by which, the truth of (1.124) is apparent. Noticing

$$\hat{a}_0^{(\alpha)} = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (t_n - t)^{-\alpha}\mathrm{d}t, \quad \hat{a}_1^{(\alpha)} = \frac{1}{\tau}\int_{t_{n-2}}^{t_{n-1}} \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_n-t)}\mathrm{d}t,$$

we have

$$\begin{aligned} \hat{a}_0^{(\alpha)} - \hat{a}_1^{(\alpha)} &= \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (t_n - t)^{-\alpha}\mathrm{d}t - \frac{1}{\tau}\int_{t_{n-2}}^{t_{n-1}} (t_n - t)^{-\alpha}\mathrm{d}t \\ &\quad + \frac{1}{\tau}\int_{t_{n-2}}^{t_{n-1}} \left[(t_n - t)^{-\alpha} - \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_n-t)}\right]\mathrm{d}t. \end{aligned}$$

Computing yields

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(t_n-t)^{-\alpha}\mathrm{d}t-\frac{1}{\tau}\int_{t_{n-2}}^{t_{n-1}}(t_n-t)^{-\alpha}\mathrm{d}t=\frac{2-2^{1-\alpha}}{1-\alpha}\tau^{-\alpha}.$$

In addition, by Lemma 1.7.1, we have

$$\begin{aligned}&\left|\frac{1}{\tau}\int_{t_{n-2}}^{t_{n-1}}\left[(t_n-t)^{-\alpha}-\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t)}\right]\mathrm{d}t\right|\\&\leqslant\frac{1}{\tau}\int_{t_{n-2}}^{t_{n-1}}\left|(t_n-t)^{-\alpha}-\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t)}\right|\mathrm{d}t\\&\leqslant\epsilon.\end{aligned}$$

Hence when $\epsilon\leqslant\frac{2-2^{1-\alpha}}{1-\alpha}\tau^{-\alpha}$,

$$\hat{a}_0^{(\alpha)}-\hat{a}_1^{(\alpha)}\geqslant\frac{2-2^{1-\alpha}}{1-\alpha}\tau^{-\alpha}-\epsilon\geqslant 0.$$

Using (1.123), we have

$$\begin{aligned}\hat{a}_{n-k}^{(\alpha)}&=\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t)}\mathrm{d}t\\&=\frac{1}{\tau}\int_{t_{k-1}}^{t_k}(t_n-t)^{-\alpha}\mathrm{d}t-\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\left[(t_n-t)^{-\alpha}-\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t)}\right]\mathrm{d}t\\&=\frac{\tau^{-\alpha}}{1-\alpha}a_{n-k}^{(\alpha)}-\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\left[(t_n-t)^{-\alpha}-\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t)}\right]\mathrm{d}t,\\&\qquad\qquad 1\leqslant k\leqslant n-1.\end{aligned}$$

Further, it follows

$$\left|\hat{a}_{n-k}^{(\alpha)}-\frac{\tau^{-\alpha}}{1-\alpha}a_{n-k}^{(\alpha)}\right|\leqslant\epsilon,\quad 1\leqslant k\leqslant n-1,$$

namely,

$$\frac{\tau^{-\alpha}}{1-\alpha}a_k^{(\alpha)}-\epsilon\leqslant\hat{a}_k^{(\alpha)}\leqslant\frac{\tau^{-\alpha}}{1-\alpha}a_k^{(\alpha)}+\epsilon,\quad 1\leqslant k\leqslant n-1. \tag{1.126}$$

Combining with Lemma 1.6.1, (1.125) can be obtained. The proof ends. □

For the common taken ϵ, the inequality $\epsilon \leqslant \frac{2-2^{1-\alpha}}{1-\alpha}\tau^{-\alpha}$ always holds.

The truncation error of ${}^{\mathcal{F}}D_t^{\alpha}f(t_n)$ to approximate ${}_0^C D_t^{\alpha}f(t)|_{t=t_n}$ is given in the following theorem.

Theorem 1.7.1. *Suppose $f \in C^2[t_0, t_n]$, then we have*

$$\begin{aligned}&\left|{}_0^C D_t^{\alpha}f(t)|_{t=t_n} - {}^{\mathcal{F}}D_t^{\alpha}f(t_n)\right|\\ &\leqslant \frac{1}{2\Gamma(1-\alpha)}\left[\frac{1}{4} + \frac{\alpha}{(1-\alpha)(2-\alpha)}\right]\max_{t_0\leqslant t\leqslant t_n}\left|f''(t)\right|\cdot\tau^{2-\alpha} + \frac{\epsilon t_n}{\Gamma(1-\alpha)}\max_{t_0\leqslant t\leqslant t_n}\left|f'(t)\right|. \end{aligned}\tag{1.127}$$

Proof. Some direct calculations yield

$$\begin{aligned}&{}_0^C D_t^{\alpha}f(t)|_{t=t_n} - {}^{\mathcal{F}}D_t^{\alpha}f(t_n)\\
&= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} f'(t)(t_n-t)^{-\alpha}\mathrm{d}t + \int_{t_{n-1}}^{t_n} f'(t)(t_n-t)^{-\alpha}\mathrm{d}t\right]\\
&\quad - \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L'_{1,k}(t)\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t)}\mathrm{d}t\right.\\
&\quad \left. + \int_{t_{n-1}}^{t_n} L'_{1,n}(t)(t_n-t)^{-\alpha}\mathrm{d}t\right]\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}[f'(t)-L'_{1,k}(t)](t_n-t)^{-\alpha}\mathrm{d}t\\
&\quad + \frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L'_{1,k}(t)\left[(t_n-t)^{-\alpha} - \sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t)}\right]\mathrm{d}t\\
&\equiv \mathrm{I}_n + \mathrm{II}_n.\end{aligned}\tag{1.128}$$

By Theorem 1.6.1, for the first term in (1.128), we have

$$|\mathrm{I}_n| \leqslant \frac{1}{2\Gamma(1-\alpha)}\left[\frac{1}{4} + \frac{\alpha}{(1-\alpha)(2-\alpha)}\right]\max_{t_0\leqslant t\leqslant t_n}\left|f''(t)\right|\cdot\tau^{2-\alpha}.\tag{1.129}$$

For the second term in (1.128), we have

$$\begin{aligned}|\mathrm{II}_n| &\leqslant \frac{1}{\Gamma(1-\alpha)}\max_{t_0\leqslant t\leqslant t_n}\left|f'(t)\right|\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\left|(t_n-t)^{-\alpha} - \sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_n-t)}\right|\mathrm{d}t\\
&\leqslant \frac{1}{\Gamma(1-\alpha)}\max_{t_0\leqslant t\leqslant t_n}\left|f'(t)\right|\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\epsilon\,\mathrm{d}t\\
&\leqslant \frac{\epsilon\, t_n}{\Gamma(1-\alpha)}\max_{t_0\leqslant t\leqslant t_n}\left|f'(t)\right|.\end{aligned}\tag{1.130}$$

Substituting (1.129) and (1.130) into (1.128), we have (1.127) immediately. The proof ends. □

1.7.2 Fast L2-1_σ approximation

With the help of Lemma 1.7.1, we can give a fast algorithm of L2-1_σ interpolation approximation[101].

Take $\sigma = 1 - \frac{\alpha}{2}$, $\hat{\tau} = \sigma\tau$. Table 1.2 lists the values of N_{exp} needed to approximate $t^{-\alpha}$ with different parameters α, τ, ϵ. It shows the number of exponentials needed is very limited and no more than 200 usually.

Table 1.2: N_{exp} needed to approximate $t^{-\alpha}$ ($t \in (\hat{\tau}, T)$) with different parameters α, τ, ϵ, when $T = 1$, $\hat{\tau} = \sigma\tau$.

		τ				
α	ϵ	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
0.1	10^{-6}	31	37	43	48	54
	10^{-8}	40	48	55	62	69
	10^{-10}	48	57	66	75	84
	10^{-12}	58	68	79	89	100
	10^{-14}	66	78	90	103	115
0.5	10^{-6}	32	38	43	49	55
	10^{-8}	41	49	56	63	70
	10^{-10}	50	59	68	77	85
	10^{-12}	59	70	80	91	102
	10^{-14}	68	80	92	105	117
0.9	10^{-6}	33	39	45	50	56
	10^{-8}	43	50	57	65	72
	10^{-10}	51	60	69	78	87
	10^{-12}	61	72	82	93	104
	10^{-14}	70	82	95	107	119

Reformulating the fractional derivative as the sum of the integrals over the subintervals and approximating $f(t)$ using $L_{2,k}(t)$ $(1 \leqslant k \leqslant n-1)$ and $L_{1,n}(t)$, respectively, by Lemma 1.7.1, we have

$$\begin{aligned}
{}_0^C D_t^\alpha f(t)|_{t=t_{n-1+\sigma}} &= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{f'(t)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t + \int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{f'(t)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t\right] \\
&\approx \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L'_{2,k}(t)\left(\sum_{l=1}^{N_{\text{exp}}}\omega_l e^{-s_l(t_{n-1+\sigma}-t)}\right)\mathrm{d}t\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{t_{n-1}}^{t_{n-1+\sigma}} L'_{1,n}(t)\,(t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t\Bigg] \qquad (1.131)\\
&= \frac{1}{\Gamma(1-\alpha)}\Bigg[\sum_{l=1}^{N_{\exp}} \omega_l\Bigg(\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L'_{2,k}(t)e^{-s_l(t_{n-1+\sigma}-t)}\mathrm{d}t\Bigg)\\
&\quad + \int_{t_{n-1}}^{t_{n-1+\sigma}} L'_{1,n}(t)(t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t\Bigg]\\
&= \frac{1}{\Gamma(1-\alpha)}\Bigg[\sum_{l=1}^{N_{\exp}} \omega_l F_l^n + \frac{\sigma^{1-\alpha}\tau^{-\alpha}}{1-\alpha}\big(f(t_n)-f(t_{n-1})\big)\Bigg]\\
&\equiv {}^{\mathcal{F}}\triangle_\tau^\alpha f(t_{n-1+\sigma}),
\end{aligned}
$$

where

$$
\begin{cases}
F_l^1 = 0, \quad 1 \leqslant l \leqslant N_{\exp},\\
F_l^n = \displaystyle\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L'_{2,k}(t)e^{-s_l(t_{n-1+\sigma}-t)}\mathrm{d}t, \quad 1 \leqslant l \leqslant N_{\exp}, \quad n \geqslant 2.
\end{cases}
$$

It is noted that F_l^n can be evaluated by the following recursive relation:

$$
\begin{aligned}
F_l^n &= \sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} L'_{2,k}(t)e^{-s_l(t_{n-1+\sigma}-t)}\mathrm{d}t\\
&= \sum_{k=1}^{n-2}\int_{t_{k-1}}^{t_k} L'_{2,k}(t)e^{-s_l(t_{n-1+\sigma}-t)}\mathrm{d}t + \int_{t_{n-2}}^{t_{n-1}} L'_{2,n-1}(t)e^{-s_l(t_{n-1+\sigma}-t)}\mathrm{d}t\\
&= e^{-s_l\tau}\sum_{k=1}^{n-2}\int_{t_{k-1}}^{t_k} L'_{2,k}(t)e^{-s_l(t_{n-2+\sigma}-t)}\mathrm{d}t + \int_{t_{n-2}}^{t_{n-1}} L'_{2,n-1}(t)e^{-s_l(t_{n-1+\sigma}-t)}\mathrm{d}t\\
&= e^{-s_l\tau}F_l^{n-1} + \int_{t_{n-2}}^{t_{n-1}}\Bigg[\frac{t_{n-\frac12}-t}{\tau}\,\delta_t f^{n-\frac32} + \frac{t-t_{n-\frac32}}{\tau}\,\delta_t f^{n-\frac12}\Bigg]e^{-s_l(t_{n-1+\sigma}-t)}\mathrm{d}t\\
&= e^{-s_l\tau}F_l^{n-1} + [f(t_{n-1})-f(t_{n-2})]\int_0^1\Big(\frac32-\xi\Big)e^{-s_l(\sigma+1-\xi)\tau}\mathrm{d}\xi\\
&\quad + [f(t_n)-f(t_{n-1})]\int_0^1\Big(\xi-\frac12\Big)e^{-s_l(\sigma+1-\xi)\tau}\mathrm{d}\xi, \quad 2 \leqslant n \leqslant N.
\end{aligned}
$$

Denote

$$A_l = \int_0^1 \left(\frac{3}{2} - \xi\right) e^{-s_l(\sigma+1-\xi)\tau} \mathrm{d}\xi, \quad B_l = \int_0^1 \left(\xi - \frac{1}{2}\right) e^{-s_l(\sigma+1-\xi)\tau} \mathrm{d}\xi.$$

Obviously, both A_l and B_l are positive. Hence we obtain the following algorithm: For $n = 1, 2, \ldots, N$, compute

$$\begin{cases} {}^{\mathcal{F}}\triangle_\tau^\alpha f(t_{n-1+\sigma}) = \dfrac{1}{\Gamma(1-\alpha)} \left[\displaystyle\sum_{l=1}^{N_{\exp}} \omega_l F_l^n + \dfrac{\sigma^{1-\alpha}\tau^{-\alpha}}{1-\alpha} (f(t_n) - f(t_{n-1})) \right], & (1.132) \\ F_l^1 = 0, \quad 1 \leqslant l \leqslant N_{\exp}, & (1.133) \\ F_l^n = e^{-s_l\tau} F_l^{n-1} + A_l[f(t_{n-1}) - f(t_{n-2})] + B_l[f(t_n) - f(t_{n-1})], & \\ \qquad\qquad 1 \leqslant l \leqslant N_{\exp}, \quad n \geqslant 2. & (1.134) \end{cases}$$

The computational complexity of the algorithm (1.132)–(1.134) is $O(NN_{\exp})$. When N is very large, the computational complexity of the algorithm is much smaller compared with L2-1_σ algorithm (1.81), of which the computational complexity is $O(N^2)$. For this reason, we call the algorithm (1.132)–(1.134) a fast algorithm based on L2-1_σ interpolation approximation, or, a **fast L2-1_σ approximation**.

Denote

$$d_0^{(1,\alpha)} = \frac{\sigma^{1-\alpha}\tau^{-\alpha}}{\Gamma(2-\alpha)}, \tag{1.135}$$

when $n \geqslant 2$,

$$\begin{cases} d_0^{(n,\alpha)} = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\sum_{l=1}^{N_{\exp}} \omega_l B_l + \dfrac{\sigma^{1-\alpha}\tau^{-\alpha}}{\Gamma(2-\alpha)}, & (1.136) \\ d_k^{(n,\alpha)} = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\sum_{l=1}^{N_{\exp}} \omega_l (e^{-s_l t_{k-1}} A_l + e^{-s_l t_k} B_l), \quad 1 \leqslant k \leqslant n-2, & (1.137) \\ d_{n-1}^{(n,\alpha)} = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l t_{n-2}} A_l. & (1.138) \end{cases}$$

A direct evaluation for (1.131) yields the following results:

When $n = 1$,

$$\begin{aligned} {}^{\mathcal{F}}\triangle_\tau^\alpha f(t_{n-1+\sigma}) &= \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_\sigma} \frac{L_{1,1}'(t)}{(t_\sigma - t)^\alpha} \mathrm{d}t \\ &= \frac{\sigma^{1-\alpha}\tau^{-\alpha}}{\Gamma(2-\alpha)} [f(t_1) - f(t_0)] = d_0^{(1,\alpha)} [f(t_1) - f(t_0)]. \end{aligned}$$

When $n \geqslant 2$,

$$
\begin{aligned}
&{}^{\mathcal{F}}\triangle_\tau^\alpha f(t_{n-1+\sigma})\\
&= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\left(\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_{n-1+\sigma}-t)}\right)L'_{2,k}(t)\mathrm{d}t\right.\\
&\quad\left.+\int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{L'_{1,n}(t)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t\right]\\
&= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\left(\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_{n-1+\sigma}-t)}\right)\left(\delta_t f^{k-\frac{1}{2}}\frac{t_{k+\frac{1}{2}}-t}{\tau}\right.\right.\\
&\quad\left.\left.+\delta_t f^{k+\frac{1}{2}}\frac{t-t_{k-\frac{1}{2}}}{\tau}\right)\mathrm{d}t+\int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{\delta_t f^{n-\frac{1}{2}}}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t\right]\\
&= \frac{1}{\Gamma(1-\alpha)}\left\{\left[\int_{t_0}^{t_1}\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_{n-1+\sigma}-t)}\frac{t_{\frac{3}{2}}-t}{\tau}\mathrm{d}t\right]\delta_t f^{\frac{1}{2}}\right.\\
&\quad+\sum_{k=2}^{n-1}\left[\int_{t_{k-1}}^{t_k}\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_{n-1+\sigma}-t)}\frac{t_{k+\frac{1}{2}}-t}{\tau}\mathrm{d}t\right.\\
&\quad\left.+\int_{t_{k-2}}^{t_{k-1}}\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_{n-1+\sigma}-t)}\frac{t-t_{k-\frac{3}{2}}}{\tau}\mathrm{d}t\right]\delta_t f^{k-\frac{1}{2}}\\
&\quad+\left[\int_{t_{n-2}}^{t_{n-1}}\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l(t_{n-1+\sigma}-t)}\frac{t-t_{n-\frac{3}{2}}}{\tau}\mathrm{d}t\right.\\
&\quad\left.\left.+\int_{t_{n-1}}^{t_{n-1+\sigma}}(t_{n-1+\sigma}-t)^{-\alpha}\mathrm{d}t\right]\delta_t f^{n-\frac{1}{2}}\right\}\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{l=1}^{N_{\exp}}\omega_l e^{-s_l t_{n-2}}A_l[f(t_1)-f(t_0)]\\
&\quad+\frac{1}{\Gamma(1-\alpha)}\sum_{k=2}^{n-1}\sum_{l=1}^{N_{\exp}}\omega_l(e^{-s_l t_{n-k-1}}A_l+e^{-s_l t_{n-k}}B_l)\,[f(t_k)-f(t_{k-1})]\\
&\quad+\left(\frac{1}{\Gamma(1-\alpha)}\sum_{l=1}^{N_{\exp}}\omega_l B_l+\frac{\sigma^{1-\alpha}\tau^{-\alpha}}{\Gamma(2-\alpha)}\right)[f(t_n)-f(t_{n-1})]\\
&= \sum_{k=1}^{n} d_{n-k}^{(n,\alpha)}[f(t_k)-f(t_{k-1})].
\end{aligned}
$$

The following result is given in [101].

Theorem 1.7.2. *Suppose $f \in C^3[t_0, t_n]$, then it holds*

$$\begin{aligned} &\left|{}_0^C D_t^\alpha f(t)|_{t=t_{n-1+\sigma}} - {}^{\mathcal{F}}\triangle_\tau^\alpha f(t_{n-1+\sigma})\right| \\ &\leqslant \frac{(4\sigma-1)\sigma^{-\alpha}}{12\Gamma(2-\alpha)} \max_{t_0\leqslant t\leqslant t_n} \left|f'''(t)\right|\tau^{3-\alpha} + \frac{5\epsilon t_n}{4\Gamma(1-\alpha)} \max_{t_0\leqslant t\leqslant t_n} \left|f'(t)\right|, \end{aligned} \tag{1.139}$$

where $\sigma = 1 - \frac{\alpha}{2}$, $0 < \alpha < 1$.

Proof. Denote

$$R^n = {}_0^C D_t^\alpha f(t)|_{t=t_{n-1+\sigma}} - {}^{\mathcal{F}}\triangle_\tau^\alpha f(t_{n-1+\sigma}).$$

With the help of (1.131), we have

$$\begin{aligned} R^n &= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} \frac{f'(t)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t + \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{f'(t)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t\right] \\ &\quad - \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\left(\sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_{n-1+\sigma}-t)}\right)L_{2,k}'(t)\mathrm{d}t \right. \\ &\quad \left. + \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{L_{1,n}'(t)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t\right] \\ &= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} \frac{f'(t)-L_{2,k}'(t)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t + \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{f'(t)-L_{1,n}'(t)}{(t_{n-1+\sigma}-t)^\alpha}\mathrm{d}t\right] \\ &\quad + \frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\left[(t_{n-1+\sigma}-t)^{-\alpha} - \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_{n-1+\sigma}-t)}\right]L_{2,k}'(t)\mathrm{d}t \\ &\equiv \mathrm{I}_n + \mathrm{II}_n. \end{aligned} \tag{1.140}$$

According to Theorem 1.6.4, we have

$$|\mathrm{I}_n| \leqslant \frac{(4\sigma-1)\sigma^{-\alpha}}{12\Gamma(2-\alpha)} \max_{t_0\leqslant t\leqslant t_n}\left|f'''(t)\right|\tau^{3-\alpha}. \tag{1.141}$$

Using Lemma 1.7.1, we have

$$\begin{aligned} |\mathrm{II}_n| &\leqslant \frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\left|(t_{n-1+\sigma}-t)^{-\alpha} - \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_{n-1+\sigma}-t)}\right| \cdot \left|L_{2,k}'(t)\right|\mathrm{d}t \\ &\leqslant \frac{\epsilon}{\Gamma(1-\alpha)}\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\left|L_{2,k}'(t)\right|\mathrm{d}t \end{aligned}$$

$$\leqslant \frac{\epsilon}{\Gamma(1-\alpha)}\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\left|\delta_t f^{k-\frac{1}{2}}\frac{t_{k+\frac{1}{2}}-t}{\tau}+\delta_t f^{k+\frac{1}{2}}\frac{t-t_{k-\frac{1}{2}}}{\tau}\right|\mathrm{d}t$$

$$\leqslant \frac{\epsilon}{\Gamma(1-\alpha)}\left(\max_{1\leqslant k\leqslant n}|\delta_t f^{k-\frac{1}{2}}|\right)\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\left(\left|\frac{t_{k+\frac{1}{2}}-t}{\tau}\right|+\left|\frac{t-t_{k-\frac{1}{2}}}{\tau}\right|\right)\mathrm{d}t$$

$$= \frac{\epsilon}{\Gamma(1-\alpha)}\frac{5}{4}(n-1)\tau\max_{1\leqslant k\leqslant n}|\delta_t f^{k-\frac{1}{2}}|$$

$$\leqslant \frac{5\epsilon t_n}{4\Gamma(1-\alpha)}\max_{t_0\leqslant t\leqslant t_n}|f'(t)|. \tag{1.142}$$

Substituting (1.141) and (1.142) into (1.140), we have (1.139). This completes the proof. □

Lemma 1.7.3. [101] *The coefficient* $\{d_k^{(n,\alpha)} \mid 0\leqslant k\leqslant n-1\}$ *defined by* (1.135)–(1.138) *satisfies the following relations:*

When $n=1$,

$$d_0^{(n,\alpha)}>0.$$

When $n\geqslant 2$,

(I)

$$d_1^{(n,\alpha)}>d_2^{(n,\alpha)}>d_3^{(n,\alpha)}>\cdots>d_{n-1}^{(n,\alpha)}; \tag{1.143}$$

(II) *If* $\epsilon<\frac{2(1-\sigma)}{\sigma(7\sigma-1)(1+\sigma)^\alpha}\tau^{-\alpha}$, *then we have*

$$(2\sigma-1)d_0^{(n,\alpha)}-\sigma d_1^{(n,\alpha)}>0, \tag{1.144}$$

$$d_0^{(n,\alpha)}>d_1^{(n,\alpha)}>0, \tag{1.145}$$

$$d_{n-1}^{(n,\alpha)}\geqslant\frac{1}{2t_n^\alpha\Gamma(1-\alpha)}. \tag{1.146}$$

Proof. When $n=1$, it is obvious that $d_0^{(n,\alpha)}>0$.

When $n\geqslant 2$, (1.143) is true by using (1.137)–(1.138). In addition, (1.145) is true provided that (1.144) holds. Now we prove (1.144).

As the beginning, a relation between $d_k^{(n,\alpha)}$ and $\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}c_k^{(n,\alpha)}$ is illustrated. For $n=2,3,\ldots$, we have

$$\left|d_k^{(n,\alpha)}-\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}c_k^{(n,\alpha)}\right|$$

$$\leqslant\frac{\epsilon}{\Gamma(1-\alpha)}\cdot\begin{cases}\frac{1}{\tau}\int_{t_{n-2}}^{t_{n-1}}\frac{|t-t_{n-\frac{3}{2}}|}{\tau}\mathrm{d}t, & k=0,\\ \frac{1}{\tau}\int_{t_{k-1}}^{t_k}\frac{|t_{k+\frac{1}{2}}-t|}{\tau}\mathrm{d}t+\frac{1}{\tau}\int_{t_{k-2}}^{t_{k-1}}\frac{|t-t_{k-\frac{3}{2}}|}{\tau}\mathrm{d}t, & 1\leqslant k\leqslant n-2,\\ \frac{1}{\tau}\int_{t_0}^{t_1}\frac{|t_{\frac{3}{2}}-t|}{\tau}\mathrm{d}t, & k=n-1,\end{cases}$$

$$= \frac{\epsilon}{\Gamma(1-\alpha)} \cdot \begin{cases} \frac{1}{4}, & k = 0, \\ \frac{5}{4}, & 1 \leqslant k \leqslant n-2, \\ 1, & k = n-1. \end{cases} \tag{1.147}$$

When $n = 2$, because of

$$(2\sigma - 1)c_0^{(2,\alpha)} - \sigma c_1^{(2,\alpha)} \geqslant \frac{(2\sigma-1)(1-\sigma)}{2\sigma(1+\sigma)^{\alpha-1}}$$

and (1.147), we have

$$\begin{aligned}
&(2\sigma - 1)d_0^{(2,\alpha)} - \sigma d_1^{(2,\alpha)} \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[(2\sigma-1)c_0^{(2,\alpha)} - \sigma c_1^{(2,\alpha)}] + (2\sigma-1)\left[d_0^{(2,\alpha)} - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}c_0^{(2,\alpha)}\right] \\
&\quad - \sigma\left[d_1^{(2,\alpha)} - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}c_1^{(2,\alpha)}\right] \\
&\geqslant \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \cdot \frac{(2\sigma-1)(1-\sigma)}{2\sigma(1+\sigma)^{\alpha-1}} - (2\sigma-1)\frac{\epsilon}{4\Gamma(1-\alpha)} - \sigma\frac{\epsilon}{\Gamma(1-\alpha)} \\
&= \frac{1}{\Gamma(1-\alpha)}\left[\frac{\tau^{-\alpha}(1-\sigma)}{2\sigma(1+\sigma)^{\alpha-1}} - \frac{6\sigma-1}{4}\epsilon\right] \\
&\geqslant 0.
\end{aligned}$$

When $n \geqslant 3$, in view of

$$(2\sigma - 1)c_0^{(n,\alpha)} - \sigma c_1^{(n,\alpha)} \geqslant \frac{(2\sigma-1)(1-\sigma)}{2\sigma(1+\sigma)^{\alpha}}$$

and (1.147), we have

$$\begin{aligned}
&(2\sigma - 1)d_0^{(n,\alpha)} - \sigma d_1^{(n,\alpha)} \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[(2\sigma-1)c_0^{(n,\alpha)} - \sigma c_1^{(n,\alpha)}] + (2\sigma-1)\left[d_0^{(n,\alpha)} - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}c_0^{(n,\alpha)}\right] \\
&\quad - \sigma\left[d_1^{(n,\alpha)} - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}c_1^{(n,\alpha)}\right] \\
&\geqslant \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \cdot \frac{(2\sigma-1)(1-\sigma)}{2\sigma(1+\sigma)^{\alpha}} - (2\sigma-1)\frac{\epsilon}{4\Gamma(1-\alpha)} - \sigma\frac{5\epsilon}{4\Gamma(1-\alpha)} \\
&= \frac{1}{\Gamma(1-\alpha)}\left[\frac{\tau^{-\alpha}(1-\sigma)}{2\sigma(1+\sigma)^{\alpha}} - \frac{7\sigma-1}{4}\epsilon\right] \\
&\geqslant 0.
\end{aligned}$$

In addition,

$$d_{n-1}^{(n,\alpha)} \geqslant \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}c_{n-1}^{(n,\alpha)} - \frac{\epsilon}{\Gamma(1-\alpha)}$$

$$\geqslant \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \cdot (1-\alpha)n^{-\alpha} - \frac{\epsilon}{\Gamma(1-\alpha)}$$
$$\geqslant \frac{1}{\Gamma(1-\alpha)}(t_n^{-\alpha} - \epsilon) \geqslant \frac{t_n^{-\alpha}}{2\Gamma(1-\alpha)}.$$

The proof ends. □

1.7.3 Fast H2N2 approximation

In what follows, we will give a fast algorithm for the H2N2 interpolation approximation of Caputo fractional derivative of order $\gamma \in (1,2)$[71].

In this subsection, the superscript $(\gamma-1)$ in $N_{\text{exp}}^{(\gamma-1)}$, $s_l^{(\gamma-1)}$, $\omega_l^{(\gamma-1)}$ will be omitted for brevity.

Applying Lemma 1.7.1 and the quadratic interpolation polynomials $H_{2,0}(t)$ and $N_{2,k}(t)$, we have

$$\begin{aligned}
&{}_0^C D_t^\gamma f(t)|_{t=t_{n-\frac{1}{2}}} \\
&= \frac{1}{\Gamma(2-\gamma)}\left[\int_{t_0}^{t_{\frac{1}{2}}} f''(t)(t_{n-\frac{1}{2}}-t)^{1-\gamma}\mathrm{d}t + \sum_{k=1}^{n-1}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} f''(t)(t_{n-\frac{1}{2}}-t)^{1-\gamma}\mathrm{d}t\right] \\
&\approx \frac{1}{\Gamma(2-\gamma)}\left[\int_{t_0}^{t_{\frac{1}{2}}} H''_{2,0}(t)\sum_{l=1}^{N_{\text{exp}}}\omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)}\mathrm{d}t\right. \\
&\quad + \sum_{k=1}^{n-2}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N''_{2,k}(t)\sum_{l=1}^{N_{\text{exp}}}\omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)}\mathrm{d}t \\
&\quad \left. + \int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} N''_{2,n-1}(t)(t_{n-\frac{1}{2}}-t)^{1-\gamma}\mathrm{d}t\right] \qquad (1.148) \\
&= \frac{1}{\Gamma(2-\gamma)}\left[\sum_{l=1}^{N_{\text{exp}}}\omega_l F_l^n + \delta_t^2 f^{n-1}\int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}}(t_{n-\frac{1}{2}}-t)^{1-\gamma}\mathrm{d}t\right] \\
&\equiv {}^{\mathcal{F}}\hat{D}^\gamma f(t_{n-\frac{1}{2}}), \quad 2 \leqslant n \leqslant N, \qquad (1.149)
\end{aligned}$$

where

$$F_l^n = \int_{t_0}^{t_{\frac{1}{2}}} H''_{2,0}(t)e^{-s_l(t_{n-\frac{1}{2}}-t)}\mathrm{d}t + \sum_{k=1}^{n-2}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N''_{2,k}(t)e^{-s_l(t_{n-\frac{1}{2}}-t)}\mathrm{d}t,$$

$$1 \leqslant l \leqslant N_{\exp}, \quad 2 \leqslant n \leqslant N.$$

The evaluation of F_l^n can be carried out using the following recursive algorithm:

$$\begin{aligned}
F_l^n &= \int_{t_0}^{t_{\frac{1}{2}}} H_{2,0}''(t) e^{-s_l(t_{n-\frac{1}{2}}-t)} \mathrm{d}t + \sum_{k=1}^{n-2} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N_{2,k}''(t) e^{-s_l(t_{n-\frac{1}{2}}-t)} \mathrm{d}t \\
&= e^{-s_l\tau}\left[\int_{t_0}^{t_{\frac{1}{2}}} H_{2,0}''(t) e^{-s_l(t_{n-\frac{3}{2}}-t)} \mathrm{d}t + \sum_{k=1}^{n-3} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N_{2,k}''(t) e^{-s_l(t_{n-\frac{3}{2}}-t)} \mathrm{d}t\right] \\
&\quad + \int_{t_{n-\frac{5}{2}}}^{t_{n-\frac{3}{2}}} N_{2,n-2}''(t) e^{-s_l(t_{n-\frac{1}{2}}-t)} \mathrm{d}t \qquad (1.150) \\
&= e^{-s_l\tau} F_l^{n-1} + \delta_t^2 f^{n-2} \int_{t_{n-\frac{5}{2}}}^{t_{n-\frac{3}{2}}} e^{-s_l(t_{n-\frac{1}{2}}-t)} \mathrm{d}t \\
&= e^{-s_l\tau} F_l^{n-1} + B_l(\delta_t f^{n-\frac{3}{2}} - \delta_t f^{n-\frac{5}{2}}), \quad 3 \leqslant n \leqslant N, \qquad (1.151)
\end{aligned}$$

where

$$F_l^2 = \int_{t_0}^{t_{\frac{1}{2}}} H_{2,0}''(t) e^{-s_l(t_{\frac{3}{2}}-t)} \mathrm{d}t = \frac{2}{s_l\tau}(e^{-s_l\tau} - e^{-\frac{3}{2}s_l\tau})(\delta_t f^{\frac{1}{2}} - f'(t_0)), \qquad (1.152)$$

$$B_l = \frac{1}{\tau}\int_{t_{n-\frac{5}{2}}}^{t_{n-\frac{3}{2}}} e^{-s_l(t_{n-\frac{1}{2}}-t)} \mathrm{d}t = \frac{1}{s_l\tau}(e^{-s_l\tau} - e^{-2s_l\tau}). \qquad (1.153)$$

From (1.149)–(1.153), the following algorithm is obtained to evaluate ${}_0^C D_t^\gamma f(t)|_{t=t_{n-\frac{1}{2}}}$:

$$\begin{cases}
{}^F\hat{D}^\gamma f(t_{n-\frac{1}{2}}) = \dfrac{1}{\Gamma(2-\gamma)}\left[\displaystyle\sum_{l=1}^{N_{\exp}} \omega_l F_l^n + \frac{\tau^{2-\gamma}}{2-\gamma}\delta_t^2 f^{n-1}\right], \quad 2 \leqslant n \leqslant N, & (1.154) \\
F_l^2 = \dfrac{2}{\tau}\displaystyle\int_{t_0}^{t_{\frac{1}{2}}} e^{-s_l(t_{\frac{3}{2}}-t)} \mathrm{d}t(\delta_t f^{\frac{1}{2}} - f'(t_0)), \quad 1 \leqslant l \leqslant N_{\exp}, & (1.155) \\
F_l^n = e^{-s_l\tau} F_l^{n-1} + B_l(\delta_t f^{n-\frac{3}{2}} - \delta_t f^{n-\frac{5}{2}}), \quad 1 \leqslant l \leqslant N_{\exp}, \quad 3 \leqslant n \leqslant N. & (1.156)
\end{cases}$$

In what follows, we try to analyze the truncation error by using ${}^{\mathcal{F}}\hat{D}^\gamma f(t_{n-\frac{1}{2}})$ to approximate ${}_0^C D_t^\gamma f(t)|_{t=t_{n-\frac{1}{2}}}$.

Theorem 1.7.3. *Let* $f \in C^3[t_0, t_n]$. *Denote*

$$\hat{R}^n = {}_0^C D_t^\gamma f(t)|_{t=t_{n-\frac{1}{2}}} - {}^{\mathcal{F}}\hat{D}^\gamma f(t_{n-\frac{1}{2}}).$$

Then, when $n \geqslant 2$,

$$\begin{aligned} |\hat{R}^n| \leqslant& \left[\frac{1}{8\Gamma(2-\gamma)} + \frac{1}{12\Gamma(3-\gamma)} + \frac{\gamma-1}{2\Gamma(4-\gamma)}\right] \max_{t_0 \leqslant t \leqslant t_n} |f'''(t)| \tau^{3-\gamma} \\ &+ \frac{\epsilon\, t_{n-\frac{3}{2}}}{\Gamma(2-\gamma)} \max_{t_0 \leqslant t \leqslant t_n} |f''(t)|. \end{aligned} \tag{1.157}$$

Proof. Notice the fact that

$$\begin{aligned} \hat{R}^n =& \frac{1}{\Gamma(2-\gamma)}\left[\int_{t_0}^{t_{\frac{1}{2}}} f''(t)(t_{n-\frac{1}{2}}-t)^{1-\gamma} \mathrm{d}t + \sum_{k=1}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} f''(t)(t_{n-\frac{1}{2}}-t)^{1-\gamma} \mathrm{d}t\right] \\ &- \frac{1}{\Gamma(2-\gamma)}\left[\int_{t_0}^{t_{\frac{1}{2}}} H_{2,0}''(t) \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)} \mathrm{d}t\right. \\ &+ \sum_{k=1}^{n-2} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N_{2,k}''(t) \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)} \mathrm{d}t \\ &+ \left.\int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} N_{2,n-1}''(t)(t_{n-\frac{1}{2}}-t)^{1-\gamma} \mathrm{d}t\right] \\ =& \frac{1}{\Gamma(2-\gamma)}\left[\int_{t_0}^{t_{\frac{1}{2}}} (f''(t) - H_{2,0}''(t))(t_{n-\frac{1}{2}}-t)^{1-\gamma} \mathrm{d}t\right. \\ &+ \left.\sum_{k=1}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (f''(t) - N_{2,k}''(t))(t_{n-\frac{1}{2}}-t)^{1-\gamma} \mathrm{d}t\right] \\ &+ \frac{1}{\Gamma(2-\gamma)}\left[\int_{t_0}^{t_{\frac{1}{2}}} H_{2,0}''(t)\left((t_{n-\frac{1}{2}}-t)^{1-\gamma} - \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)}\right)\mathrm{d}t\right. \\ &+ \left.\sum_{k=1}^{n-2} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N_{2,k}''(t)\left((t_{n-\frac{1}{2}}-t)^{1-\gamma} - \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)}\right)\mathrm{d}t\right] \\ \equiv& \mathrm{I}_n + \mathrm{II}_n. \end{aligned} \tag{1.158}$$

On the one hand, using Theorem 1.6.6, we have

$$|\mathrm{I}_n| \leqslant \left[\frac{1}{8\Gamma(2-\gamma)} + \frac{1}{12\Gamma(3-\gamma)} + \frac{\gamma-1}{2\Gamma(4-\gamma)}\right] \max_{t_0\leqslant t\leqslant t_n} |f'''(t)|\tau^{3-\gamma}. \tag{1.159}$$

On the other hand, combining (1.100)–(1.101) with (1.102)–(1.103), we have

$$\begin{aligned} |\mathrm{II}_n| &\leqslant \frac{\epsilon}{\Gamma(2-\gamma)}\left[\int_{t_0}^{t_{\frac{1}{2}}} |H''_{2,0}(t)|\mathrm{d}t + \sum_{k=1}^{n-2}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} |N''_{2,k}(t)|\mathrm{d}t\right] \\ &= \frac{\epsilon}{\Gamma(2-\gamma)}\left[\int_{t_0}^{t_{\frac{1}{2}}} |f''(\zeta_0)|\mathrm{d}t + \sum_{k=1}^{n-2}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} |f''(\zeta_k)|\mathrm{d}t\right] \\ &\leqslant \frac{\epsilon\, t_{n-\frac{3}{2}}}{\Gamma(2-\gamma)} \max_{t_0\leqslant t\leqslant t_n} |f''(t)|, \end{aligned} \tag{1.160}$$

where $\zeta_0 \in (t_0, t_1)$, $\zeta_k \in (t_{k-1}, t_{k+1})$, $1 \leqslant k \leqslant n-2$. Using (1.158)–(1.160), we can get (1.157). The proof ends. □

Denote

$$\begin{cases} \tilde{b}_0^{(n,\gamma)} = \dfrac{1}{\tau}\displaystyle\int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\gamma}\mathrm{d}t = \dfrac{\tau^{1-\gamma}}{2-\gamma}, & (1.161) \\ \tilde{b}_{n-k-1}^{(n,\gamma)} = \dfrac{1}{\tau}\displaystyle\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)}\mathrm{d}t = \int_0^1 \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(n-k-\xi)\tau}\mathrm{d}\xi, & \\ \qquad 1 \leqslant k \leqslant n-2, & (1.162) \\ \tilde{b}_{n-1}^{(n,\gamma)} = \dfrac{2}{\tau}\displaystyle\int_{t_0}^{t_{\frac{1}{2}}} \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)}\mathrm{d}t = \int_0^1 \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(n-\frac{1}{2}-\frac{\xi}{2})\tau}\mathrm{d}\xi. & (1.163) \end{cases}$$

Then it follows from (1.148) that

$$\begin{aligned} &{}^{\mathcal{F}}\hat{D}^{\gamma} f(t_{n-\frac{1}{2}}) \\ &= \frac{1}{\Gamma(2-\gamma)}\left[\tilde{b}_{n-1}^{(n,\gamma)} \cdot (\delta_t f^{\frac{1}{2}} - f'(t_0)) + \sum_{k=1}^{n-1} \tilde{b}_{n-k-1}^{(n,\gamma)} \cdot (\delta_t f^{k+\frac{1}{2}} - \delta_t f^{k-\frac{1}{2}})\right] \\ &= \frac{1}{\Gamma(2-\gamma)}\left[\tilde{b}_0^{(n,\gamma)}\delta_t f^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(\tilde{b}_{n-k-1}^{(n,\gamma)} - \tilde{b}_{n-k}^{(n,\gamma)})\delta_t f^{k-\frac{1}{2}} - \tilde{b}_{n-1}^{(n,\gamma)} f'(t_0)\right]. \end{aligned} \tag{1.164}$$

From the definitions of $\tilde{b}_k^{(n,\gamma)}$ and $\hat{b}_k^{(n,\gamma)}$, we have

$$\tilde{b}_0^{(n,\gamma)} - \hat{b}_0^{(n,\gamma)} = 0; \quad |\tilde{b}_k^{(n,\gamma)} - \hat{b}_k^{(n,\gamma)}| \leqslant \epsilon, \quad 1 \leqslant k \leqslant n-1.$$

The coefficients in the formula (1.164) satisfy the following lemma.

Lemma 1.7.4.

$$\tilde{b}_1^{(n,\gamma)} > \tilde{b}_2^{(n,\gamma)} > \tilde{b}_3^{(n,\gamma)} > \cdots > \tilde{b}_{n-1}^{(n,\gamma)}.$$

When $\epsilon < \frac{2-2^{2-\gamma}}{2-\gamma}\tau^{1-\gamma}$,

$$\tilde{b}_0^{(n,\gamma)} > \tilde{b}_1^{(n,\gamma)}.$$

Proof. We rewrite (1.162) as

$$\begin{aligned}\tilde{b}_k^{(n,\gamma)} &= \frac{1}{\tau}\int_{t_{n-k-\frac{3}{2}}}^{t_{n-k-\frac{1}{2}}} \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)} \mathrm{d}t \\ &= \int_0^1 \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l(k+1-\xi)\tau} \mathrm{d}\xi, \quad 1 \leqslant k \leqslant n-2. \end{aligned} \tag{1.165}$$

Then the following inequalites

$$\tilde{b}_1^{(n,\gamma)} > \tilde{b}_2^{(n,\gamma)} > \cdots > \tilde{b}_{n-1}^{(n,\gamma)}$$

are followed immediately by (1.165) and (1.163). In addition,

$$\begin{aligned}\tilde{b}_0^{(n,\gamma)} - \tilde{b}_1^{(n,\gamma)} &= \hat{b}_0^{(n,\gamma)} - \tilde{b}_1^{(n,\gamma)} \\ &= (\hat{b}_0^{(n,\gamma)} - \hat{b}_1^{(n,\gamma)}) + (\hat{b}_1^{(n,\gamma)} - \tilde{b}_1^{(n,\gamma)}) \\ &\geqslant (\hat{b}_0^{(n,\gamma)} - \hat{b}_1^{(n,\gamma)}) - \epsilon \\ &= \frac{\tau^{1-\gamma}}{2-\gamma} - (2^{2-\gamma}-1)\frac{\tau^{1-\gamma}}{2-\gamma} - \epsilon \\ &= (2-2^{2-\gamma})\frac{\tau^{1-\gamma}}{2-\gamma} - \epsilon > 0.\end{aligned}$$

This completes the proof. □

1.8 Finite difference methods for FODEs

1.8.1 Method based on G-L approximation

Problem 1.8.1. Solve the following initial value problem:

$$\begin{cases} {}_0\mathbf{D}_t^{\alpha} y(t) = f(t), \quad 0 < t \leqslant T, & (1.166) \\ y(0) = 0, & (1.167) \end{cases}$$

where $\alpha \in (0,1)$.

Define the function

$$\hat{u}(t) = \begin{cases} 0, & t < 0, \\ y(t), & 0 \leqslant t \leqslant T, \\ v(t), & T < t < 2T, \\ 0, & t \geqslant 2T, \end{cases}$$

where $v(t)$ is a smooth function satisfying $v^{(k)}(T) = y^{(k)}(T)$ and $v^{(k)}(2T) = 0$, $k = 0,1,2$. Suppose $\hat{u} \in \mathscr{C}^{1+\alpha}(\mathcal{R})$. It should be pointed out that the aim of the introduction of function $v(t)$ is only to provide a sufficient condition for using Theorem 1.4.2 without any practical calculation.

Take a positive integer N and denote $\tau = \frac{T}{N}$, $t_k = k\tau$, $k = 0,1,2,\dots,N$.

Considering equation (1.166) at $t = t_n$, we have

$${}_0\mathbf{D}_t^{\alpha} y(t)|_{t=t_n} = f(t_n), \quad 1 \leqslant n \leqslant N.$$

It follows from the G-L formula (1.19) that

$$\tau^{-\alpha} \sum_{k=0}^{n} g_k^{(\alpha)} y(t_{n-k}) = f(t_n) + (r_1)^n, \quad 1 \leqslant n \leqslant N, \tag{1.168}$$

where, by Theorem 1.4.2, there is a positive constant c_1 such that

$$|(r_1)^n| \leqslant c_1\tau, \quad 1 \leqslant n \leqslant N. \tag{1.169}$$

Noticing

$$y(t_0) = 0, \tag{1.170}$$

omitting the small term $(r_1)^n$ in (1.168) and replacing the exact solution $y(t_n)$ with the numerical one y^n produce the difference scheme for solving (1.166)–(1.167) as follows:

$$\begin{cases} \tau^{-\alpha} \sum\limits_{k=0}^{n} g_k^{(\alpha)} y^{n-k} = f(t_n), \quad 1 \leqslant n \leqslant N, & (1.171) \\ y^0 = 0. & (1.172) \end{cases}$$

The stability of the difference scheme

Theorem 1.8.1. *Suppose* $\{y^n \mid n = 0,1,2,\dots,N\}$ *is the solution of the difference scheme* (1.171)–(1.172)*, then it holds*

$$|y^k| \leqslant \frac{5}{(1-\alpha)2^{\alpha}} k^{\alpha}\tau^{\alpha} \max_{1\leqslant m\leqslant k} |f(t_m)|, \quad 1 \leqslant k \leqslant N. \tag{1.173}$$

Proof. Noticing that when $0 < \alpha < 1$, $g_0^{(\alpha)} = 1$, $g_k^{(\alpha)} < 0$, $k \geqslant 1$, rewrite (1.171) as follows:

$$y^n = \sum_{k=1}^{n-1}(-g_k^{(\alpha)})y^{n-k} + \tau^\alpha f(t_n), \quad 1 \leqslant n \leqslant N. \tag{1.174}$$

Next, the induction method will be used to prove (1.173).

When $n = 1$, it follows from (1.174) that

$$|y^1| = \tau^\alpha |f(t_1)| \leqslant \frac{5}{(1-\alpha)2^\alpha}\tau^\alpha |f(t_1)|.$$

Thus, (1.173) is true for $k = 1$. Now assume that (1.173) is true for $k = 1, 2, \ldots, n-1$, then it follows from (1.174) that

$$\begin{aligned}
|y^n| &\leqslant \sum_{k=1}^{n-1}(-g_k^{(\alpha)})|y^{n-k}| + \tau^\alpha |f(t_n)| \\
&\leqslant \sum_{k=1}^{n-1}(-g_k^{(\alpha)})\left[\frac{5}{(1-\alpha)2^\alpha}(n-k)^\alpha \tau^\alpha \max_{1\leqslant m \leqslant n-k}|f(t_m)|\right] + \tau^\alpha |f(t_n)| \\
&\leqslant \left[\sum_{k=1}^{n-1}(-g_k^{(\alpha)})\frac{5}{(1-\alpha)2^\alpha}n^\alpha + 1\right]\tau^\alpha \max_{1\leqslant m\leqslant n}|f(t_m)| \\
&= \left\{\left[\sum_{k=1}^{\infty}(-g_k^{(\alpha)}) - \sum_{k=n}^{\infty}(-g_k^{(\alpha)})\right]\frac{5}{(1-\alpha)2^\alpha}n^\alpha + 1\right\}\tau^\alpha \max_{1\leqslant m\leqslant n}|f(t_m)| \\
&\leqslant \left\{\left[1 - \frac{1-\alpha}{5}\cdot\left(\frac{2}{n}\right)^\alpha\right]\frac{5}{(1-\alpha)2^\alpha}n^\alpha + 1\right\}\tau^\alpha \max_{1\leqslant m\leqslant n}|f(t_m)| \\
&= \frac{5}{(1-\alpha)2^\alpha}n^\alpha \tau^\alpha \max_{1\leqslant m\leqslant n}|f(t_m)|,
\end{aligned}$$

where Lemma 1.4.3 is used in the penultimate step above. Hence, (1.173) is also true for $k = n$.

By the principle of induction, the theorem is true. The proof ends. □

The convergence of the difference scheme

Theorem 1.8.2. *Suppose* $\{y(t_n) \mid n = 0, 1, 2, \ldots, N\}$ *and* $\{y^n \mid n = 0, 1, 2, \ldots, N\}$ *are solutions of the problem* (1.166)–(1.167) *and the difference scheme* (1.171)–(1.172), *respectively. Let*

$$e^n = y(t_n) - y^n, \quad n = 0, 1, 2, \ldots, N,$$

then it holds

$$|e^n| \leqslant \frac{5c_1}{(1-\alpha)2^\alpha}T^\alpha \tau, \quad 1 \leqslant n \leqslant N.$$

Proof. Subtracting (1.171)–(1.172) from (1.168) and (1.170), respectively, yields the system of error equations as follows:

$$\begin{cases} \tau^{-\alpha} \sum_{k=0}^{n} g_k^{(\alpha)} e^{n-k} = (r_1)^n, \quad 1 \leqslant n \leqslant N, \\ e^0 = 0. \end{cases}$$

Noticing (1.169), the application of Theorem 1.8.1 produces

$$|e^n| \leqslant \frac{5}{(1-\alpha)2^\alpha} n^\alpha \tau^\alpha \max_{1 \leqslant m \leqslant n} |(r_1)^m| \leqslant \frac{5c_1}{(1-\alpha)2^\alpha} T^\alpha \tau, \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

Problem 1.8.2. Solve the initial value problem

$$\begin{cases} {}_0\mathbf{D}_t^\gamma y(t) = f(t), \quad 0 < t \leqslant T, & (1.175) \\ y(0) = 0, \quad y'(0) = 0, & (1.176) \end{cases}$$

where $\gamma \in (1, 2)$.

Define the function

$$\hat{u}(t) = \begin{cases} 0, & t < 0, \\ y(t), & 0 \leqslant t \leqslant T, \\ v(t), & T < t < 2T, \\ 0, & t \geqslant 2T, \end{cases}$$

where $v(t)$ is a smooth function satisfying $v^{(k)}(T) = y^{(k)}(T)$, $v^{(k)}(2T) = 0$, $k = 0, 1, 2, 3$. Suppose $\hat{u} \in \mathscr{C}^{1+\gamma}(\mathcal{R})$ and $y \in C^3[0, T]$.

Let

$$z(t) = y'(t), \quad \alpha = \gamma - 1,$$

then from (1.175)–(1.176), the differential equation for $z(t)$ is obtained as follows:

$$\begin{cases} {}_0\mathbf{D}_t^\alpha z(t) = f(t), \quad 0 < t \leqslant T, & (1.177) \\ z(0) = 0. & (1.178) \end{cases}$$

Denote

$$Y^n = y(t_n), \quad Z^n = z(t_n), \quad 0 \leqslant n \leqslant N,$$
$$Y^{n-\frac{1}{2}} = \frac{1}{2}(Y^n + Y^{n-1}), \quad \delta_t Y^{n-\frac{1}{2}} = \frac{1}{\tau}(Y^n - Y^{n-1}), \quad 1 \leqslant n \leqslant N.$$

Considering equation (1.177) at $t = t_n$, we have

$$ {}_0\mathbf{D}_t^\alpha z(t)|_{t=t_n} = f(t_n), \quad 1 \leqslant n \leqslant N.$$

It follows from the G-L formula (1.19) that

$$\begin{cases} \tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)} Z^{n-k} = f(t_n) + (r_2)^n, \quad 1 \leqslant n \leqslant N, & (1.179)\\ Z^0 = 0, & (1.180)\end{cases}$$

where, by Theorem 1.4.2, there is a positive constant c_2 such that

$$\left|(r_2)^n\right| \leqslant c_2\tau, \quad 1 \leqslant n \leqslant N. \tag{1.181}$$

It follows from (1.179) and (1.180) that

$$\tau^{-\alpha}\sum_{k=0}^{n-1} g_k^{(\alpha)} \frac{Z^{n-k} + Z^{n-k-1}}{2} = f^{n-\frac{1}{2}} + \frac{1}{2}\left[(r_2)^n + (r_2)^{n-1}\right], \quad 1 \leqslant n \leqslant N,$$

where $(r_2)^0 = 0, f^{n-\frac{1}{2}} = \frac{1}{2}[f(t_n) + f(t_{n-1})]$. Substituting

$$\delta_t Y^{n-k-\frac{1}{2}} = \frac{1}{2}(Z^{n-k} + Z^{n-k-1}) + O(\tau^2)$$

into the equality above gives

$$\tau^{1-\gamma}\sum_{k=0}^{n-1} g_k^{(\gamma-1)} \delta_t Y^{n-k-\frac{1}{2}} = f^{n-\frac{1}{2}} + (r_3)^{n-\frac{1}{2}}, \quad 1 \leqslant n \leqslant N, \tag{1.182}$$

and there is a positive constant c_3 such that

$$\left|(r_3)^{n-\frac{1}{2}}\right| \leqslant c_3\tau, \quad 1 \leqslant n \leqslant N. \tag{1.183}$$

Noticing

$$y(t_0) = 0, \tag{1.184}$$

omitting the small term $(r_3)^{n-\frac{1}{2}}$ in (1.182) and replacing the exact solution $y(t_n)$ with its numerical one y^n arrive at the difference scheme for solving the problem (1.175)–(1.176) as follows:

$$\begin{cases} \tau^{1-\gamma}\sum_{k=0}^{n-1} g_k^{(\gamma-1)} \delta_t y^{n-k-\frac{1}{2}} = f^{n-\frac{1}{2}}, \quad 1 \leqslant n \leqslant N, & (1.185)\\ y^0 = 0. & (1.186)\end{cases}$$

The stability of the difference scheme

Theorem 1.8.3. *Suppose* $\{y^n \mid n = 0, 1, 2, \ldots, N\}$ *is the solution of the difference scheme* (1.185)–(1.186), *then it holds*

$$\left|\delta_t y^{k-\frac{1}{2}}\right| \leqslant \frac{5}{(2-\gamma)2^{\gamma-1}} t_k^{\gamma-1} \max_{1\leqslant m\leqslant k}\left|f^{m-\frac{1}{2}}\right|, \quad 1 \leqslant k \leqslant N \tag{1.187}$$

and

$$|y^k| \leqslant \frac{5}{(2-\gamma)2^{\gamma-1}} t_k^\gamma \max_{1\leqslant m\leqslant k} |f^{m-\frac{1}{2}}|, \quad 1 \leqslant k \leqslant N. \tag{1.188}$$

Proof. The inequality (1.187) can be obtained similarly to Theorem 1.8.1. Noticing

$$y^k = y^0 + \tau \sum_{m=1}^{k} \delta_t y^{m-\frac{1}{2}} = \tau \sum_{m=1}^{k} \delta_t y^{m-\frac{1}{2}},$$

the application of the inequality (1.187) will produce (1.188). The proof ends. □

The convergence of the difference scheme

Theorem 1.8.4. *Suppose* $\{y(t_n) \mid n = 0,1,2,\ldots,N\}$ *and* $\{y^n \mid n = 0,1,2,\ldots,N\}$ *are solutions of the problem* (1.175)–(1.176) *and the difference scheme* (1.185)–(1.186), *respectively. Let*

$$e^n = y(t_n) - y^n, \quad n = 0,1,2,\ldots,N,$$

then it holds

$$|e^n| \leqslant \frac{5c_3}{(2-\gamma)2^{\gamma-1}} T^\gamma \tau, \quad 1 \leqslant n \leqslant N.$$

Proof. Subtracting (1.185)–(1.186) from (1.182) and (1.184), respectively, yields the system of error equations as follows:

$$\begin{cases} \tau^{1-\gamma} \sum\limits_{k=0}^{n-1} g_k^{(\gamma-1)} \delta_t e^{n-k-\frac{1}{2}} = (r_3)^{n-\frac{1}{2}}, & 1 \leqslant n \leqslant N, \\ e^0 = 0. \end{cases}$$

Noticing (1.183), the application of Theorem 1.8.3 arrives at

$$|e^n| \leqslant \frac{5}{(2-\gamma)2^{\gamma-1}} t_n^\gamma \max_{1\leqslant m\leqslant n} |(r_3)^{m-\frac{1}{2}}| \leqslant \frac{5c_3}{(2-\gamma)2^{\gamma-1}} T^\gamma \tau, \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

Problem 1.8.3. Solve the boundary value problem

$$\begin{cases} {}_0\mathbf{D}_t^\gamma y(t) = f(t), \quad 0 < t < T, & (1.189) \\ y(0) = 0, \quad y(T) = B, & (1.190) \end{cases}$$

where $\gamma \in (1,2)$.

Define the function

$$\hat{u}(t)=\begin{cases}0, & t<0,\\ y(t), & 0\leqslant t\leqslant T,\\ v(t), & T<t\leqslant 2T,\\ 0, & t>2T,\end{cases}$$

where $v(t)$ is a smooth function satisfying $v^{(k)}(T)=y^{(k)}(T)$, $v^{(k)}(2T)=0$, $k=0,1,2,3$ and suppose $\hat{u}\in\mathscr{C}^{1+\gamma}(\mathcal{R})$.

Considering equation (1.189) at $t=t_n$, we have

$${}_0\mathbf{D}_t^{\gamma}y(t)|_{t=t_n}=f(t_n),\quad 1\leqslant n\leqslant N-1.$$

It follows from the shifted G-L formula (1.19) that

$$\tau^{-\gamma}\sum_{k=0}^{n+1}g_k^{(\gamma)}y(t_{n-k+1})=f(t_n)+(r_4)^n,\quad 1\leqslant n\leqslant N-1,\tag{1.191}$$

where, by Theorem 1.4.2, there is a positive constant c_4 such that

$$\left|(r_4)^n\right|\leqslant c_4\tau,\quad 1\leqslant n\leqslant N-1.\tag{1.192}$$

Noticing the boundary conditions

$$y(t_0)=0,\quad y(t_N)=B,\tag{1.193}$$

omitting the small term $(r_4)^n$ in (1.191) and replacing the exact solution $y(t_n)$ with its numerical one y^n arrive at the difference scheme for solving (1.189)–(1.190) as follows:

$$\begin{cases}\tau^{-\gamma}\sum\limits_{k=0}^{n+1}g_k^{(\gamma)}y^{n-k+1}=f(t_n),\quad 1\leqslant n\leqslant N-1, & (1.194)\\ y^0=0,\quad y^N=B. & (1.195)\end{cases}$$

The stability of the difference scheme

Theorem 1.8.5. *Suppose $\{y^n\mid n=0,1,2,\ldots,N\}$ is the solution of the difference scheme*

$$\begin{cases}\tau^{-\gamma}\sum\limits_{k=0}^{n+1}g_k^{(\gamma)}y^{n-k+1}=f(t_n),\quad 1\leqslant n\leqslant N-1, & (1.196)\\ y^0=0,\quad y^N=0, & (1.197)\end{cases}$$

then it holds

$$\|y\|_\infty\leqslant\frac{45}{(\gamma-1)(2-\gamma)(3-\gamma)}\left(\frac{T}{4}\right)^{\gamma}\|f\|_\infty,\tag{1.198}$$

where

$$\|y\|_\infty=\max_{1\leqslant n\leqslant N-1}\left|y^n\right|,\quad \|f\|_\infty=\max_{1\leqslant n\leqslant N-1}\left|f(t_n)\right|.$$

Proof. Noticing that $g_0^{(\gamma)} = 1$, $g_1^{(\gamma)} = -\gamma$, $g_2^{(\gamma)} > g_3^{(\gamma)} > \cdots > 0$ when $1 < \gamma < 2$, rewrite (1.196) as follows:

$$(-g_1^{(\gamma)})y^n = \sum_{\substack{k=0\\k\neq 1}}^{n} g_k^{(\gamma)} y^{n-k+1} - \tau^\gamma f(t_n), \quad 1 \leqslant n \leqslant N-1. \tag{1.199}$$

Suppose $\|y\|_\infty = |y^{n_0}|$, where $n_0 \in \{1, 2, \ldots, N-1\}$.

Letting $n = n_0$ in (1.199) and taking the absolute values of the both hand sides, an application of the triangle inequality yields

$$\begin{aligned}(-g_1^{(\gamma)})\|y\|_\infty &\leqslant \sum_{\substack{k=0\\k\neq 1}}^{n_0} g_k^{(\gamma)}\|y\|_\infty + \tau^\gamma\|f\|_\infty\\ &\leqslant \sum_{\substack{k=0\\k\neq 1}}^{N-1} g_k^{(\gamma)}\|y\|_\infty + \tau^\gamma\|f\|_\infty,\end{aligned}$$

that is,

$$\left(-\sum_{k=0}^{N-1} g_k^{(\gamma)}\right)\|y\|_\infty \leqslant \tau^\gamma\|f\|_\infty.$$

In view of

$$\sum_{k=0}^{\infty} g_k^{(\gamma)} = 0,$$

it follows

$$-\sum_{k=0}^{N-1} g_k^{(\gamma)} = \sum_{k=N}^{\infty} g_k^{(\gamma)} > 0.$$

Thus,

$$\|y\|_\infty \leqslant \frac{\tau^\gamma}{\sum_{k=N}^{\infty} g_k^{(\gamma)}}\|f\|_\infty.$$

The application of Lemma 1.4.4 gives

$$\begin{aligned}\|y\|_\infty &\leqslant \frac{\tau^\gamma}{\frac{(\gamma-1)(2-\gamma)(3-\gamma)}{45}(\frac{4}{N})^\gamma}\|f\|_\infty\\ &= \frac{45}{(\gamma-1)(2-\gamma)(3-\gamma)}\left(\frac{T}{4}\right)^\gamma\|f\|_\infty.\end{aligned}$$

The proof ends. □

The convergence of the difference scheme

Theorem 1.8.6. *Suppose* $\{y(t_n) \mid n = 0, 1, \ldots, N\}$ *and* $\{y^n \mid n = 0, 1, \ldots, N\}$ *are solutions of the problem* (1.189)–(1.190) *and the difference scheme* (1.194)–(1.195), *respectively. Let*

$$e^n = y(t_n) - y^n, \quad n = 0, 1, \ldots, N,$$

then it holds

$$\|e\|_\infty \leqslant \frac{45}{(\gamma-1)(2-\gamma)(3-\gamma)} \left(\frac{T}{4}\right)^\gamma c_4 \tau.$$

Proof. Subtracting (1.194)–(1.195) from (1.191) and (1.193), respectively, yields the system of error equations as follows:

$$\begin{cases} \tau^{-\gamma} \sum\limits_{k=0}^{n+1} g_k^{(\gamma)} e^{n-k+1} = (r_4)^n, \quad 1 \leqslant n \leqslant N-1, \\ e^0 = 0, \quad e^N = 0. \end{cases}$$

Noticing (1.192), the application of Theorem 1.8.5 gives

$$\begin{aligned} \|e\|_\infty &\leqslant \frac{45}{(\gamma-1)(2-\gamma)(3-\gamma)} \left(\frac{T}{4}\right)^\gamma \max_{1 \leqslant n \leqslant N-1} \left|(r_4)^n\right| \\ &\leqslant \frac{45}{(\gamma-1)(2-\gamma)(3-\gamma)} \left(\frac{T}{4}\right)^\gamma c_4 \tau. \end{aligned}$$

The proof ends. □

1.8.2 Method based on L1 approximation

Problem 1.8.4. Solve the initial value problem

$$\begin{cases} {}_0^C D_t^\alpha y(t) = f(t), \quad 0 < t \leqslant T, & (1.200) \\ y(0) = A, & (1.201) \end{cases}$$

where $\alpha \in (0, 1)$.

Suppose $y \in C^2[0, T]$. Considering equation (1.200) at $t = t_n$, we have

$${}_0^C D_t^\alpha y(t)|_{t=t_n} = f(t_n), \quad 1 \leqslant n \leqslant N.$$

It follows from Theorem 1.6.1 that

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[a_0^{(\alpha)} y(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) y(t_k) - a_{n-1}^{(\alpha)} y(t_0) \right]$$

$$= f(t_n) + (r_5)^n, \quad 1 \leqslant n \leqslant N, \tag{1.202}$$

and there is a positive constant c_5 such that

$$|(r_5)^n| \leqslant c_5 \tau^{2-\alpha}, \quad 1 \leqslant n \leqslant N. \tag{1.203}$$

Noticing

$$y(t_0) = A, \tag{1.204}$$

omitting the small term $(r_5)^n$ in (1.202) and replacing the exact solution $y(t_n)$ with the numerical one y^n arrive at the difference scheme for solving (1.200)–(1.201) as follows:

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)} y^n - \displaystyle\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})y^k - a_{n-1}^{(\alpha)} y^0\right] = f(t_n), \quad 1 \leqslant n \leqslant N, & (1.205) \\ y^0 = A. & (1.206) \end{cases}$$

The stability of the difference scheme

Theorem 1.8.7. *Suppose* $\{y^n \mid n = 0, 1, 2, \ldots, N\}$ *is the solution of the difference scheme* (1.205)–(1.206), *then it holds*

$$|y^k| \leqslant |y^0| + \Gamma(1-\alpha) \max_{1 \leqslant l \leqslant k} |t_l^\alpha f(t_l)|, \quad 1 \leqslant k \leqslant N. \tag{1.207}$$

Proof. Reformulate (1.205) as follows:

$$\begin{aligned} a_0^{(\alpha)} y^n &= \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})y^k + a_{n-1}^{(\alpha)} y^0 + \tau^\alpha \Gamma(2-\alpha) f(t_n) \\ &= \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})y^k + a_{n-1}^{(\alpha)}\left[y^0 + \frac{\tau^\alpha}{a_{n-1}^{(\alpha)}}\Gamma(2-\alpha) f(t_n)\right]. \end{aligned}$$

Taking the absolute value on both hand sides of the equality above, the application of Lemma 1.6.1, the triangle inequality and

$$\frac{\tau^\alpha}{a_{n-1}^{(\alpha)}}\Gamma(2-\alpha) \leqslant \frac{\tau^\alpha n^\alpha}{1-\alpha}\Gamma(2-\alpha) = t_n^\alpha \Gamma(1-\alpha)$$

lead to

$$\begin{aligned} a_0^{(\alpha)} |y^n| \leqslant &\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})|y^k| \\ &+ a_{n-1}^{(\alpha)}[|y^0| + t_n^\alpha \Gamma(1-\alpha)|f(t_n)|], \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{1.208}$$

Next the induction method will be applied to prove the truth of (1.207).

When $n = 1$, it follows from (1.208) that

$$a_0^{(\alpha)}|y^1| \leqslant a_0^{(\alpha)}(|y^0| + \tau^\alpha \Gamma(1-\alpha)|f(t_1)|).$$

Obviously, (1.207) is true for $k = 1$. Now suppose (1.207) is true for $k = 1, 2, \ldots, n-1$, then it follows from (1.208) that

$$\begin{aligned} a_0^{(\alpha)}|y^n| &\leqslant \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})\Big[|y^0| + \Gamma(1-\alpha)\max_{1\leqslant l\leqslant k}|t_l^\alpha f(t_l)|\Big] \\ &\quad + a_{n-1}^{(\alpha)}[|y^0| + t_n^\alpha \Gamma(1-\alpha)|f(t_n)|] \\ &\leqslant \left\{\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) + a_{n-1}^{(\alpha)}\right\}\Big[|y^0| + \Gamma(1-\alpha)\max_{1\leqslant l\leqslant n}|t_l^\alpha f(t_l)|\Big] \\ &= a_0^{(\alpha)}\Big[|y^0| + \Gamma(1-\alpha)\max_{1\leqslant l\leqslant n}|t_l^\alpha f(t_l)|\Big]. \end{aligned}$$

Therefore,

$$|y^n| \leqslant |y^0| + \Gamma(1-\alpha)\max_{1\leqslant l\leqslant n}|t_l^\alpha f(t_l)|,$$

hence, (1.207) is also true for $k = n$.

By the principle of induction, the theorem is proved. The proof ends. □

The convergence of the difference scheme

Theorem 1.8.8. *Suppose* $\{y(t_n) \mid n = 0, 1, 2, \ldots, N\}$ *and* $\{y^n \mid n = 0, 1, 2, \ldots, N\}$ *are solutions of the problem* (1.200)–(1.201) *and the difference scheme* (1.205)–(1.206), *respectively. Let*

$$e^n = y(t_n) - y^n, \quad n = 0, 1, 2, \ldots, N,$$

then it holds

$$|e^n| \leqslant c_5 T^\alpha \Gamma(1-\alpha)\tau^{2-\alpha}, \quad 1 \leqslant n \leqslant N.$$

Proof. Subtracting (1.205)–(1.206) from (1.202) and (1.204), respectively, yields the system of error equations as follows:

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)}e^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})e^k - a_{n-1}^{(\alpha)}e^0\right] = (r_5)^n, \quad 1 \leqslant n \leqslant N, \\ e^0 = 0. \end{cases}$$

Noticing (1.203), the application of Theorem 1.8.7 gives

$$|e^n| \leqslant |e^0| + T^\alpha \Gamma(1-\alpha)\max_{1\leqslant l\leqslant n}|(r_5)^l| \leqslant c_5 T^\alpha \Gamma(1-\alpha)\tau^{2-\alpha}, \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

Problem 1.8.5. Solve the initial value problem

$$\begin{cases} {}_0^C D_t^\gamma y(t) = f(t), \quad 0 < t \leqslant T, & (1.209) \\ y(0) = A, \quad y'(0) = B, & (1.210) \end{cases}$$

where $\gamma \in (1,2)$.

Suppose $y \in C^3[0,T]$. Considering equation (1.209) at $t = t_n$, we have

$${}_0^C D_t^\gamma y(t)|_{t=t_n} = f(t_n), \quad 0 \leqslant n \leqslant N,$$

hence,

$$\frac{1}{2}\left[{}_0^C D_t^\gamma y(t)|_{t=t_n} + {}_0^C D_t^\gamma y(t)|_{t=t_{n-1}}\right] = \frac{1}{2}[f(t_n) + f(t_{n-1})], \quad 1 \leqslant n \leqslant N.$$

Denote $Y^n = y(t_n)$. It follows from Theorem 1.6.2 that

$$\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\delta_t Y^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t Y^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)} B\right]$$
$$= \frac{1}{2}[f(t_n) + f(t_{n-1})] + (r_6)^{n-\frac{1}{2}}, \quad 1 \leqslant n \leqslant N, \tag{1.211}$$

and there is a positive constant c_6 such that

$$\left|(r_6)^{n-\frac{1}{2}}\right| \leqslant c_6 \tau^{3-\gamma}, \quad 1 \leqslant n \leqslant N. \tag{1.212}$$

Noticing the initial value condition

$$Y^0 = A, \tag{1.213}$$

omitting the small term $(r_6)^{n-\frac{1}{2}}$ in (1.211) and replacing the exact solution Y^n with its numerical one y^n arrive at the difference scheme for solving (1.209)–(1.210) as follows:

$$\begin{cases} \dfrac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\delta_t y^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t y^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)} B\right] = f^{n-\frac{1}{2}}, \\ \qquad\qquad 1 \leqslant n \leqslant N, & (1.214) \\ y^0 = A, & (1.215) \end{cases}$$

where $f^{n-\frac{1}{2}} = \frac{1}{2}[f(t_n) + f(t_{n-1})]$.

The stability of the difference scheme

Theorem 1.8.9. *Suppose $\{y^n \mid n = 0,1,2,\ldots,N\}$ is the solution of the difference scheme* (1.214)–(1.215)*, then it holds*

$$|y^n| \leqslant |A| + T\left[|B| + \Gamma(2-\gamma)\max_{1\leqslant l\leqslant n}\left|t_l^{\gamma-1} f^{l-\frac{1}{2}}\right|\right], \quad 1 \leqslant n \leqslant N.$$

Proof. Rewrite (1.214) as follows:

$$
\begin{aligned}
b_0^{(\gamma)}\delta_t y^{n-\frac{1}{2}} &= \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t y^{k-\frac{1}{2}} + b_{n-1}^{(\gamma)}B + \tau^{\gamma-1}\Gamma(3-\gamma)f^{n-\frac{1}{2}} \\
&= \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t y^{k-\frac{1}{2}} + b_{n-1}^{(\gamma)}\left[B + \frac{\tau^{\gamma-1}}{b_{n-1}^{(\gamma)}}\Gamma(3-\gamma)f^{n-\frac{1}{2}}\right].
\end{aligned}
$$

Taking the absolute value on both hand sides of the equality above and noticing

$$
\frac{\tau^{\gamma-1}}{b_{n-1}^{(\gamma)}}\Gamma(3-\gamma) \leqslant \frac{\tau^{\gamma-1}n^{\gamma-1}}{2-\gamma}\Gamma(3-\gamma) = t_n^{\gamma-1}\Gamma(2-\gamma),
$$

the application of the triangle inequality yields

$$
\begin{aligned}
b_0^{(\gamma)}|\delta_t y^{n-\frac{1}{2}}| \leqslant &\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})|\delta_t y^{k-\frac{1}{2}}| \\
&+ b_{n-1}^{(\gamma)}[|B| + t_n^{\gamma-1}\Gamma(2-\gamma)|f^{n-\frac{1}{2}}|], \quad 1 \leqslant n \leqslant N.
\end{aligned} \tag{1.216}
$$

Next, the method of induction will be used to show the truth of

$$
|\delta_t y^{k-\frac{1}{2}}| \leqslant |B| + \Gamma(2-\gamma)\max_{1\leqslant l\leqslant k}|t_l^{\gamma-1}f^{l-\frac{1}{2}}|, \quad 1 \leqslant k \leqslant N. \tag{1.217}
$$

The result of (1.216) with $n = 1$ reveals that (1.217) is true for $k = 1$. Now suppose (1.217) is true for $k = 1, 2, \ldots, n-1$, it follows from (1.216) that

$$
\begin{aligned}
&b_0^{(\gamma)}|\delta_t y^{n-\frac{1}{2}}| \\
&\leqslant \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\Big[|B| + \Gamma(2-\gamma)\max_{1\leqslant l\leqslant k}|t_l^{\gamma-1}f^{l-\frac{1}{2}}|\Big] \\
&\quad + b_{n-1}^{(\gamma)}[|B| + t_n^{\gamma-1}\Gamma(2-\gamma)|f^{n-\frac{1}{2}}|] \\
&\leqslant \left[\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)}) + b_{n-1}^{(\gamma)}\right]\cdot\Big[|B| + \Gamma(2-\gamma)\max_{1\leqslant l\leqslant n}|t_l^{\gamma-1}f^{l-\frac{1}{2}}|\Big] \\
&= b_0^{(\gamma)}\Big[|B| + \Gamma(2-\gamma)\max_{1\leqslant l\leqslant n}|t_l^{\gamma-1}f^{l-\frac{1}{2}}|\Big].
\end{aligned}
$$

Therefore,

$$
|\delta_t y^{n-\frac{1}{2}}| \leqslant |B| + \Gamma(2-\gamma)\max_{1\leqslant l\leqslant n}|t_l^{\gamma-1}f^{l-\frac{1}{2}}|,
$$

which is precisely the result of (1.217) with $k = n$.

Noticing

$$
y^n = y^0 + \tau\sum_{k=1}^{n}\delta_t y^{k-\frac{1}{2}},
$$

it follows from (1.217) that

$$
\begin{aligned}
|y^n| &\leqslant |y^0| + \tau \sum_{k=1}^{n} |\delta_t y^{k-\frac{1}{2}}| \\
&\leqslant |A| + \tau \sum_{k=1}^{n} \Big[|B| + \Gamma(2-\gamma) \max_{1\leqslant l\leqslant k} |t_l^{\gamma-1} f^{l-\frac{1}{2}}| \Big] \\
&\leqslant |A| + T\Big[|B| + \Gamma(2-\gamma) \max_{1\leqslant l\leqslant n} |t_l^{\gamma-1} f^{l-\frac{1}{2}}| \Big], \quad 1 \leqslant n \leqslant N.
\end{aligned}
$$

The proof ends. □

The convergence of the difference scheme

Theorem 1.8.10. *Suppose* $\{y(t_n) \mid n = 0, 1, 2, \dots, N\}$ *and* $\{y^n \mid n = 0, 1, 2, \dots, N\}$ *are solutions of the problem* (1.209)–(1.210) *and the difference scheme* (1.214)–(1.215), *respectively. Let*

$$
e^n = y(t_n) - y^n, \quad n = 0, 1, 2, \dots, N,
$$

then it holds

$$
|e^n| \leqslant c_6 T^\gamma \Gamma(2-\gamma) \tau^{3-\gamma}, \quad 1 \leqslant n \leqslant N.
$$

Proof. Subtracting (1.214)–(1.215) from (1.211) and (1.213), respectively, produces the system of error equations as follows:

$$
\begin{cases}
\dfrac{\tau^{1-\gamma}}{\Gamma(3-\gamma)} \left[b_0^{(\gamma)} \delta_t e^{n-\frac{1}{2}} - \displaystyle\sum_{k=1}^{n-1} (b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)}) \delta_t e^{k-\frac{1}{2}} \right] = (r_6)^{n-\frac{1}{2}}, \quad 1 \leqslant n \leqslant N, \\
e^0 = 0.
\end{cases}
$$

Noticing (1.212), the application of Theorem 1.8.9 yields

$$
|e^n| \leqslant T^\gamma \Gamma(2-\gamma) \max_{1\leqslant l\leqslant n} |(r_6)^{l-\frac{1}{2}}| \leqslant c_6 T^\gamma \Gamma(2-\gamma) \tau^{3-\gamma}, \quad 1 \leqslant n \leqslant N.
$$

The proof ends. □

1.8.3 Method based on L2-1_σ approximation

Consider another numerical method for solving the Problem 1.8.4, i. e.,

$$
\begin{cases}
{}_0^C D_t^\alpha y(t) = f(t), \quad 0 < t \leqslant T, & (1.218) \\
y(0) = A, & (1.219)
\end{cases}
$$

where $\alpha \in (0,1)$.

Suppose $y \in C^3[0,T]$. Considering equation (1.218) at $t = t_{n-1+\sigma}$, by Theorem 1.6.4, we have

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} [y(t_{n-k}) - y(t_{n-k-1})] = f(t_{n-1+\sigma}) + (r_7)^n, \quad 1 \leqslant n \leqslant N \tag{1.220}$$

and there is a positive constant c_7 such that

$$\left|(r_7)^n\right| \leqslant c_7 \tau^{3-\alpha}, \quad 1 \leqslant n \leqslant N. \tag{1.221}$$

Noticing the initial value condition (1.219) and omitting the small term $(r_7)^n$ in (1.220), a difference scheme for solving (1.218)–(1.219) can be derived as follows:

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \displaystyle\sum_{k=0}^{n-1} c_k^{(n,\alpha)} (y^{n-k} - y^{n-k-1}) = f(t_{n-1+\sigma}), \quad 1 \leqslant n \leqslant N, & (1.222) \\ y^0 = A. & (1.223) \end{cases}$$

The stability of the difference scheme

Theorem 1.8.11. *Suppose $\{y^n \mid n = 0,1,\ldots,N\}$ is the solution of the difference scheme*

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \displaystyle\sum_{k=0}^{n-1} c_k^{(n,\alpha)} (y^{n-k} - y^{n-k-1}) = g^n, \quad 1 \leqslant n \leqslant N, & (1.224) \\ y^0 = A, & (1.225) \end{cases}$$

then it holds

$$|y^k| \leqslant |y^0| + \Gamma(1-\alpha) \max_{1\leqslant m\leqslant k} |t_m^\alpha g^m|, \quad 1 \leqslant k \leqslant N.$$

Proof. Denote $s = \tau^\alpha \Gamma(2-\alpha)$. It follows from Lemma 1.6.3 that

$$\frac{s}{c_{n-1}^{(n,\alpha)}} \leqslant \frac{n^\alpha \tau^\alpha}{1-\alpha} \Gamma(2-\alpha) = t_n^\alpha \Gamma(1-\alpha).$$

Reformulate (1.224) as

$$\begin{aligned} c_0^{(n,\alpha)} y^n &= \sum_{k=0}^{n-1} c_k^{(n,\alpha)} y^{n-k-1} - \sum_{k=1}^{n-1} c_k^{(n,\alpha)} y^{n-k} + s g^n \\ &= \sum_{k=1}^{n} c_{k-1}^{(n,\alpha)} y^{n-k} - \sum_{k=1}^{n-1} c_k^{(n,\alpha)} y^{n-k} + s g^n \\ &= \sum_{k=1}^{n-1} (c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)}) y^{n-k} + c_{n-1}^{(n,\alpha)} y^0 + s g^n, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Taking the absolute values on both hand sides of the equality above and noticing Lemma 1.6.3, the application of the triangle inequality yields

$$\begin{aligned} c_0^{(n,\alpha)}|y^n| &\leqslant \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})|y^{n-k}| + c_{n-1}^{(n,\alpha)}|y^0| + s|g^n| \\ &= \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})|y^{n-k}| + c_{n-1}^{(n,\alpha)}\Big(|y^0| + \frac{s}{c_{n-1}^{(n,\alpha)}}|g^n|\Big) \\ &\leqslant \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})|y^{n-k}| + c_{n-1}^{(n,\alpha)}(|y^0| + \Gamma(1-\alpha)t_n^\alpha|g^n|), \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{1.226}$$

The following result

$$|y^k| \leqslant |y^0| + \Gamma(1-\alpha)\max_{1\leqslant m\leqslant k}|t_m^\alpha g^m|, \quad 1 \leqslant k \leqslant N$$

can be obtained from (1.226) by the method of induction. The proof ends. □

The convergence of the difference scheme

Theorem 1.8.12. *Suppose* $\{y(t_n) \mid n = 0, 1, \dots, N\}$ *and* $\{y^n \mid n = 0, 1, \dots, N\}$ *are solutions of the problem* (1.218)–(1.219) *and the difference scheme* (1.222)–(1.223), *respectively. Let*

$$e^n = y(t_n) - y^n, \quad 0 \leqslant n \leqslant N,$$

then it holds

$$|e^n| \leqslant c_7 T^\alpha \Gamma(1-\alpha)\tau^{3-\alpha}, \quad 1 \leqslant n \leqslant N.$$

Proof. The system of error equations is

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\displaystyle\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(e^{n-k} - e^{n-k-1}) = (r_7)^n, \quad 1 \leqslant n \leqslant N, \\ e^0 = 0. \end{cases}$$

Noticing (1.221) and applying Theorem 1.8.11 will easily produce the desired result. The proof ends. □

Next, we try to solve Problem 1.8.5, i. e.,

$$\begin{cases} {}_0^C D_t^\gamma y(t) = f(t), \quad 0 < t \leqslant T, & (1.227) \\ y(0) = A, \quad y'(0) = B & (1.228) \end{cases}$$

using the L2-1_σ method, where $\gamma \in (1, 2)$.

Let $z(t) = y'(t)$, $\alpha = \gamma - 1$, then the problem (1.227)–(1.228) for $y(t)$ can be converted to the initial value problem for $z(t)$ as follows:

$$\begin{cases} {}_0^C D_t^\alpha z(t) = f(t), \quad 0 < t \leqslant T, & (1.229) \\ z(0) = B. & (1.230) \end{cases}$$

It can be found that (1.229)–(1.230) is in the same form with Problem 1.8.4, which can be solved using the L2-1_σ method. Once the approximate values z^n of $z(t_n)$, $n = 0, 1, 2, \ldots, N$ are obtained, with the help of the composite trapezoid formula or composite Simpson formula in the numerical integral, the approximate values y^n of $y(t_n)$, $n = 0, 1, 2, \ldots, N$ are available in view of

$$y(t_n) = y(t_0) + \int_{t_0}^{t_n} y'(\eta)\mathrm{d}\eta = A + \int_{t_0}^{t_n} z(\eta)\mathrm{d}\eta.$$

1.9 A simple classification of the fractional partial differential equations

Li et al. made a simple classification for the FPDEs in their review article[45].

First, consider the time FPDE:

$${}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t),$$

where the time-fractional derivative is often defined in Caputo sense. According to the value of α, a categorization can be given in Table 1.3.

Table 1.3: The classification of ${}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t)$ with $\alpha \in (0,2]$.

α	Math. type	Phys. sense
(0, 1)	Time-fractional parabolic equation	Temporal subdiffusion
1	Parabolic equation	Diffusion
(1, 2)	Time-fractional hyperbolic equation	Temporal superdiffusion or temporal fractional wave
2	Hyperbolic equation	Wave

Second, consider the space FPDE:

$$u_t(x,t) = \frac{\partial^\beta u(x,t)}{\partial |x|^\beta},$$

where the space-fractional derivative $\frac{\partial^\beta u(x,t)}{\partial |x|^\beta}$ is often in Riemann–Liouville sense or Riesz sense. According to the value of β, a categorization can be described as that in Table 1.4.

Table 1.4: The classification of $u_t(x,t) = \frac{\partial^\beta u(x,t)}{\partial |x|^\beta}$ with $\beta \in (1,2]$.

β	Math. type	Phys. sense
(0, 1)	Space-fractional hyperbolic equation	Fractional advection
1	Hyperbolic equation	Advection
(1, 2)	Space-fractional parabolic equation	Fractional diffusion
2	Parabolic equation	Diffusion

Finally, consider the time-space FPDE:

$$ {}_0^C D_t^\alpha u(x,t) = \frac{\partial^\beta u(x,t)}{\partial |x|^\beta}, $$

where the time derivative is in Caputo sense, and the space derivative is in Riesz sense, or in other distinct senses, such as R-L sense, fractional Laplacian sense, etc. A classification can be formed as that in Table 1.5.

Table 1.5: The classification of ${}_0^C D_t^\alpha u(x,t) = \frac{\partial^\beta u(x,t)}{\partial |x|^\beta}$ with $\alpha \in (0,2)$, $\beta \in (0,2)$.

α	β	Math. type	Phys. sense
(0, 1)	(0, 1)	Time-space-fractional hyperbolic equation	Temporal subdiffusion and fractional advection
	(1, 2)	Time-space-fractional parabolic equation	Temporal subdiffusion and fractional diffusion
(1,2)	(0, 1)	Space-time-fractional parabolic equation	Temporal superdiffusion and fractional advection
	(1, 2)	Time-space-fractional hyperbolic equation	Temporal superdiffusion and fractional diffusion

In this book, we mainly present the finite difference methods for three types of FPDEs, namely time-fractional, space-fractional and time-space-fractional PDEs.

In Chapter 2, we study the finite difference methods for the initial-boundary value problem of the time-fractional subdiffusion equation

$$ {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) + f(x,t), $$

where $\alpha \in (0,1)$; Chapter 3 shows the finite difference methods for the initial-boundary value problem of the time-fractional wave equation

$$ {}_0^C D_t^{\gamma} u(x,t) = u_{xx}(x,t) + f(x,t), $$

where $\gamma \in (1,2)$; In chapter 4, we introduce the finite difference methods for the initial-boundary value problem of the space-fractional partial differential equation

$$ u_t(x,t) = K_1 \, {}_0\mathbf{D}_x^{\beta} u(x,t) + K_2 \, {}_x\mathbf{D}_L^{\beta} u(x,t) + f(x,t), $$

where $\beta \in (1,2)$; Chapter 5 considers the finite difference methods for the initial-boundary value problem of the time-space-fractional differential equation

$$ {}_0^C D_t^{\alpha} u(x,t) = \frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + f(x,t), $$

where $\alpha \in (0,1)$, $\beta \in (1,2)$; In chapter 6, the finite difference methods are concerned for solving a class of initial-boundary value problem of time distributed-order subdiffusion equation

$$ \mathcal{D}_t^{w} u(x,t) = u_{xx}(x,t) + f(x,t), $$

where $\mathcal{D}_t^{w} u(x,t) = \int_0^1 w(\alpha) \, {}_0^C D_t^{\alpha} u(x,t) \mathrm{d}\alpha$.

The two dimensional problems are also considered in each chapter.

1.10 Supplementary remarks and discussions

1. In this chapter, four kinds of definitions of fractional derivatives were introduced. The analytical solutions to the linear FODEs with two types of fractional derivatives were described, which provides readers a general idea on the characteristics of solutions to fractional differential equations (FDEs). Regarding the definitions and properties of fractional derivatives, readers can refer to [63]. Fractional Laplace operators were not covered in this book. Interested readers may refer to [44].

2. Several numerical ways to approximate the fractional derivatives were introduced in this chapter, together with their numerical accuracy and applications into solving FODEs.

3. The G-L fractional derivative is in a limit form, which is equivalent to the R-L fractional derivative. Hence it is natural to approximate the R-L fractional derivative using the G-L formula with an appropriate step size. The standard G-L formula has the accuracy of order one[58] and when it is directly used to solve the space-fractional differential equations, the resultant difference scheme is unstable[58]. Then the shifted G-L formula was proposed by Meerschaert and Tadjeran, the asymptotic

expansion for which was derived by Tadjeran et al. in [86]. Two research teams, conducted by Deng and Sun, respectively, discussed a series of high-order numerical differentiation formulae with the aid of different weighted combinations of shifted G-L formulae[35, 88, 117]. It should be noted that for these high-order formulae, the appropriate smoothness of functions is required. Generally speaking, the possible highest order accuracy that an approximation can reach depends on the regularity of the function. For the Riesz fractional derivative, the asymptotic expansion of the central difference quotient formula[14, 15, 116] can be similarly obtained using the method in [35, 88, 117]. The asymptotic expansion of numerical differentiation formulae is the powerful tool to derive various high-order formulae.

4. The G-L approximation of Riemann-Liouville fractional derivatives is considered on the whole domain $\mathcal{R}$. When the G-L formula was used to solve the FDEs on bounded domains, the solutions to FDEs have to be extended to be defined on the whole domain $\mathcal{R}$ and the extended solution is supposed to have the certain smoothness. If the condition is not satisfied, the consistency and convergence of the corresponding difference schemes cannot be ensured.

Suppose $p > 0$. Define the function

$$f(x) = \begin{cases} x^p, & x \geqslant 0, \\ 0, & x < 0. \end{cases}$$

Denote $x_i = ih$. Noticing

$${}_0\mathbf{D}_x^{\alpha} f(x) = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha},$$

a direct calculation gives

$${}_0\mathbf{D}_x^{\alpha} f(x_1) - h^{-\alpha} \sum_{k=0}^{1} g_k^{(\alpha)} f(x_{1-k}) = \left[\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} - 1 \right] h^{p-\alpha},$$

$${}_0\mathbf{D}_x^{\alpha} f(x_1) - h^{-\alpha} \sum_{k=0}^{2} g_k^{(\alpha)} f(x_{2-k}) = \left[\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} - (2^p - \alpha) \right] h^{p-\alpha}.$$

If $p = 1$, $\alpha = 0.5$, then when $x \geqslant 0$, $f(x) = x$, both the G-L formula and the shifted G-L formula have only the accuracy of order $O(h^{1/2})$. If $p = 0$, $\alpha = 0.5$, these two formulae only have the accuracy of order $O(h^{-1/2})$. Therefore, both the G-L formula and the shifted G-L formula are not uniformly convergent in order one. For some problems, the approximation errors may be large and a completely inconsistent difference scheme may be produced using these two formulae; hence, the solution of difference schemes may not be convergent to the solution of the differential equations.

For the L1 formula, when $0 < \alpha < 1$, the second-order derivative of function f is required to be continuous. Otherwise, the expecting result cannot be obtained.

5. The Caputo fractional derivative is actually an integral with a weak singular kernel. It is a quite natural idea[61] to use the piecewise linear interpolation polynomial

to establish a weighted numerical integral and the resultant formula is called the L1 formula or L1 approximation. A rough estimation on the numerical accuracy of this formula is only the first order[2, 105, 119]. In [99], Wu and Sun proved that the L1 formula for the half-order fractional derivative has the accuracy of order 3/2. Later, in [84], they proved that for the α-th ($\alpha \in (0,1)$) order Caputo fractional derivative, the numerical accuracy of the L1 formula is order of $2-\alpha$, where the rigorous expression of errors was derived. Combining with the method of order reduction, the numerical differentiation formula for the γ-th ($\gamma \in (1,2)$) order Caputo fractional derivative is also established, which has the numerical accuracy of order $3-\gamma$. Lin and Xu[50] also proved that the numerical accuracy of the L1 formula to approximate the α-th ($\alpha \in (0,1)$) order Caputo fractional derivative is order of $2-\alpha$ using the series method. In addition, some improved methods were investigated in [1, 31, 113].

6. About the estimation on the coefficient $\{g_k^{(\alpha)}\}_{k=0}^{\infty}$ in the G-L formula, where $\alpha \in (0,1)$, the inequality $\sum_{n=k}^{\infty}|g_n^{(\alpha)}| \geqslant \frac{1-\alpha}{5}\left(\frac{2}{k}\right)^{\alpha}$ has been provided in Lemma 1.4.3. In fact, the similar result $\sum_{n=k}^{\infty}|g_n^{(\alpha)}| \geqslant \frac{1}{k^{\alpha}\Gamma(1-\alpha)}$ can also be obtained and used to make the analysis on the related difference schemes. More details can be found in [9].

7. Alikhanov[1] developed an L2-1_σ method for the fractional derivative of order α ($\alpha \in (0,1)$), which improved L1-2 method[31]. The authors of [19] provided the L2-1_σ method for the multi-term fractional derivatives. The authors of [81, 83] used the method of order reduction to give an L2-1_σ method for the fractional derivative of order γ ($\gamma \in (1,2)$). In [16], the authors presented the L2-1_σ approximation for the variable order fractional derivatives. The advantage of the L2-1_σ method is that a temporal second-order convergent difference scheme can be obtained when applied to numerically solve the time-fractional differential equations.

8. H2N2 approximation for the fractional derivative of order γ ($\gamma \in (1,2)$) was obtained directly by the quadratic interpolation polynomial[71], which avoids using linear interpolation polynomial as an intermediate transition. The application of H2N2 formula to solve the fractional wave equation is similar to that of L1 formula to solve the fractional subdiffusion equation.

9. Fractional derivatives are historically memorized. The value of the current moment depends on the value of all moments since the initial time. It is necessary to find a fast calculation method. Many fast approximations can be obtained by applying the sum of exponential functions to approach the kernel of fractional derivatives[41]. In [41] and [101], the authors presented the fast L1 approximation and fast L2-1_σ approximation for the Caputo derivative of order α ($\alpha \in (0,1)$), respectively. In [71], a fast algorithm based on H2N2 approximation for the fractional Caputo derivative of order γ ($\gamma \in (1,2)$) was investigated. Gao and Yang[32], Sun and Sun[79] discussed the fast L2-1_σ approximations for the multi-term time Caputo derivatives of orders belonging to $(0,1)$ and $(1,2)$, respectively. For the fast algorithm of Riemann-Liouville derivative, please refer to [78].

10. The numerical approximation of fractional derivatives with the initial singularity has been paid some attention in recent years. Stynes et al. [74] gave a numerical formula on the graded mesh. Shen et al. [72] further provided a fast approximation for this kind of fractional derivatives.

Exercises 1

1.1 Compute
(1) ${}_a\mathbf{D}_t^\alpha (t-a)^{-1/2}$;
(2) ${}_a\mathbf{D}_t^\alpha (t-a)^{1/2}$;
(3) ${}_a\mathbf{D}_t^\alpha (t-a)^{2}$;
(4) ${}_0^CD_t^\alpha t^{1/2}$;
(5) ${}_0^CD_t^\alpha t^{2}$;
(6) $\frac{\partial^\alpha (t-t^2)}{\partial |t|^\alpha}$.

1.2 Suppose the function $f(t)$ can be expanded into the following Taylor series:

$$f(t)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n,$$

with the convergence radius R, $R>0, f(0)\neq 0$. Solve the following problem:

$$\begin{cases} {}_0\mathbf{D}_t^\alpha y(t)+t^{1-\alpha}y(t)=f(t), & t>0, \quad \alpha\in(0,1),\\ y(0)=0. \end{cases}$$

1.3 Suppose the function $f(t)$ can be expanded into

$$f(t)=t^{2-\gamma}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n,$$

with the convergence radius R, $R>0, f(0)\neq 0$. Solve the following problem:

$$\begin{cases} {}_0^CD_t^\gamma y(t)=f(t), & t>0, \quad \gamma\in(1,2),\\ y(0)=A, \quad y'(0)=B. \end{cases}$$

1.4 Suppose $f\in\mathscr{C}^{2+\alpha}(\mathcal{R})$. Try to prove

$$A_{h,p}^\alpha f(t)=c_1^{(\alpha,p)}{}_{-\infty}\mathbf{D}_t^\alpha f(t+h)+\big(1-c_1^{(\alpha,p)}\big){}_{-\infty}\mathbf{D}_t^\alpha f(t)+O(h^2).$$

1.5 Suppose $f\in C^3[t_0,t_n]$, $\gamma\in(1,2)$, $\alpha=\gamma-1$, $g(t)=f'(t)$. Try to estimate $P_n, Q_1, Q_2, \ldots, Q_n, R_n$ in the following equalities:

$$\begin{aligned} &{}_0^CD_t^\gamma f(t)|_{t=t_{n-\frac12}}={}_0^CD_t^\alpha g(t)|_{t=t_{n-\frac12}}\\ &=\frac{1}{\Gamma(1-\alpha)}\left[\int_{t_0}^{t_{\frac12}} g'(t)(t_{n-\frac12}-t)^{-\alpha}\mathrm{d}t+\sum_{k=1}^{n-1}\int_{t_{k-\frac12}}^{t_{k+\frac12}} g'(t)(t_{n-\frac12}-t)^{-\alpha}\mathrm{d}t\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)}\Bigg[\frac{2}{\tau}\big(g(t_{\frac{1}{2}}) - g(t_0)\big)\cdot \int_{t_0}^{t_{\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{-\alpha}\mathrm{d}t \\
&\quad + \sum_{k=1}^{n-1}\frac{1}{\tau}\big(g(t_{k+\frac{1}{2}}) - g(t_{k-\frac{1}{2}})\big)\cdot \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{-\alpha}\mathrm{d}t\Bigg] + P_n \\
&= \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\Bigg[\hat{b}_{n-1}^{(n,\gamma)}\cdot\big(g(t_{\frac{1}{2}}) - f'(t_0)\big) + \sum_{k=1}^{n-1}\hat{b}_{n-k-1}^{(n,\gamma)}\cdot\big(g(t_{k+\frac{1}{2}}) - g(t_{k-\frac{1}{2}})\big)\Bigg] + P_n \\
&= \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\Bigg[\hat{b}_0^{(n,\gamma)} g(t_{n-\frac{1}{2}}) - \sum_{k=1}^{n-1}(\hat{b}_{n-k-1}^{(n,\gamma)} - \hat{b}_{n-k}^{(n,\gamma)}) g(t_{k-\frac{1}{2}}) - \hat{b}_{n-1}^{(n,\gamma)} f'(t_0)\Bigg] + P_n \\
&= \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\Bigg[\hat{b}_0^{(n,\gamma)}(\delta_t f^{n-\frac{1}{2}} + Q_n) \\
&\quad - \sum_{k=1}^{n-1}(\hat{b}_{n-k-1}^{(n,\gamma)} - \hat{b}_{n-k}^{(n,\gamma)})(\delta_t f^{k-\frac{1}{2}} + Q_k) - \hat{b}_{n-1}^{(n,\gamma)} f'(t_0)\Bigg] + P_n \\
&= \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\Bigg[\hat{b}_0^{(n,\gamma)}\delta_t f^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(\hat{b}_{n-k-1}^{(n,\gamma)} - \hat{b}_{n-k}^{(n,\gamma)})\delta_t f^{k-\frac{1}{2}} - \hat{b}_{n-1}^{(n,\gamma)} f'(t_0)\Bigg] + R_n.
\end{aligned}
$$

2 Difference methods for the time-fractional subdiffusion equations

It has been found that there are no Gauss statistics for diffusion processes in many complex systems and the second Fick's law is no longer true. Specially, the linearity between the mean square displacement of particles in Brown motion and the time variable in classical diffusion processes is not satisfied. This kind of diffusion is often called the anomalous diffusion, whose striking feature lies in the power law dependence of the mean square displacement of particles on the time variable. The time-fractional diffusion-wave equation is a typical tool to depict the phenomenon, which is derived by replacing the first-order time derivative with the α-th order fractional derivative. It has been widely used in various fields such as physics, control, signal and image processing, mechanics and dynamic systems, biology, environmental science, materials, economic and multidisciplinary in engineering fields. When $0 < \alpha < 1$, the corresponding equation is called the time-fractional subdiffusion equation; When $1 < \alpha < 2$, it is called the time-fractional wave equation; For $\alpha = 1$, the classical diffusion equation is recovered. In this book, Chapter 2 and Chapter 3 will introduce the difference methods for these two kinds of time-fractional differential equations, respectively, and the unique solvability, stability together with the convergence of the presented difference schemes will be shown. In this chapter, for the 1D time-fractional subdiffusion equation, three distinct ways, G-L formula, L1 formula and L2-1_σ formula, will be used to approximate the time-fractional derivative, respectively, and the spatial derivative will be handled with the second-order or compact approximation. The fast L1 approximation method and fast L2-1_σ approximation method are presented. The L1 formula and L2-1_σ formula are also mentioned for a class of multiterm time-fractional subdiffusion problems. Finally, for the 2D problem, the alternating direction implicit (ADI) difference method will be described. The whole chapter is divided into 12 sections.

2.1 The second-order method in space based on G-L approximation for 1D problem

Consider the following initial-boundary value problem of the time-fractional subdiffusion equations:

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], & (2.1)\\ u(x,0) = 0, \quad x \in (0,L), & (2.2)\\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), \quad t \in [0,T], & (2.3) \end{cases}$$

where $\alpha \in (0,1)$, the functions f, μ, ν are all given and $\mu(0) = \nu(0) = 0$.

The mesh partition is firstly done. Take two positive integers M and N. Let $h = L/M$, $\tau = T/N$. Denote $x_i = ih\ (0 \leqslant i \leqslant M)$, $t_k = k\tau\ (0 \leqslant k \leqslant N)$, $\Omega_h = \{x_i \mid 0 \leqslant i \leqslant M\}$,

https://doi.org/10.1515/9783110616064-002

$\Omega_\tau = \{t_k \mid 0 \leqslant k \leqslant N\}$. Define the following mesh function spaces:

$$\mathcal{U}_h = \{u \mid u = (u_0, u_1, \ldots, u_M)\}, \quad \mathring{\mathcal{U}}_h = \{u \mid u \in \mathcal{U}_h, u_0 = u_M = 0\}.$$

For any mesh function $u \in \mathcal{U}_h$, denote

$$\delta_x u_{i-\frac{1}{2}} = \frac{1}{h}(u_i - u_{i-1}), \quad \delta_x^2 u_i = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}),$$
$$(\mathcal{A}u)_i = \begin{cases} \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}), & 1 \leqslant i \leqslant M-1, \\ u_i, & i = 0, M. \end{cases}$$

It is easy to know that $(\mathcal{A}u)_i = (\mathcal{I} + \frac{h^2}{12}\delta_x^2)u_i$ for $1 \leqslant i \leqslant M-1$, with $\mathcal{I}$ an identity operator. For brevity, henceforth denote $(\mathcal{A}u)_i$ by $\mathcal{A}u_i$.

For any mesh functions $u, v \in \mathring{\mathcal{U}}_h$, define the following inner products and norms:

$$(u, v) = h\sum_{i=1}^{M-1} u_i v_i, \quad \|u\| = \sqrt{(u, u)},$$
$$(\delta_x u, \delta_x v) = h\sum_{i=1}^{M}(\delta_x u_{i-\frac{1}{2}})(\delta_x v_{i-\frac{1}{2}}), \quad \|\delta_x u\| = \sqrt{(\delta_x u, \delta_x u)},$$
$$(\delta_x^2 u, \delta_x^2 v) = h\sum_{i=1}^{M-1}(\delta_x^2 u_i)(\delta_x^2 v_i), \quad \|\delta_x^2 u\| = \sqrt{(\delta_x^2 u, \delta_x^2 u)},$$
$$\|u\|_\infty = \max_{0\leqslant i\leqslant M} |u_i|.$$

Lemma 2.1.1. [75] *For any mesh function $u \in \mathring{\mathcal{U}}_h$, it holds*

$$\|u\|_\infty \leqslant \frac{\sqrt{L}}{2}\|\delta_x u\|, \quad \|u\| \leqslant \frac{L}{\sqrt{6}}\|\delta_x u\|,$$
$$\|\delta_x u\| \leqslant \frac{2}{h}\|u\|, \quad \frac{1}{3}\|u\|^2 \leqslant \|\mathcal{A}u\|^2 \leqslant \|u\|^2.$$

Suppose $u, v \in \mathring{\mathcal{U}}_h$. Let

$$I(u, v) \equiv (\mathcal{A}u, -\delta_x^2 v) = -h\sum_{i=1}^{M-1}(\mathcal{A}u_i)\delta_x^2 v_i.$$

Then

$$I(u, v) = -h\sum_{i=1}^{M-1}\left(u_i + \frac{h^2}{12}\delta_x^2 u_i\right)\delta_x^2 v_i$$
$$= h\sum_{i=1}^{M}(\delta_x u_{i-\frac{1}{2}})(\delta_x v_{i-\frac{1}{2}}) - \frac{h^2}{12}h\sum_{i=1}^{M-1}(\delta_x^2 u_i)(\delta_x^2 v_i)$$

$$= (\delta_x u, \delta_x v) - \frac{h^2}{12}(\delta_x^2 u, \delta_x^2 v).$$

It is easy to verify

$$\frac{2}{3}\|\delta_x u\|^2 \leqslant I(u,u) \leqslant \|\delta_x u\|^2. \tag{2.4}$$

Hence $I(u,v)$ is an inner product defined on $\mathring{\mathcal{U}}_h$. Denote

$$(u,v)_{1,A} = I(u,v), \quad \|\delta_x u\|_A = \sqrt{(u,u)_{1,A}}\,. \tag{2.5}$$

The following lemma is easily deduced from (2.4).

Lemma 2.1.2. *For any mesh function $u \in \mathring{\mathcal{U}}_h$, we have*

$$\frac{2}{3}\|\delta_x u\|^2 \leqslant \|\delta_x u\|_A^2 \leqslant \|\delta_x u\|^2.$$

Another lemma that follows is prepared for the spatial approximation.

Lemma 2.1.3. [75]

(I) *If function $g \in C^4[x_{i-1}, x_{i+1}]$, then*

$$g''(x_i) = \frac{g(x_{i-1}) - 2g(x_i) + g(x_{i+1})}{h^2} - \frac{h^2}{6}\int_0^1 [g^{(4)}(x_i + \lambda h) + g^{(4)}(x_i - \lambda h)](1-\lambda)^3 \mathrm{d}\lambda.$$

(II) *Denote $\zeta(\lambda) = (1-\lambda)^3[5 - 3(1-\lambda)^2]$. If function $g \in C^6[x_{i-1}, x_{i+1}]$, then*

$$\frac{g''(x_{i+1}) + 10g''(x_i) + g''(x_{i-1})}{12} = \frac{g(x_{i+1}) - 2g(x_i) + g(x_{i-1})}{h^2} + \frac{h^4}{360}\int_0^1 [g^{(6)}(x_i - \lambda h) + g^{(6)}(x_i + \lambda h)]\zeta(\lambda)\mathrm{d}\lambda.$$

Define mesh functions

$$U_i^n = u(x_i, t_n),\ f_i^n = f(x_i, t_n), \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N.$$

2.1.1 Derivation of the difference scheme

For any fixed $x \in [0, L]$, define a function

$$\hat{u}(x,t) = \begin{cases} 0, & t < 0, \\ u(x,t), & 0 \leqslant t \leqslant T, \\ v(x,t), & T < t < 2T, \\ 0, & t \geqslant 2T, \end{cases}$$

where $v(x,t)$ is a smooth function satisfying $\frac{\partial^k v(x,t)}{\partial t^k}|_{t=T} = \frac{\partial^k u(x,t)}{\partial t^k}|_{t=T}$, $\frac{\partial^k v(x,t)}{\partial t^k}|_{t=2T} = 0$, $k = 0,1,2$. Suppose $\hat{u}(x,\cdot) \in \mathscr{C}^{1+\alpha}(\mathcal{R})$ and $u(\cdot,t) \in C^4[0,L]$.

Considering equation (2.1) at the point (x_i, t_n), one has

$$ {}_0^C D_t^\alpha u(x_i,t_n) = u_{xx}(x_i,t_n) + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{2.6} $$

Noticing the relationship between the Caputo derivative and the R-L derivative under the zero initial condition (2.2) and applying Theorem 1.4.2, one can obtain

$$ {}_0^C D_t^\alpha u(x_i,t_n) = {}_0\mathbf{D}_t^\alpha u(x_i,t_n) = \tau^{-\alpha} \sum_{k=0}^{n} g_k^{(\alpha)} U_i^{n-k} + O(\tau). \tag{2.7} $$

By Lemma 2.1.3, it is clear that

$$ u_{xx}(x_i,t_n) = \delta_x^2 U_i^n + O(h^2). \tag{2.8} $$

Substituting (2.7) and (2.8) into (2.6) gives

$$ \tau^{-\alpha} \sum_{k=0}^{n} g_k^{(\alpha)} U_i^{n-k} = \delta_x^2 U_i^n + f_i^n + (r_1)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{2.9} $$

where there is a positive constant c_1 such that

$$ |(r_1)_i^n| \leqslant c_1(\tau + h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{2.10} $$

Noticing the initial-boundary value conditions (2.2)–(2.3), one has

$$ \begin{cases} U_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, & (2.11) \\ U_0^n = \mu(t_n), \quad U_M^n = v(t_n), \quad 0 \leqslant n \leqslant N. & (2.12) \end{cases} $$

Omitting the small term $(r_1)_i^n$ in (2.9) and replacing the exact solution U_i^n with its numerical one u_i^n, a difference scheme for solving (2.1)–(2.3) can be produced as

$$ \begin{cases} \tau^{-\alpha} \sum_{k=0}^{n} g_k^{(\alpha)} u_i^{n-k} = \delta_x^2 u_i^n + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.13) \\ u_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, & (2.14) \\ u_0^n = \mu(t_n), \quad u_M^n = v(t_n), \quad 0 \leqslant n \leqslant N. & (2.15) \end{cases} $$

Next, the unique solvability, unconditional stability and convergence of this difference scheme will be analyzed.

2.1.2 Solvability of the difference scheme

Theorem 2.1.1. *The difference scheme* (2.13)–(2.15) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The value of u^0 is obviously determined by (2.14)–(2.15). Assume that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n can be obtained from (2.13) and (2.15). To show its unique solvability, it suffices to verify that the corresponding homogeneous one

$$\begin{cases} \tau^{-\alpha} u_i^n = \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (2.16) \\ u_0^n = u_M^n = 0 & (2.17) \end{cases}$$

has only the trivial solution.

Suppose $\|u^n\|_\infty = |u_{i_n}^n|$, where $i_n \in \{1, 2, \ldots, M-1\}$. Rewrite (2.16) as

$$\left(1 + 2\frac{\tau^\alpha}{h^2}\right) u_i^n = \frac{\tau^\alpha}{h^2}(u_{i-1}^n + u_{i+1}^n), \quad 1 \leqslant i \leqslant M-1.$$

Letting $i = i_n$ in the equality above and taking the absolute value of both hand sides, an application of the triangle inequality yields

$$\left(1 + 2\frac{\tau^\alpha}{h^2}\right) \|u^n\|_\infty \leqslant 2\frac{\tau^\alpha}{h^2} \|u^n\|_\infty,$$

hence $\|u^n\|_\infty = 0$, which implies $u^n = 0$.

By the principle of induction, the theorem is true. The proof ends. □

2.1.3 Stability of the difference scheme

Theorem 2.1.2. *Suppose $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ is the solution of the difference scheme*

$$\begin{cases} \tau^{-\alpha} \sum_{k=0}^{n} g_k^{(\alpha)} v_i^{n-k} = \delta_x^2 v_i^n + f_i^n, \quad 1 \leqslant i \leqslant M-1, \ 1 \leqslant n \leqslant N, & (2.18) \\ v_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.19) \\ v_0^n = 0, \ v_M^n = 0, \quad 0 \leqslant n \leqslant N. & (2.20) \end{cases}$$

Then it holds

$$\|v^k\|_\infty \leqslant \frac{5}{1-\alpha}\|v^0\|_\infty + \frac{5}{(1-\alpha)2^\alpha} k^\alpha \tau^\alpha \max_{1 \leqslant m \leqslant k} \|f^m\|_\infty, \quad 1 \leqslant k \leqslant N, \tag{2.21}$$

where $\|f^m\|_\infty = \max_{1 \leqslant i \leqslant M-1} |f_i^m|$.

Proof. Reformulate (2.18) as follows:

$$\left(1+2\frac{\tau^\alpha}{h^2}\right)v_i^n=\sum_{k=1}^{n}(-g_k^{(\alpha)})v_i^{n-k}+\frac{\tau^\alpha}{h^2}(v_{i-1}^n+v_{i+1}^n)+\tau^\alpha f_i^n,$$
$$1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N. \tag{2.22}$$

Suppose $\|v^n\|_\infty=|v_{i_n}^n|$, where $i_n\in\{1,2,\dots,M-1\}$. Letting $i=i_n$ in (2.22) and taking the absolute value of both hand sides, noticing $-g_k^{(\alpha)}\geqslant 0$ $(1\leqslant k\leqslant n)$ and using the triangle inequality, one can get

$$\begin{aligned}&\left(1+2\frac{\tau^\alpha}{h^2}\right)\|v^n\|_\infty\\&\leqslant\sum_{k=1}^{n}(-g_k^{(\alpha)})\|v^{n-k}\|_\infty+\frac{\tau^\alpha}{h^2}(\|v^n\|_\infty+\|v^n\|_\infty)+\tau^\alpha\|f^n\|_\infty,\end{aligned}$$

which simplifies to give

$$\|v^n\|_\infty\leqslant\sum_{k=1}^{n}(-g_k^{(\alpha)})\|v^{n-k}\|_\infty+\tau^\alpha\|f^n\|_\infty,\quad 1\leqslant n\leqslant N. \tag{2.23}$$

Next, the mathematical induction method will be used to show the truth of (2.21). Let

$$A_n=\frac{5}{1-\alpha}\|v^0\|_\infty+\frac{5}{(1-\alpha)2^\alpha}n^\alpha\tau^\alpha\max_{1\leqslant m\leqslant n}\|f^m\|_\infty,\quad 1\leqslant n\leqslant N.$$

When $n=1$, it follows from (2.23) that

$$\|v^1\|_\infty\leqslant(-g_1^{(\alpha)})\|v^0\|_\infty+\tau^\alpha\|f^1\|_\infty=\alpha\|v^0\|_\infty+\tau^\alpha\|f^1\|_\infty\leqslant A_1,$$

that is, (2.21) is obvious for $k=1$. Suppose (2.21) is true for $k=1,2,\dots,n-1(n\geqslant 2)$ and consider the case of $k=n$. By (2.23), one has

$$\begin{aligned}\|v^n\|_\infty&\leqslant\sum_{k=1}^{n-1}(-g_k^{(\alpha)})\|v^{n-k}\|_\infty+(-g_n^{(\alpha)})\|v^0\|_\infty+\tau^\alpha\|f^n\|_\infty\\&\leqslant\sum_{k=1}^{n-1}(-g_k^{(\alpha)})A_{n-k}+\alpha\left(\frac{2}{n+1}\right)^{\alpha+1}\|v^0\|_\infty+\tau^\alpha\|f^n\|_\infty\\&\leqslant\sum_{k=1}^{n-1}(-g_k^{(\alpha)})A_n+\alpha\left(\frac{2}{n}\right)^{\alpha}\|v^0\|_\infty+\tau^\alpha\|f^n\|_\infty\\&=\left[\sum_{k=1}^{\infty}(-g_k^{(\alpha)})-\sum_{k=n}^{\infty}(-g_k^{(\alpha)})\right]A_n+\alpha\left(\frac{2}{n}\right)^{\alpha}\|v^0\|_\infty+\tau^\alpha\|f^n\|_\infty\\&\leqslant\left[1-\frac{1-\alpha}{5}\left(\frac{2}{n}\right)^{\alpha}\right]A_n+\alpha\left(\frac{2}{n}\right)^{\alpha}\|v^0\|_\infty+\tau^\alpha\|f^n\|_\infty\end{aligned}$$

$$
\begin{aligned}
&= A_n - \frac{1-\alpha}{5}\left(\frac{2}{n}\right)^{\alpha}\left[A_n - \frac{5\alpha}{1-\alpha}\|v^0\|_\infty - \frac{5}{1-\alpha}\left(\frac{n}{2}\right)^{\alpha}\tau^{\alpha}\|f^n\|_\infty\right] \\
&\leqslant A_n,
\end{aligned}
$$

where Lemma 1.4.1, Lemma 1.4.3 and $(\frac{2}{n+1})^{\alpha+1} < (\frac{2}{n})^{\alpha+1} \leqslant (\frac{2}{n})^{\alpha}$ when $n \geqslant 2$ have been used. So (2.21) is also true for $k = n$.

By the principle of induction, (2.21) is true for $k = 1, 2, \ldots, N$. The proof ends. □

Theorem 2.1.2 says that the difference scheme (2.13)–(2.15) is unconditionally stable with respect to both the initial value and the source term.

2.1.4 Convergence of the difference scheme

We are now ready to show the convergence of the difference scheme (2.13)–(2.15).

Theorem 2.1.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (2.1)–(2.3) *and the difference scheme* (2.13)–(2.15), *respectively. Let*

$$
e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N.
$$

Then it holds

$$
\|e^n\|_\infty \leqslant \frac{5}{(1-\alpha)2^{\alpha}} T^{\alpha} c_1(\tau + h^2), \quad 1 \leqslant n \leqslant N.
$$

Proof. Subtracting (2.13)–(2.15) from (2.9), (2.11)–(2.12), respectively, the system of error equations is produced as

$$
\begin{cases}
\tau^{-\alpha}\displaystyle\sum_{k=0}^{n} g_k^{(\alpha)} e_i^{n-k} = \delta_x^2 e_i^n + (r_1)_i^n, & 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\
e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\
e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N.
\end{cases}
$$

Noticing (2.10), by Theorem 2.1.2, it is easy to see that

$$
\begin{aligned}
\|e^n\|_\infty &\leqslant \frac{5}{(1-\alpha)2^{\alpha}} n^{\alpha}\tau^{\alpha} \max_{1\leqslant m\leqslant n}\|(r_1)^m\|_\infty \\
&\leqslant \frac{5}{(1-\alpha)2^{\alpha}} T^{\alpha} c_1(\tau + h^2), \quad 1 \leqslant n \leqslant N.
\end{aligned}
$$

The proof is completed. □

2.2 The fourth-order method in space based on G-L approximation for 1D problem

This section will explore a new fourth-order method in space for solving the problem (2.1)–(2.3).

For any $x \in [0, L]$, define a function $\hat{u}(x, t)$ like that in Section 2.1 and suppose $\hat{u}(x, \cdot) \in \mathscr{C}^{1+\alpha}(\mathcal{R})$ and $u(\cdot, t) \in C^6[0, L]$.

2.2.1 Derivation of the difference scheme

Considering equation (2.1) at the point (x_i, t_n), one has

$$ {}_0^C D_t^\alpha u(x_i, t_n) = u_{xx}(x_i, t_n) + f_i^n, \quad 0 \leqslant i \leqslant M,\ 1 \leqslant n \leqslant N. $$

Performing the operator $\mathcal{A}$ to both hand sides of the equality above yields

$$ \mathcal{A}\, {}_0^C D_t^\alpha u(x_i, t_n) = \mathcal{A} u_{xx}(x_i, t_n) + \mathcal{A} f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. $$

By Theorem 1.4.2 and Lemma 2.1.3, noticing (2.2), one can obtain

$$ \mathcal{A}\left(\tau^{-\alpha} \sum_{k=0}^{n} g_k^{(\alpha)} U_i^{n-k}\right) = \delta_x^2 U_i^n + \mathcal{A} f_i^n + (r_2)_i^n, \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{2.24} $$

where there exists a positive constant c_2 such that

$$ \left|(r_2)_i^n\right| \leqslant c_2(\tau + h^4), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{2.25} $$

Noticing the initial-boundary value conditions (2.2)–(2.3), one has

$$ \begin{cases} U_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, & (2.26) \\ U_0^n = \mu(t_n), \quad U_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.27) \end{cases} $$

Neglecting the small term $(r_2)_i^n$ in (2.24) and replacing the exact solution U_i^n with its numerical one u_i^n, another difference scheme for solving (2.1)–(2.3) is obtained as

$$ \begin{cases} \mathcal{A}\left(\tau^{-\alpha} \sum_{k=0}^{n} g_k^{(\alpha)} u_i^{n-k}\right) = \delta_x^2 u_i^n + \mathcal{A} f_i^n, & \\ \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.28) \\ u_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, & (2.29) \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.30) \end{cases} $$

Next, we will analyze the difference scheme (2.28)–(2.30).

2.2.2 Solvability of the difference scheme

Theorem 2.2.1. *The difference scheme* (2.28)–(2.30) *is uniquely solvable.*

Proof. Denote

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The initial value u^0 is uniquely determined by (2.29)–(2.30). Suppose the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n can be obtained from (2.28) and (2.30). To prove its unique solvability, it suffices to show that the corresponding homogeneous one

$$\begin{cases} \tau^{-\alpha} \mathcal{A} u_i^n = \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (2.31) \\ u_0^n = u_M^n = 0 & (2.32) \end{cases}$$

has only the trivial solution.

To this end, taking the inner product on both hand sides of (2.31) with $-\delta_x^2 u^n$ arrives at

$$\tau^{-\alpha}(\mathcal{A}u^n, -\delta_x^2 u^n) = -(\delta_x^2 u^n, \delta_x^2 u^n),$$

that is

$$\tau^{-\alpha} \|\delta_x u^n\|_A^2 = -\|\delta_x^2 u^n\|^2 \leqslant 0.$$

Thus $\|\delta_x u^n\|_A = 0$ and moreover $\|\delta_x u^n\| = 0$ by Lemma 2.1.2. Noticing (2.32), it follows $u^n = 0$.

By the principle of induction, the theorem is true. The proof ends. □

2.2.3 Stability of the difference scheme

Theorem 2.2.2. *Suppose* $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of difference scheme*

$$\begin{cases} \mathcal{A}\left(\tau^{-\alpha} \sum\limits_{k=0}^{n} g_k^{(\alpha)} v_i^{n-k}\right) = \delta_x^2 v_i^n + g_i^n, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.33) \\ v_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.34) \\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N. & (2.35) \end{cases}$$

Then it holds

$$\|\delta_x v^n\|^2 \leqslant \frac{15}{2(1-\alpha)} \left(\|\delta_x v^0\|^2 + \frac{1}{2^{\alpha+1}} t_n^\alpha \max_{1 \leqslant m \leqslant n} \|g^m\|^2 \right), \quad 1 \leqslant n \leqslant N,$$

where

$$\|g^m\|^2 = h\sum_{i=1}^{M-1}(g_i^m)^2.$$

Proof. Making the inner product on both hand sides of (2.33) with $-\delta_x^2 v^n$ and noticing (2.5) lead to

$$\tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)}(v^{n-k},v^n)_{1,A} = -\|\delta_x^2 v^n\|^2 - h\sum_{i=1}^{M-1}(\delta_x^2 v_i^n)g_i^n \leqslant \frac{1}{4}\|g^n\|^2,$$

which can be rewritten to produce

$$\begin{aligned} g_0^{(\alpha)}(v^n,v^n)_{1,A} &\leqslant \sum_{k=1}^{n}(-g_k^{(\alpha)})(v^n,v^{n-k})_{1,A} + \frac{1}{4}\tau^\alpha\|g^n\|^2 \\ &\leqslant \frac{1}{2}\sum_{k=1}^{n}(-g_k^{(\alpha)})[(v^n,v^n)_{1,A} + (v^{n-k},v^{n-k})_{1,A}] \\ &\quad + \frac{1}{4}\tau^\alpha\|g^n\|^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Noticing $\sum_{k=1}^{n}(-g_k^{(\alpha)}) \leqslant g_0^{(\alpha)} = 1$, it follows

$$(v^n,v^n)_{1,A} \leqslant \sum_{k=1}^{n}(-g_k^{(\alpha)})(v^{n-k},v^{n-k})_{1,A} + \frac{1}{2}\tau^\alpha\|g^n\|^2, \quad 1 \leqslant n \leqslant N,$$

that is,

$$\|\delta_x v^n\|_A^2 \leqslant \sum_{k=1}^{n}(-g_k^{(\alpha)})\|\delta_x v^{n-k}\|_A^2 + \frac{1}{2}\tau^\alpha\|g^n\|^2, \quad 1 \leqslant n \leqslant N.$$

By induction, similar to the proof of Theorem 2.1.2, one can get

$$\|\delta_x v^n\|_A^2 \leqslant \frac{5}{1-\alpha}\|\delta_x v^0\|_A^2 + \frac{5}{(1-\alpha)2^\alpha}t_n^\alpha \cdot \frac{1}{2}\max_{1\leqslant m\leqslant n}\|g^m\|^2, \quad 1 \leqslant n \leqslant N.$$

The application of Lemma 2.1.2 into the inequality above reaches the desired result. The proof is completed. □

Theorem 2.2.2 shows that the difference scheme (2.28)–(2.30) is unconditionally stable with respect to both the initial value and the source term.

2.2.4 Convergence of the difference scheme

Theorem 2.2.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (2.1)–(2.3) *and the difference scheme* (2.28)–(2.30), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N.$$

Then it holds

$$\|e^n\|_\infty \leqslant \frac{L}{4}\sqrt{\frac{15T^\alpha}{(1-\alpha)2^\alpha}}\, c_2(\tau + h^4), \quad 1 \leqslant n \leqslant N.$$

Proof. The subtraction of (2.28)–(2.30) from (2.24), (2.26)–(2.27), respectively, gives the system of error equations in the form of

$$\begin{cases} \mathcal{A}\left(\tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)} e_i^{n-k}\right) = \delta_x^2 e_i^n + (r_2)_i^n, & 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Utilizing Theorem 2.2.2 and noticing (2.25) lead to

$$\begin{aligned} \|\delta_x e^n\|^2 &\leqslant \frac{15}{2(1-\alpha)}\left(\|\delta_x e^0\|^2 + \frac{1}{2^{\alpha+1}} t_n^\alpha \max_{1\leqslant m\leqslant n}\|(r_2)^m\|^2\right) \\ &\leqslant \frac{15}{4(1-\alpha)2^\alpha} T^\alpha L c_2^2 (\tau + h^4)^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Moreover, if follows from Lemma 2.1.1 that

$$\|e^n\|_\infty \leqslant \frac{\sqrt{L}}{2}\|\delta_x e^n\| \leqslant \frac{L}{4}\sqrt{\frac{15T^\alpha}{(1-\alpha)2^\alpha}}\, c_2(\tau + h^4), \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

2.3 The second-order method in space based on L1 approximation for 1D problem

Consider

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], & (2.36) \\ u(x,0) = \varphi(x), \quad x \in (0,L), & (2.37) \\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), \quad t \in [0,T], & (2.38) \end{cases}$$

where $\alpha \in (0,1)$, the functions f, φ, μ, ν are all given and $\varphi(0) = \mu(0)$, $\varphi(L) = \nu(0)$.

Take the same mesh partition and notations as those in Section 2.1. Suppose $u \in C^{(4,2)}([0,L] \times [0,T])$.

2.3.1 Derivation of the difference scheme

Considering equation (2.36) at the point (x_i, t_n), one has

$${}_0^C D_t^\alpha u(x_i, t_n) = u_{xx}(x_i, t_n) + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.$$

Using the L1 formula (1.60) to approximate the time-fractional derivative and the second-order central difference quotient to treat the spatial derivative, by Theorem 1.6.1 and Lemma 2.1.3, one gets

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)}U_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})U_i^k - a_{n-1}^{(\alpha)}U_i^0\right]$$
$$= \delta_x^2 U_i^n + f_i^n + (r_3)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{2.39}$$

where there is a positive constant c_3 such that

$$|(r_3)_i^n| \leqslant c_3(\tau^{2-\alpha} + h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{2.40}$$

Noticing the initial-boundary value conditions (2.37)–(2.38), one has

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.41)\\ U_0^n = \mu(t_n), \quad U_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.42)\end{cases}$$

Omitting the small term $(r_3)_i^n$ in (2.39) and replacing the exact solution U_i^n with its numerical one u_i^n, a difference scheme for solving (2.36)–(2.38) is produced as

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)}u_i^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_i^k - a_{n-1}^{(\alpha)}u_i^0\right] & \\ = \delta_x^2 u_i^n + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.43)\\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.44)\\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.45)\end{cases}$$

Denote

$$s = \tau^\alpha \Gamma(2-\alpha), \quad \lambda = \frac{s}{h^2}.$$

In what follows, the unique solvability, unconditional stability and convergence will be considered.

2.3.2 Solvability of the difference scheme

Theorem 2.3.1. *The difference scheme* (2.43)–(2.45) *is uniquely solvable.*

Proof. Denote

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The value of u^0 is apparently determined by (2.44)–(2.45). Suppose the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n can be obtained from (2.43) and (2.45). To show its unique solvability, it suffices to prove the corresponding homogeneous one

$$\begin{cases} \dfrac{1}{s}u_i^n = \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (2.46)\\ u_0^n = u_M^n = 0 & (2.47)\end{cases}$$

has only the trivial solution.

Suppose $\|u^n\|_\infty = |u_{i_n}^n|$, where $i_n \in \{1, 2, \ldots, M-1\}$. Rewrite (2.46) as

$$(1+2\lambda)u_i^n = \lambda(u_{i-1}^n + u_{i+1}^n), \quad 1 \leqslant i \leqslant M-1.$$

Letting $i = i_n$ in the above equality and taking the absolute value of both hand sides, an application of the triangle inequality yields

$$(1+2\lambda)\|u^n\|_\infty \leqslant 2\lambda\|u^n\|_\infty,$$

so that $\|u^n\|_\infty = 0$, which implies $u^n = 0$.

By the principle of induction, the theorem is true. The proof ends. □

2.3.3 Stability of the difference scheme

Theorem 2.3.2. *Suppose* $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \dfrac{1}{s}\left[a_0^{(\alpha)}v_i^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})v_i^k - a_{n-1}^{(\alpha)}v_i^0\right] & \\ = \delta_x^2 v_i^n + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.48)\\ v_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.49)\\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N. & (2.50)\end{cases}$$

Then it holds

$$\|v^n\|_\infty \leqslant \|v^0\|_\infty + \Gamma(1-\alpha)\max_{1\leqslant m\leqslant n}\left\{t_m^\alpha\|f^m\|_\infty\right\}, \quad 1 \leqslant n \leqslant N,$$

where

$$\|f^m\|_\infty = \max_{1\leqslant i\leqslant M-1}|f_i^m|.$$

Proof. Reformulate (2.48) as

$$a_0^{(\alpha)}v_i^n = \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})v_i^k + a_{n-1}^{(\alpha)}v_i^0$$

$$+\lambda(v_{i-1}^n - 2v_i^n + v_{i+1}^n) + sf_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N,$$

which can be rearranged as

$$\begin{aligned}(a_0^{(\alpha)} + 2\lambda)v_i^n = & \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})v_i^k + a_{n-1}^{(\alpha)}v_i^0 \\ & + \lambda(v_{i-1}^n + v_{i+1}^n) + sf_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.\end{aligned}$$

Suppose $\|v^n\|_\infty = |v_{i_n}^n|$, where $i_n \in \{1,2,\dots,M-1\}$. Letting $i = i_n$ in the equality above and taking the absolute value of both hand sides, an application of the triangle inequality arrives at

$$\begin{aligned}(a_0^{(\alpha)} + 2\lambda)\|v^n\|_\infty \leqslant & \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})\|v^k\|_\infty + a_{n-1}^{(\alpha)}\|v^0\|_\infty \\ & + 2\lambda\|v^n\|_\infty + s\|f^n\|_\infty, \quad 1 \leqslant n \leqslant N.\end{aligned}$$

Therefore,

$$\begin{aligned}\|v^n\|_\infty \leqslant & \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})\|v^k\|_\infty \\ & + a_{n-1}^{(\alpha)}\left(\|v^0\|_\infty + \frac{s}{a_{n-1}^{(\alpha)}}\|f^n\|_\infty\right), \quad 1 \leqslant n \leqslant N.\end{aligned}$$

It follows from Lemma 1.6.1 that

$$\frac{s}{a_{n-1}^{(\alpha)}} \leqslant \frac{\tau^\alpha \Gamma(2-\alpha)}{(1-\alpha)n^{-\alpha}} = t_n^\alpha \Gamma(1-\alpha), \tag{2.51}$$

and then

$$\begin{aligned}a_0^{(\alpha)}\|v^n\|_\infty \leqslant & \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})\|v^k\|_\infty \\ & + a_{n-1}^{(\alpha)}(\|v^0\|_\infty + t_n^\alpha \Gamma(1-\alpha)\|f^n\|_\infty), \quad 1 \leqslant n \leqslant N.\end{aligned} \tag{2.52}$$

An induction on n in (2.52) will yield

$$\|v^n\|_\infty \leqslant \|v^0\|_\infty + \Gamma(1-\alpha)\max_{1 \leqslant m \leqslant n}\left\{t_m^\alpha \|f^m\|_\infty\right\}, \quad 1 \leqslant n \leqslant N.$$

The proof is completed. □

2.3.4 Convergence of the difference scheme

The next theorem is to describe the convergence of difference scheme (2.43)–(2.45).

Theorem 2.3.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (2.36)–(2.38) *and the difference scheme* (2.43)–(2.45), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N.$$

Then it holds

$$\|e^n\|_\infty \leqslant c_3 T^\alpha \Gamma(1-\alpha)(\tau^{2-\alpha} + h^2), \quad 1 \leqslant n \leqslant N.$$

Proof. Subtracting (2.43)–(2.45) from (2.39), (2.41)–(2.42), respectively, the system of error equations can be obtained as

$$\begin{cases} \dfrac{1}{s}\left[a_0^{(\alpha)} e_i^n - \displaystyle\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})e_i^k - a_{n-1}^{(\alpha)} e_i^0\right] \\ = \delta_x^2 e_i^n + (r_3)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

By Theorem 2.3.2 and the inequality (2.40), it follows

$$\begin{aligned} \|e^n\|_\infty &\leqslant \|e^0\|_\infty + t_n^\alpha \Gamma(1-\alpha) \max_{1 \leqslant m \leqslant n} \|(r_3)^m\|_\infty \\ &\leqslant t_n^\alpha \Gamma(1-\alpha) c_3 (\tau^{2-\alpha} + h^2) \\ &\leqslant c_3 T^\alpha \Gamma(1-\alpha)(\tau^{2-\alpha} + h^2), \quad 1 \leqslant n \leqslant N. \end{aligned}$$

The proof ends. □

2.4 The fast difference method based on L1 approximation for 1D problem

The aim of this section is to develop a difference scheme for the problem (2.36)–(2.38) by using the fast L1 approximation.

2.4.1 Derivation of the difference scheme

Considering (2.36) at the point (x_i, t_n), we get

$${}_0^C D_t^\alpha u(x_i, t_n) = u_{xx}(x_i, t_n) + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.$$

Using the fast L1 approximation (1.117)–(1.119) to discretize the time fractional derivative and second-order central difference quotient to approximate the spatial second-order derivative, by Theorem 1.7.1, we can obtain

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\left[\sum_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + \hat{a}_0^{(\alpha)}(U_i^n - U_i^{n-1})\right] \\ = \delta_x^2 U_i^n + f_i^n + (r_4)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.53) \\ F_{l,i}^1 = 0, \quad F_{l,i}^n = e^{-s_l\tau} F_{l,i}^{n-1} + B_l(U_i^{n-1} - U_i^{n-2}), \quad 1 \leqslant l \leqslant N_{\exp}, \\ \qquad 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N, & (2.54) \end{cases}$$

and there exists a positive constant c_4 such that

$$|(r_4)_i^n| \leqslant c_4(\tau^{2-\alpha} + h^2 + \epsilon), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{2.55}$$

Noticing the initial-boundary conditions (2.37)–(2.38), we have

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.56) \\ U_0^n = \mu(t_n), \quad U_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.57) \end{cases}$$

Omitting the small term $(r_4)_i^n$ in (2.53) and using numerical solution u_i^n to replace the exact solution U_i^n, we construct the difference scheme for the problem (2.36)–(2.38) as follows:

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\left[\sum_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + \hat{a}_0^{(\alpha)}(u_i^n - u_i^{n-1})\right] \\ = \delta_x^2 u_i^n + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.58) \\ F_{l,i}^1 = 0, \quad F_{l,i}^n = e^{-s_l\tau} F_{l,i}^{n-1} + B_l(u_i^{n-1} - u_i^{n-2}), \quad 1 \leqslant l \leqslant N_{\exp}, \\ \qquad 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N, & (2.59) \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.60) \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.61) \end{cases}$$

When $\{u_i^k \mid 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant n-1\}$ has been known, we can get $\{F_{l,i}^n \mid 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1\}$ from (2.59). Substituting it into the left-hand side of (2.58), we can get a system of linear equations in $\{u_i^n \mid 0 \leqslant i \leqslant M\}$, whose calculation cost is $O(MN_{\exp})$. While the calculation of the system of linear equations in $\{u_i^n \mid 0 \leqslant i \leqslant M\}$ from (2.43)–(2.45) is $O(MN)$. Thus, the total operations of (2.58)–(2.61) are $O(MNN_{\exp})$, and those of (2.43)–(2.45) are $O(MN^2)$. When $N >> N_{\exp}$, we have $O(MN^2) >> O(MNN_{\exp})$. We call (2.58)–(2.61) a fast difference scheme based on L1 approximation, or **a fast L1 difference scheme**.

Eliminating the intermediate variable $\{F_{l,i}^n\}$ in (2.58)–(2.61), we can get an equivalent form

$$\begin{cases} \frac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)}u_i^n - \sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)} - \hat{a}_k^{(\alpha)})u_i^{n-k} - \hat{a}_{n-1}^{(\alpha)}u_i^0\right] \\ = \delta_x^2 u_i^n + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.62) \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.63) \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.64) \end{cases}$$

2.4.2 Solvability of the difference scheme

Theorem 2.4.1. *The difference scheme* (2.58)–(2.61) *is uniquely solvable.*

Proof. Denote

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

From (2.60)–(2.61), we can know u^0. Suppose $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then we can get a system of linear equations in u^n from (2.58) and (2.61). It suffices to show that the corresponding homogeneous system

$$\begin{cases} \frac{1}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}u_i^n = \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (2.65) \\ u_0^n = u_M^n = 0 & (2.66) \end{cases}$$

has only the trivial solution.

Suppose $\|u^n\|_\infty = |u_{i_n}^n|$, where $i_n \in \{1, 2, \ldots, M-1\}$. Denote $\lambda = \frac{\Gamma(1-\alpha)}{\hat{a}_0^{(\alpha)}h^2}$. Rewrite (2.65) as

$$(1+2\lambda)u_i^n = \lambda(u_{i-1}^n + u_{i+1}^n), \quad 1 \leqslant i \leqslant M-1.$$

Let $i = i_n$ in the equality above. Taking absolute value of both hand sides and using the triangle inequality, we can obtain

$$(1+2\lambda)\|u^n\|_\infty \leqslant 2\lambda\|u^n\|_\infty.$$

Thus we have $\|u^n\|_\infty = 0$, which implies $u^n = 0$.

By induction principle, the conclusion holds. This completes the proof. □

2.4.3 Stability of the difference scheme

Theorem 2.4.2. *Suppose* $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of*

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\left[\sum_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + \hat{a}_0^{(\alpha)}(v_i^n - v_i^{n-1})\right] \\ = \delta_x^2 v_i^n + g_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.67) \\ F_{l,i}^1 = 0, \quad F_{l,i}^n = e^{-s_l\tau}F_{l,i}^{n-1} + B_l(v_i^{n-1} - v_i^{n-2}), \\ \qquad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N, & (2.68) \\ v_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.69) \\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N & (2.70) \end{cases}$$

and $\epsilon < \frac{2-2^{1-\alpha}}{1-\alpha}\tau^{-\alpha}$, *then we have*

$$\|v^n\|_\infty \leqslant \|v^0\|_\infty + 2\Gamma(1-\alpha)\max_{1\leqslant m\leqslant n}\{t_m^\alpha \|g^m\|_\infty\}, \quad 1 \leqslant n \leqslant N, \tag{2.71}$$

where

$$\|g^m\|_\infty = \max_{1\leqslant i\leqslant M-1}|g_i^m|.$$

Proof. From Lemma 1.7.2, we know

$$\hat{a}_0^{(\alpha)} > \hat{a}_1^{(\alpha)} > \hat{a}_2^{(\alpha)} > \cdots > \hat{a}_{n-1}^{(\alpha)}.$$

Eliminating the intermediate variable $\{F_{l,i}^n\}$ in (2.67) using (2.68), we have

$$\frac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)}v_i^n - \sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)} - \hat{a}_k^{(\alpha)})v_i^{n-k} - \hat{a}_{n-1}^{(\alpha)}v_i^0\right]$$
$$= \delta_x^2 v_i^n + g_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N,$$

or

$$\left(\frac{\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)} + \frac{2}{h^2}\right)v_i^n = \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}(\hat{a}_{n-k-1}^{(\alpha)} - \hat{a}_{n-k}^{(\alpha)})v_i^k + \hat{a}_{n-1}^{(\alpha)}v_i^0\right]$$
$$+ \frac{1}{h^2}(v_{i-1}^n + v_{i+1}^n) + g_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.$$

Suppose $\|v^n\|_\infty = |v_{i_n}^n|$, where $i_n \in \{1, 2, \ldots, M-1\}$. Letting $i = i_n$ in the above equation, taking absolute value of both hand sides and using the triangle inequality, we can obtain

$$\left(\frac{\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)} + \frac{2}{h^2}\right)\|v^n\|_\infty$$

$$\leqslant \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}(\hat{a}_{n-k-1}^{(\alpha)}-\hat{a}_{n-k}^{(\alpha)})\|v^k\|_\infty+\hat{a}_{n-1}^{(\alpha)}\|v^0\|_\infty\right]$$
$$+\frac{2}{h^2}\|v^n\|_\infty+\|g^n\|_\infty, \quad 1\leqslant n\leqslant N.$$

Thus we have

$$\hat{a}_0^{(\alpha)}\|v^n\|_\infty \leqslant \sum_{k=1}^{n-1}(\hat{a}_{n-k-1}^{(\alpha)}-\hat{a}_{n-k}^{(\alpha)})\|v^k\|_\infty$$
$$+\hat{a}_{n-1}^{(\alpha)}\left(\|v^0\|_\infty+\Gamma(1-\alpha)\frac{\|g^n\|_\infty}{\hat{a}_{n-1}^{(\alpha)}}\right), \quad 1\leqslant n\leqslant N.$$

An application of the induction method will yield

$$\|v^n\|_\infty \leqslant \|v^0\|_\infty+\Gamma(1-\alpha)\max_{1\leqslant m\leqslant n}\frac{\|g^m\|_\infty}{\hat{a}_{m-1}^{(\alpha)}}, \quad 1\leqslant n\leqslant N.$$

Noticing

$$\hat{a}_{m-1}^{(\alpha)} \geqslant t_m^{-\alpha}-\epsilon \geqslant \frac{1}{2}t_m^{-\alpha},$$

we can get (2.71). This completes the proof. □

2.4.4 Convergence of the difference scheme

Theorem 2.4.3. *Suppose* $\{U_i^n \mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *and* $\{u_i^n \mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *are the solutions of* (2.36)–(2.38) *and* (2.58)–(2.61), *respectively. Denote*

$$e_i^n = U_i^n - u_i^n, \quad 0\leqslant i\leqslant M,\ 0\leqslant n\leqslant N,$$

then we have

$$\|e^n\|_\infty \leqslant 2c_4T^\alpha\Gamma(1-\alpha)(\tau^{2-\alpha}+h^2+\epsilon), \quad 1\leqslant n\leqslant N.$$

Proof. Eliminating the intermediate variable $\{F_{l,i}^n\}$ in (2.53) using (2.54), we have

$$\frac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)}U_i^n-\sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)}-\hat{a}_k^{(\alpha)})U_i^{n-k}-\hat{a}_{n-1}^{(\alpha)}U_i^0\right]$$
$$=\delta_x^2U_i^n+f_i^n+(r_4)_i^n, \quad 1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N. \tag{2.72}$$

Subtracting (2.62)–(2.64) from (2.72), (2.56)–(2.57) respectively, we get the system of error equations

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)} e_i^n - \sum_{k=1}^{n-1}(\hat{a}_{n-k-1}^{(\alpha)} - \hat{a}_{n-k}^{(\alpha)})e_i^k - \hat{a}_{n-1}^{(\alpha)} e_i^0\right] \\ = \delta_x^2 e_i^n + (r_4)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Applying Theorem 2.4.2 and noticing (2.55), we have

$$\begin{aligned} \|e^n\|_\infty &\leqslant \|e^0\|_\infty + 2t_n^\alpha \Gamma(1-\alpha) \max_{1\leqslant m\leqslant n} \|(r_4)^m\|_\infty \\ &\leqslant 2t_n^\alpha \Gamma(1-\alpha) c_4 (\tau^{2-\alpha} + h^2 + \epsilon) \\ &\leqslant 2c_4 T^\alpha \Gamma(1-\alpha)(\tau^{2-\alpha} + h^2 + \epsilon), \quad 1 \leqslant n \leqslant N. \end{aligned}$$

This completes the proof. □

2.5 The fourth-order method in space based on L1 approximation for 1D problem

In this section, another higher-order difference scheme for solving (2.36)–(2.38) will be developed. Suppose the exact solution $u \in C^{(6,2)}([0,L] \times [0,T])$.

2.5.1 Derivation of the difference scheme

Considering equation (2.36) at the point (x_i, t_n), one has

$${}_0^C D_t^\alpha u(x_i,t_n) = u_{xx}(x_i,t_n) + f_i^n, \quad 0 \leqslant i \leqslant M,\ 1 \leqslant n \leqslant N.$$

Performing the operator $\mathcal{A}$ to both hand sides of the equality above gives

$$\mathcal{A}\, {}_0^C D_t^\alpha u(x_i,t_n) = \mathcal{A} u_{xx}(x_i,t_n) + \mathcal{A} f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.$$

It follows from Theorem 1.6.1 and Lemma 2.1.3 that

$$\begin{aligned} &\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \mathcal{A}\left[a_0^{(\alpha)} U_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})U_i^k - a_{n-1}^{(\alpha)} U_i^0\right] \\ &= \delta_x^2 U_i^n + \mathcal{A} f_i^n + (r_5)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \end{aligned} \tag{2.73}$$

where there exists a positive constant c_5 such that

$$|(r_5)_i^n| \leqslant c_5(\tau^{2-\alpha} + h^4), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{2.74}$$

Noticing the initial-boundary value conditions (2.37)–(2.38), one has

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.75) \\ U_0^n = \mu(t_n), \quad U_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.76) \end{cases}$$

Neglecting the small term $(r_5)_i^n$ in (2.73) and replacing the exact solution U_i^n with its numerical one u_i^n lead to another difference scheme for solving (2.36)–(2.38) as follows:

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\mathcal{A}\left[a_0^{(\alpha)}u_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_i^k - a_{n-1}^{(\alpha)}u_i^0\right] \\ = \delta_x^2 u_i^n + \mathcal{A}f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.77) \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.78) \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.79) \end{cases}$$

2.5.2 Solvability of the difference scheme

Let

$$s = \tau^\alpha \Gamma(2-\alpha).$$

Theorem 2.5.1. *The difference scheme* (2.77)–(2.79) *is uniquely solvable.*

Proof. Denote

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

Obviously, the value of u^0 is uniquely determined by (2.78) and (2.79). Suppose the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n can be obtained from (2.77) and (2.79). To prove its unique solvability, it suffices to show that its corresponding homogeneous one

$$\begin{cases} \dfrac{1}{s}\mathcal{A}u_i^n = \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (2.80) \\ u_0^n = u_M^n = 0 & (2.81) \end{cases}$$

has only the trivial solution.

Making the inner product on both hand sides of (2.80) with $-\delta_x^2 u^n$ and noticing (2.5) give

$$\frac{1}{s}(u^n, u^n)_{1,A} = -\|\delta_x^2 u^n\|^2 \leqslant 0,$$

so that $\|\delta_x u^n\|_A = 0$. By Lemma 2.1.2, it follows $\|\delta_x u^n\| = 0$, which implies $u^n = 0$ from (2.81).

By the principle of induction, the theorem is true. The proof ends. □

2.5.3 Stability of the difference scheme

Theorem 2.5.2. *Suppose* $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \dfrac{1}{s}\mathcal{A}\left[a_0^{(\alpha)}v_i^n - \displaystyle\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})v_i^k - a_{n-1}^{(\alpha)}v_i^0\right] \\ = \delta_x^2 v_i^n + g_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.82) \\ v_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.83) \\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N. & (2.84) \end{cases}$$

Then it holds

$$\|\delta_x v^n\|^2 \leqslant \frac{3}{2}\|\delta_x v^0\|^2 + \frac{3}{4}\Gamma(1-\alpha)\max_{1\leqslant m\leqslant n}\{t_m^\alpha\|g^m\|^2\}, \quad 1 \leqslant n \leqslant N, \tag{2.85}$$

where

$$\|g^m\|^2 = h\sum_{i=1}^{M-1}(g_i^m)^2.$$

Proof. Taking the inner product on both hand sides of (2.82) with $-\delta_x^2 v^n$ and noticing (2.5) produce

$$\frac{1}{s}\left[a_0^{(\alpha)}(v^n, v^n)_{1,A} - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})(v^k, v^n)_{1,A} - a_{n-1}^{(\alpha)}(v^0, v^n)_{1,A}\right]$$
$$= -\|\delta_x^2 v^n\|^2 + h\sum_{i=1}^{M-1} g_i^n(-\delta_x^2 v_i^n), \quad 1 \leqslant n \leqslant N.$$

Using the Cauchy–Schwarz inequality, one has

$$\begin{aligned} a_0^{(\alpha)}(v^n, v^n)_{1,A} &= \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})(v^k, v^n)_{1,A} + a_{n-1}^{(\alpha)}(v^0, v^n)_{1,A} \\ &\quad + s\left[-\|\delta_x^2 v^n\|^2 + h\sum_{i=1}^{M-1} g_i^n(-\delta_x^2 v_i^n)\right] \\ &\leqslant \frac{1}{2}\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})[(v^k, v^k)_{1,A} + (v^n, v^n)_{1,A}] \\ &\quad + \frac{1}{2}a_{n-1}^{(\alpha)}[(v^0, v^0)_{1,A} + (v^n, v^n)_{1,A}] + s\Big[-\|\delta_x^2 v^n\|^2 \\ &\quad + \|\delta_x^2 v^n\|^2 + \frac{1}{4}\|g^n\|^2\Big], \quad 1 \leqslant n \leqslant N, \end{aligned}$$

from which it follows that

$$a_0^{(\alpha)}(v^n, v^n)_{1,A} \leqslant \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})(v^k, v^k)_{1,A}$$

$$+ a_{n-1}^{(\alpha)}\left[(v^0, v^0)_{1,A} + \frac{s}{2a_{n-1}^{(\alpha)}}\|g^n\|^2\right], \quad 1 \leqslant n \leqslant N.$$

Moreover, it follows from (2.51) that

$$\begin{aligned} a_0^{(\alpha)}(v^n, v^n)_{1,A} \leqslant & \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})(v^k, v^k)_{1,A} \\ & + a_{n-1}^{(\alpha)}\left[(v^0, v^0)_{1,A} + \frac{1}{2}\Gamma(1-\alpha)t_n^\alpha\|g^n\|^2\right], \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{2.86}$$

An induction on n in (2.86) will lead to

$$(v^n, v^n)_{1,A} \leqslant (v^0, v^0)_{1,A} + \frac{1}{2}\Gamma(1-\alpha) \max_{1\leqslant m\leqslant n}\{t_m^\alpha\|g^m\|^2\}, \quad 1 \leqslant n \leqslant N.$$

Combining with Lemma 2.1.2, it is easy to get (2.85). The proof ends. □

2.5.4 Convergence of the difference scheme

The rest of this section focuses on the convergence of the difference scheme (2.77)–(2.79). At this point, the following theorem will be obtained.

Theorem 2.5.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (2.36)–(2.38) *and the difference scheme* (2.77)–(2.79), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N.$$

Then it holds

$$\|e^n\|_\infty \leqslant \frac{L}{4}\sqrt{3T^\alpha\Gamma(1-\alpha)}\, c_5(\tau^{2-\alpha} + h^4), \quad 1 \leqslant n \leqslant N.$$

Proof. The subtraction of (2.77)–(2.79) from (2.73), (2.75)–(2.76), respectively, produces the system of error equations

$$\begin{cases} \frac{1}{s}\mathcal{A}\left[a_0^{(\alpha)}e_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})e_i^k - a_{n-1}^{(\alpha)}e_i^0\right] \\ = \delta_x^2 e_i^n + (r_5)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

By Theorem 2.5.2 and the inequality (2.74), one has

$$\|\delta_x e^n\|^2 \leqslant \frac{3}{2}\|\delta_x e^0\|^2 + \frac{3}{4}\Gamma(1-\alpha) \max_{1\leqslant m\leqslant n}\{t_m^\alpha\|(r_5)^m\|^2\}$$

$$\leqslant \frac{3}{4} T^{\alpha}\Gamma(1-\alpha)Lc_5^2(\tau^{2-\alpha}+h^4)^2, \quad 1 \leqslant n \leqslant N.$$

Moreover, it follows from Lemma 2.1.1 that

$$\|e^n\|_{\infty} \leqslant \frac{\sqrt{L}}{2}\|\delta_x e^n\| \leqslant \frac{L}{4}\sqrt{3T^{\alpha}\Gamma(1-\alpha)}\, c_5(\tau^{2-\alpha}+h^4),\ 1 \leqslant n \leqslant N.$$

This ends the proof. □

2.6 The difference method based on L2-1_σ approximation for 1D problem

This section will still discuss the difference method for the problem (2.36)–(2.38). A new difference scheme will be built based on L2-1_σ approximation for the time fractional derivatives. Suppose the exact solution $u \in C^{(4,3)}([0,L]\times[0,T])$.

2.6.1 Derivation of the difference scheme

Denote

$$\sigma = 1-\frac{\alpha}{2}, \quad t_{n-1+\sigma} = (n-1+\sigma)\tau, \quad s = \tau^{\alpha}\Gamma(2-\alpha), \quad f_i^{n-1+\sigma} = f(x_i, t_{n-1+\sigma}).$$

Considering (2.36) at the point $(x_i, t_{n-1+\sigma})$, we have

$$\begin{aligned} {}_0^C D_t^{\alpha} u(x_i, t_{n-1+\sigma}) &= u_{xx}(x_i, t_{n-1+\sigma}) + f_i^{n-1+\sigma}, \\ & 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \end{aligned} \tag{2.87}$$

Using L2-1_σ approximation (1.81) to discretize the time fractional derivative, we have

$${}_0^C D_t^{\alpha} u(x_i, t_{n-1+\sigma}) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(U_i^{n-k}-U_i^{n-k-1}) + O(\tau^{3-\alpha}). \tag{2.88}$$

Using the linear interpolation and the second order central difference quotient to approximate the spatial second-order derivative, we have

$$\begin{aligned} u_{xx}(x_i, t_{n-1+\sigma}) &= \sigma u_{xx}(x_i, t_n) + (1-\sigma)u_{xx}(x_i, t_{n-1}) + O(\tau^2) \\ &= \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + O(h^2) + O(\tau^2). \end{aligned} \tag{2.89}$$

Inserting (2.88) and (2.89) into (2.87), we can obtain

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(U_i^{n-k}-U_i^{n-k-1})$$

$$= \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + f_i^{n-1+\sigma} + (r_6)_i^n,$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N \tag{2.90}$$

and there exists a positive constant c_6 such that

$$|(r_6)_i^n| \leqslant c_6(\tau^2 + h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{2.91}$$

Noticing the initial-boundary value conditions (2.37)–(2.38), we have

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.92)\\ U_0^n = \mu(t_n), \quad U_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.93)\end{cases}$$

Omitting the small term $(r_6)_i^n$ in (2.90) and using numerical solution u_i^n to replace the exact solution U_i^n, we construct the following difference scheme for the problem (2.36)–(2.38):

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \displaystyle\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u_i^{n-k} - u_i^{n-k-1}) = \sigma\delta_x^2 u_i^n \\ \quad + (1-\sigma)\delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.94)\\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.95)\\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.96)\end{cases}$$

2.6.2 Solvability of the difference scheme

Theorem 2.6.1. *The difference scheme* (2.94)–(2.96) *is uniquely solvable.*

Proof. Denote $u^n = (u_0^n, u_1^n, \ldots, u_M^n)$.

From (2.95)–(2.96) we can know u^0. Suppose $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then we can get a system of linear equations in u^n from (2.94) and (2.96). It suffices to show that the corresponding homogeneous system

$$\begin{cases} \dfrac{1}{s} c_0^{(n,\alpha)} u_i^n = \sigma\delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (2.97)\\ u_0^n = u_M^n = 0 & (2.98)\end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (2.97) with u^n yields

$$\frac{1}{s} c_0^{(n,\alpha)} \|u^n\|^2 = \sigma(u^n, \delta_x^2 u^n) = -\sigma\|\delta_x u^n\|^2.$$

It is easy to know that

$$u_i^n = 0, \quad 0 \leqslant i \leqslant M.$$

By induction principle, the conclusion holds. This completes the proof. □

2.6.3 An important lemma

Lemma 2.6.1. [1] *Suppose $\mathcal{V}$ is an inner space, $(\cdot,\cdot)$ is an inner product in $\mathcal{V}$ and $\|\cdot\|$ is the induced norm. In addition, assume that $0 < \alpha < 1$, and $\{c_k^{(n,\alpha)} \mid 0 \leqslant k \leqslant n-1, n \geqslant 1\}$ satisfies*

$$\begin{cases} c_0^{(n,\alpha)} > c_1^{(n,\alpha)} > c_2^{(n,\alpha)} > \cdots > c_{n-2}^{(n,\alpha)} > c_{n-1}^{(n,\alpha)} > (1-\alpha)n^{-\alpha}, & (2.99) \\ (2\sigma-1)c_0^{(n,\alpha)} - \sigma c_1^{(n,\alpha)} > 0. & (2.100) \end{cases}$$

Then, for any $u^0,\ u^1,\ \ldots, u^n \in \mathcal{V}$, we have

$$\begin{aligned} &\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k} - u^{n-k-1}, \sigma u^n + (1-\sigma)u^{n-1}) \\ &\geqslant \frac{1}{2}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|u^{n-k}\|^2 - \|u^{n-k-1}\|^2), \quad n = 1, 2, \ldots . \end{aligned}$$

Proof. The proof will be carried out in three steps.

(I) Prove

$$\begin{aligned} &\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k} - u^{n-k-1}, u^n) \\ &\geqslant \frac{1}{2}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|u^{n-k}\|^2 - \|u^{n-k-1}\|^2) \\ &\quad + \frac{1}{2c_0^{(n,\alpha)}}\left\|\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k} - u^{n-k-1})\right\|^2, \quad n = 1, 2, \ldots . \end{aligned} \tag{2.101}$$

When $n = 1$, we have

$$(u^n - u^{n-1}, u^n) = \frac{1}{2}(\|u^n\|^2 - \|u^{n-1}\|^2) + \frac{1}{2}\|u^n - u^{n-1}\|^2.$$

It is easy to know that (2.101) holds.

Next, we consider $n \geqslant 2$.

$$\begin{aligned} A &\equiv \sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k} - u^{n-k-1}, u^n) - \frac{1}{2}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|u^{n-k}\|^2 - \|u^{n-k-1}\|^2) \\ &= \sum_{k=0}^{n-1} c_k^{(n,\alpha)}\left(u^{n-k} - u^{n-k-1}, u^n - \frac{u^{n-k} + u^{n-k-1}}{2}\right) \\ &= \sum_{k=0}^{n-1} c_k^{(n,\alpha)}\left(u^{n-k} - u^{n-k-1}, \frac{u^{n-k} - u^{n-k-1}}{2} + \sum_{m=1}^{k}(u^{n-m+1} - u^{n-m})\right) \\ &= \frac{1}{2}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}\|u^{n-k} - u^{n-k-1}\|^2 \end{aligned}$$

$$\begin{aligned}
&+\sum_{k=1}^{n-1} c_k^{(n,\alpha)}\Big(u^{n-k}-u^{n-k-1}, \sum_{m=1}^{k}(u^{n-m+1}-u^{n-m})\Big)\\
=&\ \frac{1}{2}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}\|u^{n-k}-u^{n-k-1}\|^2\\
&+\sum_{m=1}^{n-1}\Big(u^{n-m+1}-u^{n-m}, \sum_{k=m}^{n-1} c_k^{(n,\alpha)}(u^{n-k}-u^{n-k-1})\Big).
\end{aligned}$$

Let

$$\begin{aligned}
w_m &= \sum_{k=m}^{n-1} c_k^{(n,\alpha)}(u^{n-k}-u^{n-k-1}), \quad m=0,1,\dots,n-1,\\
w_n &= 0,
\end{aligned}$$

then we have

$$w_m - w_{m+1} = c_m^{(n,\alpha)}(u^{n-m}-u^{n-m-1}), \quad m=0,1,\dots,n-1.$$

Thus we get

$$\begin{aligned}
A &= \frac{1}{2}\sum_{m=0}^{n-1}\frac{1}{c_m^{(n,\alpha)}}\|w_m-w_{m+1}\|^2+\sum_{m=1}^{n-1}\frac{1}{c_{m-1}^{(n,\alpha)}}(w_{m-1}-w_m, w_m)\\
&= \frac{1}{2}\frac{1}{c_0^{(n,\alpha)}}\|w_0\|^2+\frac{1}{2}\sum_{m=1}^{n-1}\Big(\frac{1}{c_m^{(n,\alpha)}}-\frac{1}{c_{m-1}^{(n,\alpha)}}\Big)\|w_m\|^2\\
&\geqslant \frac{1}{2}\frac{1}{c_0^{(n,\alpha)}}\|w_0\|^2\\
&= \frac{1}{2}\frac{1}{c_0^{(n,\alpha)}}\Big\|\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k}-u^{n-k-1})\Big\|^2.
\end{aligned}$$

(II) Prove

$$\begin{aligned}
&\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k}-u^{n-k-1}, u^{n-1})\\
\geqslant&\ \frac{1}{2}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|u^{n-k}\|^2-\|u^{n-k-1}\|^2)\\
&-\frac{1}{2(c_0^{(n,\alpha)}-c_1^{(n,\alpha)})}\Big\|\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k}-u^{n-k-1})\Big\|^2, \quad n=1,2,\dots. \qquad (2.102)
\end{aligned}$$

In fact, we have

$$\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k}-u^{n-k-1}, u^{n-1})$$

$$
\begin{aligned}
&- \frac{1}{2}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}\big(\|u^{n-k}\|^2 - \|u^{n-k-1}\|^2\big) \\
&+ \frac{1}{2(c_0^{(n,\alpha)} - c_1^{(n,\alpha)})}\Big\|\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k} - u^{n-k-1})\Big\|^2 \\
={}& \sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k} - u^{n-k-1}, u^n) \\
&- \frac{1}{2}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}\big(\|u^{n-k}\|^2 - \|u^{n-k-1}\|^2\big) \\
&+ \frac{1}{2(c_0^{(n,\alpha)} - c_1^{(n,\alpha)})}\Big\|\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k} - u^{n-k-1})\Big\|^2 \\
&- \Big(u^n - u^{n-1}, \sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k} - u^{n-k-1})\Big) \\
={}& \frac{1}{2}\sum_{m=1}^{n-1}\Big(\frac{1}{c_m^{(n,\alpha)}} - \frac{1}{c_{m-1}^{(n,\alpha)}}\Big)\|w_m\|^2 + \frac{1}{2c_0^{(n,\alpha)}}\|w_0\|^2 \\
&+ \frac{1}{2(c_0^{(n,\alpha)} - c_1^{(n,\alpha)})}\|w_0\|^2 - \frac{1}{c_0^{(n,\alpha)}}(w_0 - w_1, w_0) \\
={}& \frac{1}{2}\sum_{m=2}^{n-1}\Big(\frac{1}{c_m^{(n,\alpha)}} - \frac{1}{c_{m-1}^{(n,\alpha)}}\Big)\|w_m\|^2 \\
&+ \frac{c_1^{(n,\alpha)}}{2c_0^{(n,\alpha)}(c_0^{(n,\alpha)} - c_1^{(n,\alpha)})}\Big\|w_0 + \frac{c_0^{(n,\alpha)} - c_1^{(n,\alpha)}}{c_1^{(n,\alpha)}} w_1\Big\|^2 \\
\geqslant{}& 0.
\end{aligned}
$$

(III) Multiplying (2.101) with σ, (2.102) with $(1-\sigma)$, summing up the results and using (2.100), we have

$$
\begin{aligned}
&\sum_{k=0}^{n-1} c_k^{(n,\alpha)}\big(u^{n-k} - u^{n-k-1}, \sigma u^n + (1-\sigma)u^{n-1}\big) \\
\geqslant{}& \frac{1}{2}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}\big(\|u^{n-k}\|^2 - \|u^{n-k-1}\|^2\big), \quad n = 1, 2, \ldots.
\end{aligned}
$$

This completes the proof. □

2.6.4 Stability of the difference scheme

Theorem 2.6.2. *Suppose* $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(v_i^{n-k}-v_i^{n-k-1}) = \sigma\delta_x^2 v_i^n + (1-\sigma)\delta_x^2 v_i^{n-1} + g_i^n, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.103) \\ v_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.104) \\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N, & (2.105) \end{cases}$$

then we have

$$\|v^n\|^2 \leqslant \|v^0\|^2 + \frac{L^2}{12}\Gamma(1-\alpha)\max_{1\leqslant m\leqslant n}(t_m^\alpha\|g^m\|^2), \quad 1 \leqslant n \leqslant N, \tag{2.106}$$

$$\|\delta_x v^n\|^2 \leqslant \|\delta_x v^0\|^2 + \frac{1}{2}\Gamma(1-\alpha)\max_{1\leqslant m\leqslant n}(t_m^\alpha\|g^m\|^2), \quad 1 \leqslant n \leqslant N, \tag{2.107}$$

where

$$\|g^m\|^2 = h\sum_{i=1}^{M-1}|g_i^m|^2.$$

Proof. (I) Taking the inner product on both hand sides of (2.103) with $\sigma v^n + (1-\sigma)v^{n-1}$, we have

$$\begin{aligned} &\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(v^{n-k}-v^{n-k-1}, \sigma v^n + (1-\sigma)v^{n-1}) \\ &= (\delta_x^2(\sigma v^n + (1-\sigma)v^{n-1}), \sigma v^n + (1-\sigma)v^{n-1}) + (g^n, \sigma v^n + (1-\sigma)v^{n-1}) \\ &= -\|\delta_x(\sigma v^n + (1-\sigma)v^{n-1})\|^2 + (g^n, \sigma v^n + (1-\sigma)v^{n-1}) \\ &\leqslant -\frac{6}{L^2}\|\sigma v^n + (1-\sigma)v^{n-1}\|^2 + \left[\frac{6}{L^2}\|\sigma v^n + (1-\sigma)v^{n-1}\|^2 + \frac{L^2}{24}\|g^n\|^2\right] \\ &= \frac{L^2}{24}\|g^n\|^2, \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{2.108}$$

For the left-hand side of the inequality above, applying Lemma 2.6.1 and Lemma 1.6.3, we have

$$\begin{aligned} &\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(v^{n-k}-v^{n-k-1}, \sigma v^n + (1-\sigma)v^{n-1}) \\ &\geqslant \frac{1}{2}\cdot\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|v^{n-k}\|^2 - \|v^{n-k-1}\|^2). \end{aligned} \tag{2.109}$$

From (2.108) and (2.109), we have

$$\frac{1}{2}\cdot\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|v^{n-k}\|^2 - \|v^{n-k-1}\|^2) \leqslant \frac{L^2}{24}\|g^n\|^2, \quad 1 \leqslant n \leqslant N. \tag{2.110}$$

In view of Lemma 1.6.3, we have

$$\frac{s}{c_{n-1}^{(n,\alpha)}} \leqslant \frac{\tau^\alpha \Gamma(2-\alpha)}{(1-\alpha)n^{-\alpha}} \leqslant t_n^\alpha \Gamma(1-\alpha). \tag{2.111}$$

Rewrite (2.110) as

$$\begin{aligned} c_0^{(n,\alpha)}\|v^n\|^2 &\leqslant \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})\|v^{n-k}\|^2 + c_{n-1}^{(n,\alpha)}\|v^0\|^2 + \frac{L^2}{12}s\|g^n\|^2 \\ &= \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})\|v^{n-k}\|^2 + c_{n-1}^{(n,\alpha)}\Big(\|v^0\|^2 + \frac{L^2}{12}\frac{s}{c_{n-1}^{(n,\alpha)}}\|g^n\|^2\Big) \\ &\leqslant \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})\|v^{n-k}\|^2 + c_{n-1}^{(n,\alpha)}\Big(\|v^0\|^2 + \frac{L^2}{12}t_n^\alpha\Gamma(1-\alpha)\|g^n\|^2\Big), \\ &\qquad\qquad 1 \leqslant n \leqslant N. \end{aligned}$$

By mathematical induction, we can obtain (2.106).

(II) Taking an inner product on both hand sides of (2.103) with $-\delta_x^2(\sigma v^n + (1-\sigma)v^{n-1})$, we have

$$\begin{aligned} &\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(v^{n-k} - v^{n-k-1}, -\delta_x^2(\sigma v^n + (1-\sigma)v^{n-1})) \\ &= -\|\delta_x^2(\sigma v^n + (1-\sigma)v^{n-1})\|^2 - (g^n, \delta_x^2(\sigma v^n + (1-\sigma)v^{n-1})) \\ &\leqslant -\|\delta_x^2(\sigma v^n + (1-\sigma)v^{n-1})\|^2 + \|\delta_x^2(\sigma v^n + (1-\sigma)v^{n-1})\|^2 + \frac{1}{4}\|g^n\|^2 \\ &= \frac{1}{4}\|g^n\|^2, \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{2.112}$$

For the left-hand side of the inequality above, applying Lemma 2.6.1, we have

$$\begin{aligned} &\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(v^{n-k} - v^{n-k-1}, -\delta_x^2(\sigma v^n + (1-\sigma)v^{n-1})) \\ &= \frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\delta_x(v^{n-k} - v^{n-k-1}), \delta_x(\sigma v^n + (1-\sigma)v^{n-1})) \\ &\geqslant \frac{1}{2}\cdot\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|\delta_x v^{n-k}\|^2 - \|\delta_x v^{n-k-1}\|^2). \end{aligned} \tag{2.113}$$

From (2.112) and (2.113), we have

$$\frac{1}{2}\cdot\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|\delta_x v^{n-k}\|^2 - \|\delta_x v^{n-k-1}\|^2) \leqslant \frac{1}{4}\|g^n\|^2, \quad 1 \leqslant n \leqslant N. \tag{2.114}$$

Noticing (2.111), rewrite (2.114) as

$$\begin{aligned}
c_0^{(n,\alpha)}\|\delta_x v^n\|^2 &\leqslant \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})\|\delta_x v^{n-k}\|^2 + c_{n-1}^{(n,\alpha)}\|\delta_x v^0\|^2 + \frac{1}{2}s\|g^n\|^2 \\
&= \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})\|\delta_x v^{n-k}\|^2 + c_{n-1}^{(n,\alpha)}\Big(\|\delta_x v^0\|^2 + \frac{1}{2}\frac{s}{c_{n-1}^{(n,\alpha)}}\|g^n\|^2\Big) \\
&\leqslant \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})\|\delta_x v^{n-k}\|^2 + c_{n-1}^{(n,\alpha)}\Big(\|\delta_x v^0\|^2 + \frac{1}{2}\Gamma(1-\alpha)t_n^\alpha\|g^n\|^2\Big), \\
&\qquad 1 \leqslant n \leqslant N.
\end{aligned}$$

By mathematical induction, we can obtain (2.107).

This completes the proof. □

2.6.5 Convergence of the difference scheme

Theorem 2.6.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (2.36)–(2.38) *and the difference scheme* (2.94)–(2.96), *respectively. Denote*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then we have

$$\|\delta_x e^n\| \leqslant \sqrt{\tfrac{1}{2}\Gamma(1-\alpha)t_n^\alpha L}\, c_6(\tau^2 + h^2), \quad 1 \leqslant n \leqslant N, \tag{2.115}$$

$$\|e^n\|_\infty \leqslant \tfrac{1}{4}\sqrt{2\Gamma(1-\alpha)t_n^\alpha}\, L c_6(\tau^2 + h^2), \quad 1 \leqslant n \leqslant N. \tag{2.116}$$

Proof. Subtracting (2.94)–(2.96) from (2.90), (2.92)–(2.93) respectively, we get the system of error equations

$$\begin{cases}
\dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\displaystyle\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(e_i^{n-k} - e_i^{n-k-1}) = \sigma\delta_x^2 e_i^n + (1-\sigma)\delta_x^2 e_i^{n-1} + (r_6)_i^n, \\
\qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\
e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\
e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N.
\end{cases}$$

Applying Theorem 2.6.2 and noticing (2.91), we have

$$\begin{aligned}
\|\delta_x e^n\|^2 &\leqslant \|\delta_x e^0\|^2 + \frac{1}{2}\Gamma(1-\alpha)\max_{1\leqslant m\leqslant n}\{t_m^\alpha\|(r_6)^m\|^2\} \\
&\leqslant \frac{1}{2}\Gamma(1-\alpha)t_n^\alpha L[c_6(\tau^2 + h^2)]^2, \quad 1 \leqslant n \leqslant N.
\end{aligned}$$

Taking the square root on both hand sides of the inequality above, we can get (2.115).

Noticing Lemma 2.1.1, from (2.115), we can obtain (2.116) easily. This completes the proof. □

2.7 The fast difference method based on L2-1_σ approximation for 1D problem

In this section, a fast difference method based on L2-1_σ approximation will be given for the problem (2.36)–(2.38). Suppose the exact solution $u \in C^{(4,3)}([0,L]\times[0,T])$.

2.7.1 Derivation of the difference scheme

Denote

$$\sigma = 1-\frac{\alpha}{2}, \quad t_{n-1+\sigma} = (n-1+\sigma)\tau, \quad f_i^{n-1+\sigma} = f(x_i, t_{n-1+\sigma}).$$

Considering (2.36) at the point $(x_i, t_{n-1+\sigma})$, we have

$$ {}_0^C D_t^\alpha u(x_i,t_{n-1+\sigma}) = u_{xx}(x_i,t_{n-1+\sigma}) + f_i^{n-1+\sigma}, \quad 1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N. \tag{2.117}$$

Using the theory in Subsection 1.7.2 to discretize the time fractional derivative, we get

$$\begin{cases} {}_0^C D_t^\alpha u(x_i,t_{n-1+\sigma}) = \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\sum_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + d_0^{(1,\alpha)}(U_i^n - U_i^{n-1}) \\ \qquad\qquad + O(\tau^{3-\alpha}+\epsilon), \quad 1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N, & (2.118)\\ F_{l,i}^1 = 0, \quad 1\leqslant l\leqslant N_{\exp},\ 1\leqslant i\leqslant M-1, & (2.119)\\ F_{l,i}^n = e^{-s_l\tau}F_{l,i}^{n-1} + A_l(U_i^{n-1}-U_i^{n-2}) + B_l(U_i^n - U_i^{n-1}), \\ \qquad\qquad 1\leqslant l\leqslant N_{\exp},\ 1\leqslant i\leqslant M-1,\ 2\leqslant n\leqslant N. & (2.120)\end{cases}$$

Using the linear interpolation and the second-order central difference quotient to approximate the spatial second-order derivative, we get

$$\begin{aligned} u_{xx}(x_i,t_{n-1+\sigma}) &= \sigma u_{xx}(x_i,t_n) + (1-\sigma)u_{xx}(x_i,t_{n-1}) + O(\tau^2) \\ &= \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + O(h^2) + O(\tau^2). \end{aligned} \tag{2.121}$$

Substituting (2.118)–(2.121) into (2.117), we get

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\sum_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + d_0^{(1,\alpha)}(U_i^n - U_i^{n-1}) = \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} \\ \qquad\qquad + f_i^{n-1+\sigma} + (r_7)_i^n, \quad 1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N, & (2.122)\\ F_{l,i}^1 = 0, \quad 1\leqslant l\leqslant N_{\exp},\ 1\leqslant i\leqslant M-1, & (2.123)\\ F_{l,i}^n = e^{-s_l\tau}F_{l,i}^{n-1} + A_l(U_i^{n-1}-U_i^{n-2}) + B_l(U_i^n - U_i^{n-1}), \\ \qquad\qquad 1\leqslant l\leqslant N_{\exp},\ 1\leqslant i\leqslant M-1,\ 2\leqslant n\leqslant N & (2.124)\end{cases}$$

and there exists a positive constant c_7 such that

$$\left|(r_7)_i^n\right| \leqslant c_7(\tau^2 + h^2 + \epsilon), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{2.125}$$

Substituting (2.123)–(2.124) into (2.122) and eliminating the intermediate variable $\{F_{l,i}^n\}$, we have

$$\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(U_i^{n-k} - U_i^{n-k-1}) = \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + f_i^{n-1+\sigma} + (r_7)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.$$

Noticing the initial-boundary value conditions (2.37)–(2.38), we have

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.126)\\ U_0^n = \mu(t_n), \quad U_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.127)\end{cases}$$

Omitting the small term $(r_7)_i^n$ in (2.122) and using numerical solution u_i^n to replace the exact solution U_i^n, we construct for the problem (2.36)–(2.38) the fast difference scheme as follows:

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\sum_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + d_0^{(1,\alpha)}(u_i^n - u_i^{n-1}) = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} \\ \qquad + f_i^{n-1+\sigma}, \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.128)\\ F_{l,i}^1 = 0, \quad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1, & (2.129)\\ F_{l,i}^n = e^{-s_l\tau} F_{l,i}^{n-1} + A_l(u_i^{n-1} - u_i^{n-2}) + B_l(u_i^n - u_i^{n-1}) \\ \qquad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N, & (2.130)\\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.131)\\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.132)\end{cases}$$

Substituting (2.129)–(2.130) into (2.128), we can get the following equivalent form

$$\begin{cases} \displaystyle\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(u_i^{n-k} - u_i^{n-k-1}) = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma}, \\ \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.133)\\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.134)\\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.135)\end{cases}$$

2.7.2 Solvability of the difference scheme

Theorem 2.7.1. *The difference scheme* (2.133)–(2.135) *is uniquely solvable.*

Proof. Denote

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

From (2.134)–(2.135), we can know u^0. Suppose $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, we can get the system of linear equations in u^n from (2.133) and (2.135). It suffices to show that the homogeneous system

$$\begin{cases} d_0^{(n,\alpha)} u_i^n = \sigma \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (2.136)\\ u_0^n = u_M^n = 0 & (2.137)\end{cases}$$

has only the trivial solution.

Taking an inner product on both hand sides of (2.136) with u^n, we have

$$d_0^{(n,\alpha)} \|u^n\|^2 = \sigma(u^n, \delta_x^2 u^n) = -\sigma \|\delta_x u^n\|^2,$$

from which it is easy to know that

$$u_i^n = 0, \quad 0 \leqslant i \leqslant M.$$

By induction principle, the conclusion holds. This completes the proof. □

2.7.3 Stability of the difference scheme

Theorem 2.7.2. *Suppose* $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\sum_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + d_0^{(1,\alpha)}(v_i^n - v_i^{n-1}) = \sigma \delta_x^2 v_i^n + (1-\sigma)\delta_x^2 v_i^{n-1} + g_i^n, \\ \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.138)\\ F_{l,i}^1 = 0, \quad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1, & (2.139)\\ F_{l,i}^n = e^{-s_l \tau} F_{l,i}^{n-1} + A_l(v_i^{n-1} - v_i^{n-2}) + B_l(v_i^n - v_i^{n-1}), \\ \qquad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N, & (2.140)\\ v_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.141)\\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N. & (2.142)\end{cases}$$

Then we have

$$\|v^n\|^2 \leqslant \|v^0\|^2 + \frac{L^2}{6}\Gamma(1-\alpha) \max_{1 \leqslant m \leqslant n} (t_m^\alpha \|g^m\|^2), \quad 1 \leqslant n \leqslant N, \tag{2.143}$$

$$\|\delta_x v^n\|^2 \leqslant \|\delta_x v^0\|^2 + \Gamma(1-\alpha) \max_{1 \leqslant m \leqslant n} (t_m^\alpha \|g^m\|^2), \quad 1 \leqslant n \leqslant N, \tag{2.144}$$

where

$$\|g^m\|^2 = h \sum_{i=1}^{M-1} |g_i^m|^2.$$

Proof. Substituting (2.139)–(2.140) into (2.138) and eliminating the intermediate variable $\{F_{l,i}^n\}$, we have

$$\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(v_i^{n-k} - v_i^{n-k-1}) = \sigma\delta_x^2 v_i^n + (1-\sigma)\delta_x^2 v_i^{n-1} + g_i^n,$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{2.145}$$

(I) Taking an inner product on both hand sides of (2.145) with $\sigma v^n + (1-\sigma)v^{n-1}$, we have

$$\begin{aligned}
&\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(v^{n-k} - v^{n-k-1}, \sigma v^n + (1-\sigma)v^{n-1}) \\
&= (\delta_x^2(\sigma v^n + (1-\sigma)v^{n-1}), \sigma v^n + (1-\sigma)v^{n-1}) + (g^n, \sigma v^n + (1-\sigma)v^{n-1}) \\
&= -\|\delta_x(\sigma v^n + (1-\sigma)v^{n-1})\|^2 + (g^n, \sigma v^n + (1-\sigma)v^{n-1}) \\
&\leqslant -\frac{6}{L^2}\|\sigma v^n + (1-\sigma)v^{n-1}\|^2 + \left[\frac{6}{L^2}\|\sigma v^n + (1-\sigma)v^{n-1}\|^2 + \frac{L^2}{24}\|g^n\|^2\right] \\
&= \frac{L^2}{24}\|g^n\|^2, \quad 1 \leqslant n \leqslant N.
\end{aligned} \tag{2.146}$$

For the left-hand side of the inequality above, applying Lemma 2.6.1 and Lemma 1.7.3, we can get

$$\begin{aligned}
&\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(v^{n-k} - v^{n-k-1}, \sigma v^n + (1-\sigma)v^{n-1}) \\
&\geqslant \frac{1}{2}\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(\|v^{n-k}\|^2 - \|v^{n-k-1}\|^2).
\end{aligned} \tag{2.147}$$

From (2.146) and (2.147), we get

$$\frac{1}{2}\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(\|v^{n-k}\|^2 - \|v^{n-k-1}\|^2) \leqslant \frac{L^2}{24}\|g^n\|^2, \quad 1 \leqslant n \leqslant N. \tag{2.148}$$

Noticing (1.146), rewrite (2.148) as

$$\begin{aligned}
d_0^{(n,\alpha)}\|v^n\|^2 &\leqslant \sum_{k=1}^{n-1}(d_{k-1}^{(n,\alpha)} - d_k^{(n,\alpha)})\|v^{n-k}\|^2 + d_{n-1}^{(n,\alpha)}\|v^0\|^2 + \frac{L^2}{12}\|g^n\|^2 \\
&\leqslant \sum_{k=1}^{n-1}(d_{k-1}^{(n,\alpha)} - d_k^{(n,\alpha)})\|v^{n-k}\|^2 + d_{n-1}^{(n,\alpha)}\left[\|v^0\|^2 + \frac{L^2}{6}\Gamma(1-\alpha)t_n^\alpha\|g^n\|^2\right], \\
&\qquad 1 \leqslant n \leqslant N.
\end{aligned}$$

By induction principle, we can get (2.143).

(II) Taking an inner product on both hand sides of (2.145) with $-\delta_x^2(\sigma v^n+(1-\sigma)v^{n-1})$, we have

$$\begin{aligned}&\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(v^{n-k}-v^{n-k-1}, -\delta_x^2(\sigma v^n+(1-\sigma)v^{n-1}))\\&= -h\sum_{i=1}^{M-1}[\delta_x^2(\sigma v_i^n+(1-\sigma)v_i^{n-1})]^2 - h\sum_{i=1}^{M-1}[\delta_x^2(\sigma v_i^n+(1-\sigma)v_i^{n-1})]g_i^n\\&\leqslant \frac{1}{4}\|g^n\|^2, \quad 1\leqslant n\leqslant N.\end{aligned} \tag{2.149}$$

For the left-hand side of the inequality above, applying Lemma 2.6.1 and Lemma 1.7.3, we can get

$$\begin{aligned}&\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(v^{n-k}-v^{n-k-1}, -\delta_x^2(\sigma v^n+(1-\sigma)v^{n-1}))\\&= \sum_{k=0}^{n-1} d_k^{(n,\alpha)}(\delta_x(v^{n-k}-v^{n-k-1}), \delta_x(\sigma v^n+(1-\sigma)v^{n-1}))\\&\geqslant \frac{1}{2}\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(\|\delta_x v^{n-k}\|^2-\|\delta_x v^{n-k-1}\|^2).\end{aligned} \tag{2.150}$$

Substituting (2.150) into (2.149) gives

$$\frac{1}{2}\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(\|\delta_x v^{n-k}\|^2-\|\delta_x v^{n-k-1}\|^2)\leqslant \frac{1}{4}\|g^n\|^2, \quad 1\leqslant n\leqslant N. \tag{2.151}$$

Noticing (1.146), rewrite (2.151) as

$$\begin{aligned}d_0^{(n,\alpha)}\|\delta_x v^n\|^2 &\leqslant \sum_{k=1}^{n-1}(d_{k-1}^{(n,\alpha)}-d_k^{(n,\alpha)})\|\delta_x v^{n-k}\|^2 + d_{n-1}^{(n,\alpha)}\|\delta_x v^0\|^2 + \frac{1}{2}\|g^n\|^2\\&\leqslant \sum_{k=1}^{n-1}(d_{k-1}^{(n,\alpha)}-d_k^{(n,\alpha)})\|\delta_x v^{n-k}\|^2 + d_{n-1}^{(n,\alpha)}(\|\delta_x v^0\|^2 + \Gamma(1-\alpha)t_n^\alpha\|g^n\|^2),\\&\qquad 1\leqslant n\leqslant N.\end{aligned}$$

By induction principle, we can get (2.144). This completes the proof. □

2.7.4 Convergence of the difference scheme

Theorem 2.7.3. *Suppose* $\{U_i^n \mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *and* $\{u_i^n \mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *are solutions of the problem* (2.36)–(2.38) *and the difference scheme* (2.128)–(2.132), *respectively. Denote*

$$e_i^n = U_i^n - u_i^n, \quad 0\leqslant i\leqslant M,\ 0\leqslant n\leqslant N,$$

then we have

$$\|\delta_x e^n\| \leqslant \sqrt{\Gamma(1-\alpha)t_n^\alpha L}\, c_7(\tau^2 + h^2 + \epsilon), \quad 1 \leqslant n \leqslant N, \tag{2.152}$$

$$\|e^n\|_\infty \leqslant \frac{L}{2}\sqrt{\Gamma(1-\alpha)t_n^\alpha} c_7(\tau^2 + h^2 + \epsilon), \quad 1 \leqslant n \leqslant N. \tag{2.153}$$

Proof. Subtracting (2.128)–(2.132) from (2.122)–(2.124), (2.126)–(2.127), respectively, we get the system of error equations

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\sum_{l=1}^{N_{\text{exp}}} \omega_l F_{l,i}^n + d_0^{(1,\alpha)}(e_i^n - e_i^{n-1}) = \sigma\delta_x^2 e_i^n + (1-\sigma)\delta_x^2 e_i^{n-1} + (r_7)_i^n, \\ \qquad\qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ F_{l,i}^1 = 0, \quad 1 \leqslant l \leqslant N_{\text{exp}},\ 1 \leqslant i \leqslant M-1, \\ F_{l,i}^n = e^{-s_l\tau} F_{l,i}^{n-1} + A_l(e_i^{n-1} - e_i^{n-2}) + B_l(e_i^n - e_i^{n-1}), \\ \qquad\qquad\qquad 1 \leqslant l \leqslant N_{\text{exp}},\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Applying Theorem 2.7.2 and noticing (2.125), we have

$$\begin{aligned} \|\delta_x e^n\|^2 &\leqslant \|\delta_x e^0\|^2 + \Gamma(1-\alpha) \max_{1\leqslant m\leqslant n} \left(t_m^\alpha \|(r_7)^m\|^2\right) \\ &\leqslant \Gamma(1-\alpha)t_n^\alpha L\left[c_7(\tau^2 + h^2 + \epsilon)\right]^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Taking the square root on both hand sides of the inequality above, we can get (2.152).

From (2.152) and Lemma 2.1.1, it is easy to obtain (2.153). This completes the proof. □

2.8 The difference method based on L1 approximation for the MTTFSD equations

This section will be devoted to the investigation of difference methods for solving a class of multiterm time-fractional subdiffusion (MTTFSD) equations. To simplify the statement, take a two-term case with the constant coefficients as an example.

Consider the following two-term problem:

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) + {}_0^C D_t^{\alpha_1} u(x,t) = u_{xx}(x,t) + f(x,t), \\ \qquad\qquad x \in (0,L),\ t \in (0,T], & (2.154) \\ u(x,0) = \varphi(x), \quad x \in (0,L), & (2.155) \\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), \quad t \in [0,T], & (2.156) \end{cases}$$

where $0 < \alpha_1 < \alpha < 1$, the functions f, φ, μ, ν are all given and $\varphi(0) = \mu(0)$, $\varphi(L) = \nu(0)$.

Take the same mesh partition and notations as those in Section 2.1. In addition, suppose $u \in C^{(4,2)}([0,L] \times [0,T])$.

2.8.1 Derivation of the difference scheme

Considering equation (2.154) at the point (x_i, t_n), one has

$$ {}_0^C D_t^{\alpha} u(x_i,t_n) + {}_0^C D_t^{\alpha_1} u(x_i,t_n) = u_{xx}(x_i,t_n) + f_i^n, \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. $$

Applying the L1 formula (1.60) to approximate the time-fractional derivatives and the second-order central difference quotient to approximate the spatial derivative in the equation above yields

$$ \begin{aligned} &\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(a_0^{(\alpha)} U_i^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) U_i^k - a_{n-1}^{(\alpha)} U_i^0 \right) \\ &+ \frac{\tau^{-\alpha_1}}{\Gamma(2-\alpha_1)}\left(a_0^{(\alpha_1)} U_i^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\alpha_1)} - a_{n-k}^{(\alpha_1)}) U_i^k - a_{n-1}^{(\alpha_1)} U_i^0 \right) \\ &= \delta_x^2 U_i^n + f_i^n + (r_8)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \end{aligned} \tag{2.157} $$

where there is a positive constant c_8 such that

$$ |(r_8)_i^n| \leqslant c_8(\tau^{2-\alpha} + h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{2.158} $$

Noticing the initial-boundary value conditions (2.155)–(2.156), one has

$$ \begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.159) \\ U_0^n = \mu(t_n), \quad U_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.160) \end{cases} $$

Neglecting the small term $(r_8)_i^n$ in (2.157) and replacing the exact solution U_i^n with its numerical one u_i^n produce a difference scheme for solving (2.154)–(2.156) as follows:

$$ \begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(a_0^{(\alpha)} u_i^n - \displaystyle\sum_{k=1}^{n-1} (a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) u_i^k - a_{n-1}^{(\alpha)} u_i^0 \right) \\ \quad + \dfrac{\tau^{-\alpha_1}}{\Gamma(2-\alpha_1)}\left(a_0^{(\alpha_1)} u_i^n - \displaystyle\sum_{k=1}^{n-1} (a_{n-k-1}^{(\alpha_1)} - a_{n-k}^{(\alpha_1)}) u_i^k - a_{n-1}^{(\alpha_1)} u_i^0 \right) \\ = \delta_x^2 u_i^n + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.161) \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.162) \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.163) \end{cases} $$

Denote

$$ s = \tau^{\alpha}\Gamma(2-\alpha), \quad s_1 = \tau^{\alpha_1}\Gamma(2-\alpha_1). $$

In what follows, the difference scheme (2.161)–(2.163) will be analyzed.

2.8.2 Solvability of the difference scheme

Theorem 2.8.1. *The difference scheme* (2.161)–(2.163) *is uniquely solvable.*

Proof. Denote

$$u^n = (u_0^n, u_1^n, \dots, u_M^n).$$

The value of u^0 is obviously determined by (2.162)–(2.163). Suppose that the values of $u^0, u^1, \dots, u^{n-1}$ have been uniquely determined. From (2.161) and (2.163), we can obtain the linear system in the unknown u^n. To prove its unique solvability, it suffices to show the corresponding homogeneous one

$$\begin{cases} \left(\dfrac{a_0^{(\alpha)}}{s} + \dfrac{a_0^{(\alpha_1)}}{s_1}\right) u_i^n = \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (2.164) \\ u_0^n = u_M^n = 0 & (2.165) \end{cases}$$

has only the trivial solution.

Suppose $\|u^n\|_\infty = |u_{i_n}^n|$, where $i_n \in \{1, 2, \dots, M-1\}$. Rewrite (2.164) as

$$\left(\frac{a_0^{(\alpha)}}{s} + \frac{a_0^{(\alpha_1)}}{s_1} + \frac{2}{h^2}\right) u_i^n = \frac{1}{h^2}(u_{i-1}^n + u_{i+1}^n), \quad 1 \leqslant i \leqslant M-1.$$

Letting $i = i_n$ in the equality above and taking the absolute value of both hand sides, by the triangle inequality, we have

$$\left(\frac{a_0^{(\alpha)}}{s} + \frac{a_0^{(\alpha_1)}}{s_1} + \frac{2}{h^2}\right) \|u^n\|_\infty \leqslant \frac{2}{h^2}\|u^n\|_\infty.$$

Thus $\|u^n\|_\infty = 0$, which implies $u^n = 0$.

By the principle of induction, the difference scheme (2.161)–(2.163) is uniquely solvable. The proof ends. □

2.8.3 Stability of the difference scheme

Theorem 2.8.2. *Suppose* $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \dfrac{1}{s}\left(a_0^{(\alpha)} v_i^n - \displaystyle\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) v_i^k - a_{n-1}^{(\alpha)} v_i^0\right) \\ \quad + \dfrac{1}{s_1}\left(a_0^{(\alpha_1)} v_i^n - \displaystyle\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_1)} - a_{n-k}^{(\alpha_1)}) v_i^k - a_{n-1}^{(\alpha_1)} v_i^0\right) \\ = \delta_x^2 v_i^n + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.166) \\ v_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.167) \\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N. & (2.168) \end{cases}$$

Then it holds

$$\|v^k\|_\infty \leqslant \|v^0\|_\infty + \kappa_1 \max_{1\leqslant m\leqslant k} \|f^m\|_\infty, \quad 1 \leqslant k \leqslant N, \tag{2.169}$$

where

$$\kappa_1 = \frac{1}{2}\max\{T^\alpha \Gamma(1-\alpha), T^{\alpha_1}\Gamma(1-\alpha_1)\}, \quad \|f^m\|_\infty = \max_{1\leqslant i\leqslant M-1} |f_i^m|.$$

Proof. Rewrite equation (2.166) as

$$\begin{aligned}
&\left(\frac{a_0^{(\alpha)}}{s} + \frac{a_0^{(\alpha_1)}}{s_1} + \frac{2}{h^2}\right) v_i^n \\
&= \frac{1}{s}\left(\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) v_i^k + a_{n-1}^{(\alpha)} v_i^0\right) \\
&\quad + \frac{1}{s_1}\left(\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_1)} - a_{n-k}^{(\alpha_1)}) v_i^k + a_{n-1}^{(\alpha_1)} v_i^0\right) \\
&\quad + \frac{1}{h^2}(v_{i-1}^n + v_{i+1}^n) + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.
\end{aligned}$$

Suppose $\|v^n\|_\infty = |v_{i_n}^n|$, where $i_n \in \{1, 2, \ldots, M-1\}$. Letting $i = i_n$ in the equality above and taking the absolute value of both hand sides, by the triangle inequality, we have

$$\begin{aligned}
&\left(\frac{a_0^{(\alpha)}}{s} + \frac{a_0^{(\alpha_1)}}{s_1} + \frac{2}{h^2}\right)\|v^n\|_\infty \\
&\leqslant \frac{1}{s}\left[\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})\|v^k\|_\infty + a_{n-1}^{(\alpha)}\|v^0\|_\infty\right] \\
&\quad + \frac{1}{s_1}\left[\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_1)} - a_{n-k}^{(\alpha_1)})\|v^k\|_\infty + a_{n-1}^{(\alpha_1)}\|v^0\|_\infty\right] + \frac{2}{h^2}\|v^n\|_\infty + \|f^n\|_\infty \\
&= \sum_{k=1}^{n-1}\left[\frac{1}{s}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) + \frac{1}{s_1}(a_{n-k-1}^{(\alpha_1)} - a_{n-k}^{(\alpha_1)})\right]\|v^k\|_\infty \\
&\quad + \left[\frac{1}{s}a_{n-1}^{(\alpha)} + \frac{1}{s_1}a_{n-1}^{(\alpha_1)}\right]\|v^0\|_\infty + \frac{2}{h^2}\|v^n\|_\infty + \frac{1}{s}a_{n-1}^{(\alpha)}\cdot\frac{s}{2a_{n-1}^{(\alpha)}}\|f^n\|_\infty \\
&\quad + \frac{1}{s_1}a_{n-1}^{(\alpha_1)}\cdot\frac{s_1}{2a_{n-1}^{(\alpha_1)}}\|f^n\|_\infty, \quad 1 \leqslant n \leqslant N.
\end{aligned} \tag{2.170}$$

By virtue of Lemma 1.6.1, we conclude that

$$\frac{s}{2a_{n-1}^{(\alpha)}} \leqslant \frac{\tau^\alpha \Gamma(2-\alpha)}{2(1-\alpha)n^{-\alpha}} = \frac{t_n^\alpha \Gamma(1-\alpha)}{2}, \tag{2.171}$$

$$\frac{s_1}{2a_{n-1}^{(\alpha_1)}} \leqslant \frac{\tau^{\alpha_1}\Gamma(2-\alpha_1)}{2(1-\alpha_1)n^{-\alpha_1}} = \frac{t_n^{\alpha_1}\Gamma(1-\alpha_1)}{2}. \tag{2.172}$$

Inserting (2.171)–(2.172) into (2.170) yields

$$\begin{aligned}&\left(\frac{a_0^{(\alpha)}}{s}+\frac{a_0^{(\alpha_1)}}{s_1}\right)\|v^n\|_\infty\\&\leqslant\sum_{k=1}^{n-1}\left[\frac{1}{s}(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)})+\frac{1}{s_1}(a_{n-k-1}^{(\alpha_1)}-a_{n-k}^{(\alpha_1)})\right]\|v^k\|_\infty\\&\quad+\left(\frac{1}{s}a_{n-1}^{(\alpha)}+\frac{1}{s_1}a_{n-1}^{(\alpha_1)}\right)(\|v^0\|_\infty+\kappa_1\|f^n\|_\infty),\quad 1\leqslant n\leqslant N.\end{aligned}\tag{2.173}$$

Next, we proceed by the mathematical induction to prove the truth of (2.169).

In view of (2.173), when $n=1$,

$$\left(\frac{a_0^{(\alpha)}}{s}+\frac{a_0^{(\alpha_1)}}{s_1}\right)\|v^1\|_\infty\leqslant\left(\frac{a_0^{(\alpha)}}{s}+\frac{a_0^{(\alpha_1)}}{s_1}\right)(\|v^0\|_\infty+\kappa_1\|f^1\|_\infty),$$

thus

$$\|v^1\|_\infty\leqslant\|v^0\|_\infty+\kappa_1\|f^1\|_\infty.$$

Obviously, (2.169) is true for $k=1$.

Suppose that (2.169) is true for $k=1,2,\ldots,n-1$. From (2.173), we can obtain

$$\begin{aligned}&\left(\frac{a_0^{(\alpha)}}{s}+\frac{a_0^{(\alpha_1)}}{s_1}\right)\|v^n\|_\infty\\&\leqslant\sum_{k=1}^{n-1}\left[\frac{1}{s}(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)})+\frac{1}{s_1}(a_{n-k-1}^{(\alpha_1)}-a_{n-k}^{(\alpha_1)})\right]\Big(\|v^0\|_\infty\\&\quad+\kappa_1\max_{1\leqslant m\leqslant k}\|f^m\|_\infty\Big)+\left[\frac{1}{s}a_{n-1}^{(\alpha)}+\frac{1}{s_1}a_{n-1}^{(\alpha_1)}\right](\|v^0\|_\infty+\kappa_1\|f^n\|_\infty)\\&\leqslant\left\{\sum_{k=1}^{n-1}\left[\frac{1}{s}(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)})+\frac{1}{s_1}(a_{n-k-1}^{(\alpha_1)}-a_{n-k}^{(\alpha_1)})\right]\right.\\&\quad\left.+\left(\frac{1}{s}a_{n-1}^{(\alpha)}+\frac{1}{s_1}a_{n-1}^{(\alpha_1)}\right)\right\}\cdot\Big(\|v^0\|_\infty+\kappa_1\max_{1\leqslant m\leqslant n}\|f^m\|_\infty\Big)\\&=\left(\frac{a_0^{(\alpha)}}{s}+\frac{a_0^{(\alpha_1)}}{s_1}\right)\Big(\|v^0\|_\infty+\kappa_1\max_{1\leqslant m\leqslant n}\|f^m\|_\infty\Big).\end{aligned}$$

Hence (2.169) is also true for $k=n$.

The desired result can be obtained by the principle of induction. The proof is completed. □

2.8.4 Convergence of the difference scheme

Theorem 2.8.3. *Suppose that* $\{U_i^n \mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *and* $\{u_i^n \mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *are solutions of the problem* (2.154)–(2.156) *and the difference scheme* (2.161)–(2.163), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N.$$

Then it holds

$$\|e^n\|_\infty \leqslant c_8 \kappa_1 (\tau^{2-\alpha} + h^2), \quad 1 \leqslant n \leqslant N.$$

Proof. The subtraction of (2.161)–(2.163) from (2.157), (2.159)–(2.160), respectively, gives the system of error equations as follows:

$$\begin{cases} \dfrac{1}{s}\left[a_0^{(\alpha)} e_i^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})e_i^k - a_{n-1}^{(\alpha)} e_i^0\right] \\ \quad + \dfrac{1}{s_1}\left[a_0^{(\alpha_1)} e_i^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_1)} - a_{n-k}^{(\alpha_1)})e_i^k - a_{n-1}^{(\alpha_1)} e_i^0\right] \\ = \delta_x^2 e_i^n + (r_8)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (2.158), Theorem 2.8.2 immediately implies

$$\|e^n\|_\infty \leqslant \|e^0\|_\infty + \kappa_1 \max_{1\leqslant m\leqslant n} \|(r_8)^m\|_\infty \leqslant c_8 \kappa_1 (\tau^{2-\alpha} + h^2), \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

2.9 The difference method based on L2-1_σ approximation for the MTTFSD equations

Consider the following problem of the multiterm time-fractional subdiffusion (MTTFSD) equations:

$$\begin{cases} \sum\limits_{r=0}^{m} \lambda_r\, {}_0^C D_t^{\alpha_r} u(x,t) = u_{xx}(x,t) + f(x,t),\ x \in (0,L),\ t \in (0,T], & (2.174) \\ u(x,0) = \varphi(x), \quad x \in (0,L), & (2.175) \\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), \quad t \in [0,T], & (2.176) \end{cases}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_m$ are positive constants, $0 \leqslant \alpha_m < \alpha_{m-1} < \cdots < \alpha_0 \leqslant 1$, at least one $\alpha_r \in (0,1)$, functions f, φ, μ, ν are all given, and $\varphi(0) = \mu(0)$, $\varphi(L) = \nu(0)$. Suppose the exact solution $u \in C^{(4,3)}([0,L] \times [0,T])$.

2.9.1 Derivation of the difference scheme

Applying the theory in Subsection 1.6.4, let σ be the unique root of equation $F(\sigma) = 0$, $t_{n-1+\sigma} = (n-1+\sigma)\tau$, $f_i^{n-1+\sigma} = f(x_i, t_{n-1+\sigma})$.

Considering (2.174) at the point $(x_i, t_{n-1+\sigma})$, we have

$$\begin{aligned}&\sum_{r=0}^{m} \lambda_r\, {}_0^C D_t^{\alpha_r} u(x_i, t_{n-1+\sigma})\\ &= u_{xx}(x_i, t_{n-1+\sigma}) + f_i^{n-1+\sigma}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.\end{aligned} \tag{2.177}$$

Using the theory in Subsection 1.6.4 to discretize the time fractional derivative, we have

$$\sum_{r=0}^{m} \lambda_r {}_0^C D_t^{\alpha_r} u(x_i, t_{n-1+\sigma}) = \sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} (U_i^{n-k} - U_i^{n-k-1}) + O(\tau^{3-\alpha_0}). \tag{2.178}$$

Using the linear interpolation and the second-order central difference quotient to approximate the spatial second-order derivative, we have

$$\begin{aligned}u_{xx}(x_i, t_{n-1+\sigma}) &= \sigma u_{xx}(x_i, t_n) + (1-\sigma) u_{xx}(x_i, t_{n-1}) + O(\tau^2)\\ &= \sigma \delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + O(h^2) + O(\tau^2).\end{aligned} \tag{2.179}$$

Inserting (2.178) and (2.179) into (2.177), we have

$$\begin{aligned}&\sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} (U_i^{n-k} - U_i^{n-k-1})\\ &= \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + f_i^{n-1+\sigma} + (r_9)_i^n,\\ &\qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N\end{aligned} \tag{2.180}$$

and there exists a positive constant c_9 such that

$$\left|(r_9)_i^n\right| \leqslant c_9(\tau^2 + h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{2.181}$$

Noticing the initial-boundary value conditions (2.175)–(2.176), we have

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.182)\\ U_0^n = \mu(t_n), \quad U_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.183)\end{cases}$$

Omitting the small term $(r_9)_i^n$ in (2.180) and using numerical solution u_i^n to replace the exact solution U_i^n, we construct for the problem (2.174)–(2.176) the following difference scheme:

$$\begin{cases} \displaystyle\sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} (u_i^{n-k} - u_i^{n-k-1}) = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma}, & \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.184)\\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.185)\\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (2.186)\end{cases}$$

2.9.2 Solvability of the difference scheme

Theorem 2.9.1. *The difference scheme* (2.184)–(2.186) *is uniquely solvable.*

Proof. Denote

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

From (2.185)–(2.186) we get u^0. Suppose $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, we can get the system of linear equations in u^n from (2.184) and (2.186). It suffices to show that the homogeneous system

$$\begin{cases} \hat{c}_0^{(n,\alpha)} u_i^n = \sigma \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (2.187) \\ u_0^n = u_M^n = 0 & (2.188) \end{cases}$$

has only the trivial solution.

Taking an inner product on both hand sides of (2.187) with u^n, we have

$$\hat{c}_0^{(n,\alpha)} \|u^n\|^2 = \sigma(u^n, \delta_x^2 u^n) = -\sigma \|\delta_x u^n\|^2.$$

It is easy to know that

$$u_i^n = 0, \quad 0 \leqslant i \leqslant M.$$

By induction principle, the theorem holds. This completes the proof. □

2.9.3 Stability of the difference scheme

Theorem 2.9.2. *Suppose* $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} (v_i^{n-k} - v_i^{n-k-1}) = \sigma \delta_x^2 v_i^n + (1-\sigma)\delta_x^2 v_i^{n-1} + g_i^n, \\ \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (2.189) \\ v_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (2.190) \\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N. & (2.191) \end{cases}$$

Then we have

$$\|\delta_x v^n\|^2 \leqslant \|\delta_x v^0\|^2 + \frac{1}{2\sum_{r=0}^m \frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}} \max_{1 \leqslant l \leqslant n} \|g^l\|^2, \quad 1 \leqslant n \leqslant N, \tag{2.192}$$

where

$$\|g^l\|^2 = h \sum_{i=1}^{M-1} |g_i^l|^2.$$

Proof. Taking an inner product on both hand sides of (2.189) with $-\delta_x^2(\sigma v^n+(1-\sigma)v^{n-1})$, we have

$$\begin{aligned}&\sum_{k=0}^{n-1}\hat{c}_k^{(n,\alpha)}(v^{n-k}-v^{n-k-1},-\delta_x^2(\sigma v^n+(1-\sigma)v^{n-1}))\\&=-\|\delta_x^2(\sigma v^n+(1-\sigma)v^{n-1})\|^2-(\delta_x^2(\sigma v^n+(1-\sigma)v^{n-1}),g^n)\\&\leqslant\frac{1}{4}\|g^n\|^2,\quad 1\leqslant n\leqslant N.\end{aligned}\tag{2.193}$$

For the left-hand side of the inequality above, applying Lemmas 2.6.1, 1.6.6 and 1.6.7, we have

$$\begin{aligned}&\sum_{k=0}^{n-1}\hat{c}_k^{(n,\alpha)}(v^{n-k}-v^{n-k-1},-\delta_x^2(\sigma v^n+(1-\sigma)v^{n-1}))\\&=\sum_{k=0}^{n-1}\hat{c}_k^{(n,\alpha)}(\delta_x(v^{n-k}-v^{n-k-1}),\delta_x(\sigma v^n+(1-\sigma)v^{n-1}))\\&\geqslant\frac{1}{2}\cdot\sum_{k=0}^{n-1}\hat{c}_k^{(n,\alpha)}(\|\delta_x v^{n-k}\|^2-\|\delta_x v^{n-k-1}\|^2).\end{aligned}\tag{2.194}$$

From (2.194) and (2.193), we have

$$\frac{1}{2}\cdot\sum_{k=0}^{n-1}\hat{c}_k^{(n,\alpha)}(\|\delta_x v^{n-k}\|^2-\|\delta_x v^{n-k-1}\|^2)\leqslant\frac{1}{4}\|g^n\|^2,\quad 1\leqslant n\leqslant N,$$

that is,

$$\begin{aligned}&\hat{c}_0^{(n,\alpha)}\|\delta_x v^n\|^2\\&\leqslant\sum_{k=1}^{n-1}(\hat{c}_{k-1}^{(n,\alpha)}-\hat{c}_k^{(n,\alpha)})\|\delta_x v^{n-k}\|^2+\hat{c}_{n-1}^{(n,\alpha)}\|\delta_x v^0\|^2+\frac{1}{2}\|g^n\|^2,\quad 1\leqslant n\leqslant N.\end{aligned}$$

From Lemma 1.6.6, we know

$$\hat{c}_{n-1}^{(n,\alpha)}>\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(1-\alpha_r)}n^{-\alpha_r}\geqslant\sum_{r=0}^{m}\frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}.$$

It follows that

$$\begin{aligned}&\hat{c}_0^{(n,\alpha)}\|\delta_x v^n\|^2\\&\leqslant\sum_{k=1}^{n-1}(\hat{c}_{k-1}^{(n,\alpha)}-\hat{c}_k^{(n,\alpha)})\|\delta_x v^{n-k}\|^2+\hat{c}_{n-1}^{(n,\alpha)}\Big(\|\delta_x v^0\|^2+\frac{1}{2\hat{c}_{n-1}^{(n,\alpha)}}\|g^n\|^2\Big)\\&\leqslant\sum_{k=1}^{n-1}(\hat{c}_{k-1}^{(n,\alpha)}-\hat{c}_k^{(n,\alpha)})\|\delta_x v^{n-k}\|^2\end{aligned}$$

$$+\hat{c}_{n-1}^{(n,\alpha)}\left[\|\delta_x v^0\|^2+\frac{1}{2\sum_{r=0}^{m}\frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}}\|g^n\|^2\right],\quad 1\leqslant n\leqslant N.$$

By mathematical induction, we can get (2.192). This completes the proof. □

From Theorem 2.9.2, we can know that the difference scheme (2.184)–(2.186) is stable with respect to the initial value and right-hand function.

2.9.4 Convergence of the difference scheme

Theorem 2.9.3. *Suppose that* $\{U_i^n \mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *and* $\{u_i^n \mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *are solutions of the problem* (2.174)–(2.176) *and the difference scheme* (2.184)–(2.186), *respectively. Denote*

$$e_i^n=U_i^n-u_i^n,\quad 0\leqslant i\leqslant M,\ 0\leqslant n\leqslant N,$$

then we have

$$\|\delta_x e^n\|\leqslant\sqrt{\frac{L}{2\sum_{r=0}^{m}\frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}}}\,c_9(\tau^2+h^2),\quad 1\leqslant n\leqslant N,\tag{2.195}$$

$$\|e^n\|_\infty\leqslant\frac{1}{2}\sqrt{\frac{1}{2\sum_{r=0}^{m}\frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}}}\,Lc_9(\tau^2+h^2),\quad 1\leqslant n\leqslant N.\tag{2.196}$$

Proof. Subtracting (2.184)–(2.186) from (2.180), (2.182)–(2.183), we have the system of error equations

$$\begin{cases}\sum_{k=0}^{n-1}\hat{c}_k^{(n,\alpha)}(e_i^{n-k}-e_i^{n-k-1})=\sigma\delta_x^2e_i^n+(1-\sigma)\delta_x^2e_i^{n-1}+(r_9)_i^n,\\ \qquad\qquad 1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N,\\ e_i^0=0,\quad 1\leqslant i\leqslant M-1,\\ e_0^n=0,\quad e_M^n=0,\quad 0\leqslant n\leqslant N.\end{cases}$$

Applying Theorem 2.9.2 and noticing (2.181), we have

$$\begin{aligned}\|\delta_x e^n\|^2&\leqslant\|\delta_x e^0\|^2+\frac{1}{2\sum_{r=0}^{m}\frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}}\max_{1\leqslant m\leqslant n}\|(r_9)^m\|^2\\&\leqslant\frac{1}{2\sum_{r=0}^{m}\frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}}L[c_9(\tau^2+h^2)]^2,\quad 1\leqslant n\leqslant N.\end{aligned}$$

Taking the square root on both hand sides of the inequality above, we get (2.195). From (2.195) and Lemma 2.1.1, it is easy to obtain (2.196). This completes the proof. □

2.10 The ADI method based on G-L approximation for 2D problem

In this section, the finite difference method for solving two-dimensional time-fractional subdiffusion equations will be investigated. For this, consider the following problem:

$$\begin{cases} {}_0^C D_t^\alpha u(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) + f(x,y,t), \\ \qquad\qquad (x,y) \in \Omega,\ t \in (0,T], & (2.197) \\ u(x,y,0) = 0, \quad (x,y) \in \Omega, & (2.198) \\ u(x,y,t) = \mu(x,y,t), \quad (x,y) \in \partial\Omega,\ t \in [0,T], & (2.199) \end{cases}$$

where $\Omega = (0,L_1)\times(0,L_2)$, $\alpha \in (0,1)$, the functions f, μ are given and $\mu(x,y,0)|_{(x,y)\in\partial\Omega} = 0$.

For the numerical approximation, the mesh partition is made firstly. Assume that $h_1 = L_1/M_1$, $h_2 = L_2/M_2$ and $\tau = T/N$ for some positive integers M_1, M_2 and N. Let $x_i = ih_1$ $(0 \leqslant i \leqslant M_1)$, $y_j = jh_2$ $(0 \leqslant j \leqslant M_2)$ and $t_k = k\tau$ $(0 \leqslant k \leqslant N)$. Define $\bar{\Omega}_h = \{(x_i, y_j) \mid 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2\}$ as a set of mesh points on $\bar{\Omega}$, $\Omega_h = \bar{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega$ as sets of interior mesh points and boundary mesh points, respectively. Let $\omega = \{(i,j) \mid (x_i, y_j) \in \Omega_h\}$, $\partial\omega = \{(i,j) \mid (x_i, y_j) \in \partial\Omega_h\}$, $\bar{\omega} = \omega \cup \partial\omega$, $\Omega_\tau = \{t_k \mid 0 \leqslant k \leqslant N\}$.

Define two mesh function spaces as follows:

$$\mathcal{V}_h = \{u \mid u = \{u_{ij} \mid (i,j) \in \bar{\omega}\} \text{ is a mesh function defined on } \bar{\Omega}_h\},$$
$$\mathring{\mathcal{V}}_h = \{u \mid u \in \mathcal{V}_h; u_{ij} = 0 \text{ if } (i,j) \in \partial\omega\}.$$

For any mesh function $v \in \mathcal{V}_h$, introduce some notations:

$$\delta_x v_{i-\frac{1}{2},j} = \frac{1}{h_1}(v_{ij} - v_{i-1,j}), \quad \delta_x^2 v_{ij} = \frac{1}{h_1}(\delta_x v_{i+\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j}),$$
$$\delta_y v_{i,j-\frac{1}{2}} = \frac{1}{h_2}(v_{ij} - v_{i,j-1}), \quad \delta_y^2 v_{ij} = \frac{1}{h_2}(\delta_y v_{i,j+\frac{1}{2}} - \delta_y v_{i,j-\frac{1}{2}}),$$
$$\delta_x \delta_y v_{i-\frac{1}{2},j-\frac{1}{2}} = \frac{1}{h_1}(\delta_y v_{i,j-\frac{1}{2}} - \delta_y v_{i-1,j-\frac{1}{2}}),$$
$$\delta_x^2 \delta_y^2 v_{ij} = \frac{1}{h_1^2}(\delta_y^2 v_{i-1,j} - 2\delta_y^2 v_{ij} + \delta_y^2 v_{i+1,j}),$$

$\delta_x \delta_y^2 v_{i-\frac{1}{2},j}$, $\delta_y \delta_x^2 v_{i,j-\frac{1}{2}}$ and others can be defined similarly.

For any mesh functions $u, v \in \mathring{\mathcal{V}}_h$, define

$$(u,v) = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} u_{ij} v_{ij}, \quad \|u\| = \sqrt{(u,u)},$$

$$(\delta_x u, \delta_x v) = h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2-1} (\delta_x u_{i-\frac{1}{2},j})(\delta_x v_{i-\frac{1}{2},j}), \quad \|\delta_x u\| = \sqrt{(\delta_x u, \delta_x u)},$$

$$(\delta_y u, \delta_y v) = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2} (\delta_y u_{i,j-\frac{1}{2}})(\delta_y v_{i,j-\frac{1}{2}}), \quad \|\delta_y u\| = \sqrt{(\delta_y u, \delta_y u)},$$

$$(\delta_x\delta_y u, \delta_x\delta_y v) = h_1h_2\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}(\delta_x\delta_y u_{i-\frac{1}{2},j-\frac{1}{2}})(\delta_x\delta_y v_{i-\frac{1}{2},j-\frac{1}{2}}),$$

$$\|\delta_x\delta_y u\| = \sqrt{(\delta_x\delta_y u, \delta_x\delta_y u)}, \quad \|\nabla_h u\| = \sqrt{\|\delta_x u\|^2 + \|\delta_y u\|^2},$$

$$\|u\|_\infty = \max_{1\leqslant i\leqslant M_1-1, 1\leqslant j\leqslant M_2-1}|u_{ij}|.$$

It is easy to check that for any mesh functions $u, v \in \mathring{\mathcal{V}}_h$, it holds

$$(-\delta_x^2 u, v) \equiv h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(-\delta_x^2 u_{ij})v_{ij} = (\delta_x u, \delta_x v), \tag{2.200}$$

$$(-\delta_y^2 u, v) \equiv h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(-\delta_y^2 u_{ij})v_{ij} = (\delta_y u, \delta_y v), \tag{2.201}$$

$$(\delta_x^2\delta_y^2 u, v) \equiv h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(\delta_x^2\delta_y^2 u_{ij})v_{ij} = (\delta_x\delta_y u, \delta_x\delta_y v). \tag{2.202}$$

In addition, we denote $\mathcal{I}$ as the unit operator, or, the identity operator.

The next lemma states a relationship between two different norms.

Lemma 2.10.1. [75] For any mesh function $u \in \mathring{\mathcal{V}}_h$, we have

$$\|u\|^2 \leqslant \frac{1}{\frac{6}{L_1^2} + \frac{6}{L_2^2}}\|\nabla_h u\|^2.$$

2.10.1 Derivation of the difference scheme

Define mesh functions

$$U_{ij}^n = u(x_i, y_j, t_n), \quad f_{ij}^n = f(x_i, y_j, t_n), \quad (i,j) \in \bar{\omega}, \quad 0 \leqslant n \leqslant N.$$

For any fixed $(x, y) \in \bar{\Omega}$, define a function

$$\hat{u}(x,y,t) = \begin{cases} 0, & t < 0, \\ u(x,y,t), & 0 \leqslant t \leqslant T, \\ v(x,y,t), & T < t < 2T, \\ 0, & t \geqslant 2T, \end{cases}$$

with $v(x,y,t)$ a smooth function satisfying $\frac{\partial^k v(x,y,t)}{\partial t^k}|_{t=T} = \frac{\partial^k u(x,y,t)}{\partial t^k}|_{t=T}$ and $\frac{\partial^k v(x,y,t)}{\partial t^k}|_{t=2T} = 0$, $k = 0, 1, 2$. Assume that $\hat{u}(x,y,\cdot) \in \mathscr{C}^{1+\alpha}(\mathcal{R})$ and $u(\cdot,\cdot,t) \in C^{(4,4)}(\bar{\Omega})$.

Considering equation (2.197) at the point (x_i, y_j, t_n), one has

$${}_0^C D_t^\alpha u(x_i, y_j, t_n) = u_{xx}(x_i, y_j, t_n) + u_{yy}(x_i, y_j, t_n) + f_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N.$$

Noticing the close relationship between the Caputo derivative and the R-L derivative under the zero initial value condition (2.198), by Theorem 1.4.2 and Lemma 2.1.3, one can obtain

$$\tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)} U_{ij}^{n-k} = \delta_x^2 U_{ij}^n + \delta_y^2 U_{ij}^n + f_{ij}^n + (r_{10})_{ij}^n,$$
$$(i,j) \in \omega,\ 1 \leqslant n \leqslant N, \tag{2.203}$$

where there is a positive constant c_{10} such that

$$|(r_{10})_{ij}^n| \leqslant c_{10}(\tau + h_1^2 + h_2^2), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N.$$

Adding a small perturbation term $\tau^{2\alpha}\delta_x^2\delta_y^2(\tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)} U_{ij}^{n-k})$ into (2.203) arrives at

$$\begin{aligned}&\tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)} U_{ij}^{n-k} + \tau^{2\alpha}\delta_x^2\delta_y^2\left(\tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)} U_{ij}^{n-k}\right)\\ &= \delta_x^2 U_{ij}^n + \delta_y^2 U_{ij}^n + f_{ij}^n + (r_{11})_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N,\end{aligned} \tag{2.204}$$

where

$$(r_{11})_{ij}^n = (r_{10})_{ij}^n + \tau^{2\alpha}\left(\tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)} \delta_x^2\delta_y^2 U_{ij}^{n-k}\right)$$

and there is a positive constant c_{11} such that

$$|(r_{11})_{ij}^n| \leqslant c_{11}(\tau^{\min\{1,2\alpha\}} + h_1^2 + h_2^2), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \tag{2.205}$$

Noticing the initial-boundary value conditions (2.198)–(2.199), one has

$$\begin{cases} U_{ij}^0 = 0, \quad (i,j) \in \omega, & (2.206)\\ U_{ij}^n = \mu(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (2.207)\end{cases}$$

Neglecting the small term $(r_{11})_{ij}^n$ in (2.204) and replacing the exact solution U_{ij}^n with its numerical one u_{ij}^n, we get a difference scheme for solving (2.197)–(2.199) in the form of

$$\begin{cases} \tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)} u_{ij}^{n-k} + \tau^{2\alpha}\delta_x^2\delta_y^2\left(\tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)} u_{ij}^{n-k}\right) & \\ = \delta_x^2 u_{ij}^n + \delta_y^2 u_{ij}^n + f_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (2.208)\\ u_{ij}^0 = 0, \quad (i,j) \in \omega, & (2.209)\\ u_{ij}^n = \mu(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (2.210)\end{cases}$$

Equation (2.208) can be rewritten as

$$u_{ij}^n - \tau^{\alpha}\delta_x^2 u_{ij}^n - \tau^{\alpha}\delta_y^2 u_{ij}^n + \tau^{2\alpha}\delta_x^2\delta_y^2 u_{ij}^n$$

$$= \sum_{k=1}^{n} (-g_k^{(\alpha)})(u_{ij}^{n-k} + \tau^{2\alpha}\delta_x^2\delta_y^2 u_{ij}^{n-k}) + \tau^\alpha f_{ij}^n,$$

namely,

$$\begin{aligned}&(\mathcal{I} - \tau^\alpha \delta_x^2)(\mathcal{I} - \tau^\alpha \delta_y^2)u_{ij}^n\\ = &\sum_{k=1}^{n} (-g_k^{(\alpha)})(\mathcal{I} + \tau^{2\alpha}\delta_x^2\delta_y^2)u_{ij}^{n-k} + \tau^\alpha f_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N.\end{aligned}$$

Let

$$u_{ij}^* = (\mathcal{I} - \tau^\alpha \delta_y^2)u_{ij}^n,$$

then the difference scheme (2.208)–(2.210) can be reformulated as the following ADI form:

On each time level $t = t_n$ $(1 \leqslant n \leqslant N)$, firstly, for any fixed j from 1 to $M_2 - 1$, solve a series of linear systems in the unknown $\{u_{ij}^* \mid 0 \leqslant i \leqslant M_1\}$ in x direction

$$\begin{cases} (\mathcal{I} - \tau^\alpha \delta_x^2)u_{ij}^* = \sum\limits_{k=1}^{n} (-g_k^{(\alpha)})(\mathcal{I} + \tau^{2\alpha}\delta_x^2\delta_y^2)u_{ij}^{n-k} + \tau^\alpha f_{ij}^n, & 1 \leqslant i \leqslant M_1 - 1,\\ u_{0j}^* = (\mathcal{I} - \tau^\alpha \delta_y^2)u_{0j}^n, \quad u_{M_1,j}^* = (\mathcal{I} - \tau^\alpha \delta_y^2)u_{M_1,j}^n \end{cases}$$

to obtain the value of

$$\{u_{ij}^* \mid 1 \leqslant i \leqslant M_1 - 1\}$$

on an intermediate time level.

Then, for any fixed i from 1 to $M_1 - 1$, carry out some calculations for the unknown $\{u_{ij}^n \mid 0 \leqslant j \leqslant M_2\}$ in y direction

$$\begin{cases} (\mathcal{I} - \tau^\alpha \delta_y^2)u_{ij}^n = u_{ij}^*, \quad 1 \leqslant j \leqslant M_2 - 1,\\ u_{i0}^n = \mu(x_i, y_0, t_n), \quad u_{i,M_2}^n = \mu(x_i, y_{M_2}, t_n) \end{cases}$$

to get the desired value of

$$\{u_{ij}^n \mid 1 \leqslant j \leqslant M_2 - 1\}.$$

2.10.2 Solvability of the difference scheme

Theorem 2.10.1. *The difference scheme* (2.208)–(2.210) *is uniquely solvable.*

Proof. Denote

$$u^n = \{u_{ij}^n \mid (i,j) \in \bar{\omega}\}.$$

We proceed by the mathematical induction. The values of u^0 is obviously determined by (2.209)–(2.210).

Now assume that the values of $u^0, u^1, \dots, u^{n-1}$ have been uniquely determined, then we can obtain the linear system in the unknown u^n from (2.208) and (2.210). To show its unique solvability, it is sufficient to verify that the corresponding homogeneous one

$$\begin{cases} \tau^{-\alpha} g_0^{(\alpha)} u_{ij}^n + \tau^\alpha g_0^{(\alpha)} \delta_x^2 \delta_y^2 u_{ij}^n = \delta_x^2 u_{ij}^n + \delta_y^2 u_{ij}^n, \quad (i,j) \in \omega, & (2.211) \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega & (2.212) \end{cases}$$

has only the trivial solution.

To this end, making the inner product on both hand sides of (2.211) with u^n, respectively, and noticing (2.212), it follows from (2.200)–(2.202) that

$$\tau^{-\alpha}(u^n, u^n) + \tau^\alpha(\delta_x\delta_y u^n, \delta_x\delta_y u^n) = -[(\delta_x u^n, \delta_x u^n) + (\delta_y u^n, \delta_y u^n)].$$

Thus

$$\tau^{-\alpha}\|u^n\|^2 + \tau^\alpha\|\delta_x\delta_y u^n\|^2 = -\|\nabla_h u^n\|^2 \leqslant 0,$$

which implies $\|u^n\| = 0$. Then noticing (2.212), $u^n = 0$ can be concluded.

By the principle of induction, the difference scheme (2.208)–(2.210) is uniquely solvable. The proof ends. □

In what follows, the stability and convergence of the difference scheme (2.208)–(2.210) will be discussed.

2.10.3 Stability of the difference scheme

Theorem 2.10.2. *Suppose* $\{v_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \tau^{-\alpha} \sum\limits_{k=0}^{n} g_k^{(\alpha)} v_{ij}^{n-k} + \tau^\alpha \sum\limits_{k=0}^{n} g_k^{(\alpha)} \delta_x^2 \delta_y^2 v_{ij}^{n-k} \\ = \delta_x^2 v_{ij}^n + \delta_y^2 v_{ij}^n + f_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (2.213) \\ v_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (2.214) \\ v_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (2.215) \end{cases}$$

Then it holds

$$\begin{aligned} &\|v^n\|^2 + \tau^{2\alpha}\|\delta_x\delta_y v^n\|^2 \\ \leqslant\ &\frac{5}{1-\alpha}(\|v^0\|^2 + \tau^{2\alpha}\|\delta_x\delta_y v^0\|^2) \end{aligned}$$

$$+\frac{L_1^2L_2^2}{12(L_1^2+L_2^2)}\cdot\frac{5}{(1-\alpha)2^\alpha}t_n^\alpha\max_{1\leqslant m\leqslant n}\|f^m\|^2,\quad 1\leqslant n\leqslant N,\tag{2.216}$$

where

$$\|f^m\|^2=h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(f_{ij}^m)^2.$$

Proof. Taking an inner product on both hand sides of (2.213) with v^n, respectively, and noticing (2.215), it follows from (2.200)–(2.202) that

$$\begin{aligned}&\tau^{-\alpha}\sum_{k=0}^n g_k^{(\alpha)}(v^{n-k},v^n)+\tau^\alpha\sum_{k=0}^n g_k^{(\alpha)}(\delta_x\delta_y v^{n-k},\delta_x\delta_y v^n)\\&=-(\delta_x v^n,\delta_x v^n)-(\delta_y v^n,\delta_y v^n)+(f^n,v^n)\\&=-\|\nabla_h v^n\|^2+(f^n,v^n),\quad 1\leqslant n\leqslant N.\end{aligned}\tag{2.217}$$

By means of the Cauchy–Schwarz inequality and Lemma 2.10.1, we get

$$\begin{aligned}(f^n,v^n)&\leqslant\|f^n\|\cdot\|v^n\|\leqslant 6\Big(\frac{1}{L_1^2}+\frac{1}{L_2^2}\Big)\|v^n\|^2+\frac{1}{24(\frac{1}{L_1^2}+\frac{1}{L_2^2})}\|f^n\|^2\\&\leqslant\|\nabla_h v^n\|^2+\frac{1}{24(\frac{1}{L_1^2}+\frac{1}{L_2^2})}\|f^n\|^2,\quad 1\leqslant n\leqslant N.\end{aligned}\tag{2.218}$$

The substitution of (2.218) into (2.217) arrives at

$$\begin{aligned}&\tau^{-\alpha}\sum_{k=0}^n g_k^{(\alpha)}[(v^{n-k},v^n)+\tau^{2\alpha}(\delta_x\delta_y v^{n-k},\delta_x\delta_y v^n)]\\&\leqslant\frac{L_1^2L_2^2}{24(L_1^2+L_2^2)}\|f^n\|^2,\quad 1\leqslant n\leqslant N.\end{aligned}$$

Rearranging the above result and again using the Cauchy–Schwarz inequality lead to

$$\begin{aligned}&\|v^n\|^2+\tau^{2\alpha}\|\delta_x\delta_y v^n\|^2\\&\leqslant\sum_{k=1}^n(-g_k^{(\alpha)})[(v^{n-k},v^n)+\tau^{2\alpha}(\delta_x\delta_y v^{n-k},\delta_x\delta_y v^n)]+\frac{L_1^2L_2^2}{24(L_1^2+L_2^2)}\tau^\alpha\|f^n\|^2\\&\leqslant\sum_{k=1}^n(-g_k^{(\alpha)})\Big[\frac12(\|v^{n-k}\|^2+\|v^n\|^2)+\frac12\tau^{2\alpha}(\|\delta_x\delta_y v^{n-k}\|^2+\|\delta_x\delta_y v^n\|^2)\Big]\\&\quad+\frac{L_1^2L_2^2}{24(L_1^2+L_2^2)}\tau^\alpha\|f^n\|^2,\quad 1\leqslant n\leqslant N.\end{aligned}$$

It follows by noticing $\sum_{k=1}^n(-g_k^{(\alpha)})\leqslant g_0^{(\alpha)}=1$ that

$$\begin{aligned}&\|v^n\|^2+\tau^{2\alpha}\|\delta_x\delta_y v^n\|^2\\&\leqslant\sum_{k=1}^n(-g_k^{(\alpha)})(\|v^{n-k}\|^2+\tau^{2\alpha}\|\delta_x\delta_y v^{n-k}\|^2)\end{aligned}$$

$$+\frac{L_1^2L_2^2}{12(L_1^2+L_2^2)}\tau^\alpha\|f^n\|^2, \quad 1\leqslant n\leqslant N. \tag{2.219}$$

Starting from (2.219), an induction method will yield the desired result (2.216). The process is quite similar to that in Theorem 2.1.2 and the details are omitted here. The proof is completed. □

2.10.4 Convergence of the difference scheme

Theorem 2.10.3. *Suppose* $\{U_{ij}^n \mid (i,j)\in\bar{\omega}, 0\leqslant n\leqslant N\}$ *and* $\{u_{ij}^n \mid (i,j)\in\bar{\omega}, 0\leqslant n\leqslant N\}$ *are solutions of the problem* (2.197)–(2.199) *and the difference scheme* (2.208)–(2.210), *respectively. Let*

$$e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j)\in\bar{\omega},\ 0\leqslant n\leqslant N.$$

Then it holds

$$\|e^n\|\leqslant \kappa_2(\tau^{\min\{1,2\alpha\}}+h_1^2+h_2^2), \quad 1\leqslant n\leqslant N,$$

with

$$\kappa_2=\frac{L_1L_2}{6}\sqrt{\frac{15}{1-\alpha}\left(\frac{T}{2}\right)^\alpha\frac{L_1L_2}{L_1^2+L_2^2}}\,c_{11}.$$

Proof. Subtracting (2.208)–(2.210) from (2.204), (2.206)–(2.207), respectively, the system of error equations is obtained as

$$\begin{cases}\tau^{-\alpha}\sum\limits_{k=0}^{n} g_k^{(\alpha)}e_{ij}^{n-k}+\tau^\alpha\sum\limits_{k=0}^{n} g_k^{(\alpha)}\delta_x^2\delta_y^2e_{ij}^{n-k}\\ =\delta_x^2e_{ij}^n+\delta_y^2e_{ij}^n+(r_{11})_{ij}^n, \quad (i,j)\in\omega,\ 1\leqslant n\leqslant N,\\ e_{ij}^0=0, \quad (i,j)\in\omega,\\ e_{ij}^n=0, \quad (i,j)\in\partial\omega,\ 0\leqslant n\leqslant N.\end{cases}$$

Noticing (2.205), Theorem 2.10.2 immediately implies

$$\begin{aligned}\|e^n\|^2&\leqslant\frac{L_1^2L_2^2}{12(L_1^2+L_2^2)}\cdot\frac{5}{(1-\alpha)2^\alpha}t_n^\alpha\max_{1\leqslant m\leqslant n}\|(r_{11})^m\|^2\\&\leqslant\frac{L_1^2L_2^2}{12(L_1^2+L_2^2)}\cdot\frac{5T^\alpha}{(1-\alpha)2^\alpha}L_1L_2\Big[c_{11}(\tau^{\min\{1,2\alpha\}}+h_1^2+h_2^2)\Big]^2, \quad 1\leqslant n\leqslant N.\end{aligned}$$

The theorem follows by taking the square root on both hand sides of the above inequality. The proof ends. □

2.11 The ADI method based on L1 approximation for 2D problem

The aim of this section is to provide an ADI method based on L1 approximation for the 2D time-fractional subdiffusion problem

$$\begin{cases} {}_0^C D_t^\alpha u(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) + f(x,y,t), \\ \qquad\qquad\qquad\qquad (x,y) \in \Omega,\ t \in (0,T], & (2.220) \\ u(x,y,0) = \varphi(x,y), \quad (x,y) \in \Omega, & (2.221) \\ u(x,y,t) = \mu(x,y,t), \quad (x,y) \in \partial\Omega,\ t \in [0,T], & (2.222) \end{cases}$$

where $\Omega = (0,L_1) \times (0,L_2)$, $\alpha \in (0,1)$, the functions f, φ, μ are all given and $\mu(x,y,0)|_{(x,y)\in\partial\Omega} = \varphi(x,y)$.

Take the same mesh partition and notations as those in Section 2.10. Suppose $u \in C^{(4,4,2)}(\bar{\Omega} \times [0,T])$. In addition, denote $s = \tau^\alpha \Gamma(2-\alpha)$.

2.11.1 Derivation of the difference scheme

Considering equation (2.220) at the point (x_i, y_j, t_n), one has

$${}_0^C D_t^\alpha u(x_i,y_j,t_n) = u_{xx}(x_i,y_j,t_n) + u_{yy}(x_i,y_j,t_n) + f_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N.$$

Using the L1 formula (1.160) to handle the Caputo derivative and the central difference approximation to discretize the spatial derivative, by Theorem 1.6.1 and Lemma 2.1.3, we get

$$\begin{aligned} &\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)} U_{ij}^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) U_{ij}^k - a_{n-1}^{(\alpha)} U_{ij}^0 \right] \\ &= \delta_x^2 U_{ij}^n + \delta_y^2 U_{ij}^n + f_{ij}^n + (r_{12})_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \end{aligned} \tag{2.223}$$

where there exists a positive constant c_{12} such that

$$|(r_{12})_{ij}^n| \leqslant c_{12}(\tau^{2-\alpha} + h_1^2 + h_2^2), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N.$$

Adding a small perturbation term

$$s^2 \delta_x^2 \delta_y^2 \left[\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left(a_0^{(\alpha)} U_{ij}^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) U_{ij}^k - a_{n-1}^{(\alpha)} U_{ij}^0 \right) \right],$$

into both hand sides of (2.223) gives

$$\begin{aligned} &\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} (\mathcal{I} + s^2 \delta_x^2 \delta_y^2) \left(a_0^{(\alpha)} U_{ij}^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}) U_{ij}^k - a_{n-1}^{(\alpha)} U_{ij}^0 \right) \\ &= \delta_x^2 U_{ij}^n + \delta_y^2 U_{ij}^n + f_{ij}^n + (r_{13})_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \end{aligned} \tag{2.224}$$

where

$$(r_{13})_{ij}^n = (r_{12})_{ij}^n + s^2\left[\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\delta_x^2\delta_y^2\left(a_0^{(\alpha)}U_{ij}^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})U_{ij}^k - a_{n-1}^{(\alpha)}U_{ij}^0\right)\right]$$

and there exists a positive constant c_{13} such that

$$\left|(r_{13})_{ij}^n\right| \leqslant c_{13}(\tau^{\min\{2\alpha,2-\alpha\}} + h_1^2 + h_2^2), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \tag{2.225}$$

Noticing the initial-boundary value conditions (2.221)–(2.222), one has

$$\begin{cases} U_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (2.226)\\ U_{ij}^n = \mu(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (2.227) \end{cases}$$

Omitting the small term $(r_{13})_{ij}^n$ in (2.224) and replacing the exact solution U_{ij}^n with its numerical one u_{ij}^n, we can obtain the following difference scheme for solving (2.220)–(2.222):

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(\mathcal{I} + s^2\delta_x^2\delta_y^2)\left(a_0^{(\alpha)}u_{ij}^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_{ij}^k - a_{n-1}^{(\alpha)}u_{ij}^0\right) & \\ = \delta_x^2 u_{ij}^n + \delta_y^2 u_{ij}^n + f_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (2.228)\\ u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (2.229)\\ u_{ij}^n = \mu(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (2.230) \end{cases}$$

Rewrite equation (2.228) as

$$\begin{aligned} &(\mathcal{I} + s^2\delta_x^2\delta_y^2)u_{ij}^n - s\delta_x^2 u_{ij}^n - s\delta_y^2 u_{ij}^n \\ &= (\mathcal{I} + s^2\delta_x^2\delta_y^2)\left(\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_{ij}^k + a_{n-1}^{(\alpha)}u_{ij}^0\right) + sf_{ij}^n, \end{aligned}$$

namely,

$$\begin{aligned} &(\mathcal{I} - s\delta_x^2)(\mathcal{I} - s\delta_y^2)u_{ij}^n \\ &= (\mathcal{I} + s^2\delta_x^2\delta_y^2)\left(\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_{ij}^k + a_{n-1}^{(\alpha)}u_{ij}^0\right) + sf_{ij}^n, \\ &\qquad\qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \end{aligned}$$

Let

$$u_{ij}^* = (\mathcal{I} - s\delta_y^2)u_{ij}^n,$$

then the difference scheme (2.228)–(2.230) can be reformulated as the following ADI form:

On each time level $t = t_n$ $(1 \leqslant n \leqslant N)$, firstly, for any fixed j from 1 to $M_2 - 1$, solve a series of linear systems in the unknown $\{u^*_{ij} \mid 0 \leqslant i \leqslant M_1\}$ in x direction

$$\begin{cases} (\mathcal{I} - s\delta_x^2)u^*_{ij} = (\mathcal{I} + s^2\delta_x^2\delta_y^2)\left(\sum\limits_{k=1}^{n-1}(a^{(\alpha)}_{n-k-1} - a^{(\alpha)}_{n-k})u^k_{ij} + a^{(\alpha)}_{n-1}u^0_{ij}\right) + sf^n_{ij}, \\ \qquad\qquad 1 \leqslant i \leqslant M_1 - 1, \\ u^*_{0j} = (\mathcal{I} - s\delta_y^2)u^n_{0j}, \quad u^*_{M_1,j} = (\mathcal{I} - s\delta_y^2)u^n_{M_1,j} \end{cases}$$

to obtain the value of

$$\{u^*_{ij} \mid 1 \leqslant i \leqslant M_1 - 1\}$$

on an intermediate time level.

Then, for any fixed i from 1 to $M_1 - 1$, carry out some calculations about the unknown $\{u^n_{ij} \mid 0 \leqslant j \leqslant M_2\}$ in y direction

$$\begin{cases} (\mathcal{I} - s\delta_y^2)u^n_{ij} = u^*_{ij}, \quad 1 \leqslant j \leqslant M_2 - 1, \\ u^n_{i0} = \mu(x_i, y_0, t_n), \quad u^n_{i,M_2} = \mu(x_i, y_{M_2}, t_n) \end{cases}$$

to get the desired value of

$$\{u^n_{ij} \mid 1 \leqslant j \leqslant M_2 - 1\}.$$

2.11.2 Solvability of the difference scheme

Theorem 2.11.1. *The difference scheme* (2.228)–(2.230) *is uniquely solvable.*

Proof. Denote

$$u^n = \{u^n_{ij} \mid (i,j) \in \bar{\omega}\}.$$

The value of u^0 is uniquely determined by (2.229)–(2.230).

Suppose the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the system in u^n can be obtained from (2.228) and (2.230). To show its unique solvability, it is sufficient to prove that the corresponding homogeneous one

$$\begin{cases} \frac{1}{s}(\mathcal{I} + s^2\delta_x^2\delta_y^2)u^n_{ij} = \delta_x^2 u^n_{ij} + \delta_y^2 u^n_{ij}, \quad (i,j) \in \omega, & (2.231) \\ u^n_{ij} = 0, \quad (i,j) \in \partial\omega & (2.232) \end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (2.231) with u^n and noticing (2.232), it follows from (2.200)–(2.202) that

$$\begin{aligned} \frac{1}{s}(u^n, u^n) + s(\delta_x\delta_y u^n, \delta_x\delta_y u^n) &= -(\delta_x u^n, \delta_x u^n) - (\delta_y u^n, \delta_y u^n) \\ &= -\|\nabla_h u^n\|^2 \leqslant 0, \end{aligned}$$

which implies $\|u^n\| = 0$. It follows $u^n = 0$.

By the principle of induction, the difference scheme (2.228)–(2.230) is uniquely solvable. The proof ends. □

2.11.3 Stability of the difference scheme

For any mesh functions $u, v \in \mathring{\mathcal{V}}_h$, define

$$(u, v)_s \equiv (u, v) + s^2(\delta_x\delta_y u, \delta_x\delta_y v).$$

Theorem 2.11.2. *Suppose $\{v_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ is the solution of difference scheme*

$$\begin{cases} \dfrac{1}{s}(\mathcal{I} + s^2\delta_x^2\delta_y^2)\left(a_0^{(\alpha)}v_{ij}^n - \displaystyle\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})v_{ij}^k - a_{n-1}^{(\alpha)}v_{ij}^0\right) \\ = \delta_x^2 v_{ij}^n + \delta_y^2 v_{ij}^n + f_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (2.233) \\ v_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (2.234) \\ v_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (2.235) \end{cases}$$

Then it holds

$$(v^n, v^n)_s \leqslant (v^0, v^0)_s + \frac{L_1^2L_2^2}{12(L_1^2 + L_2^2)}\Gamma(1-\alpha)\max_{1\leqslant m\leqslant n}\{t_m^\alpha\|f^m\|^2\}, \quad 1 \leqslant n \leqslant N, \qquad (2.236)$$

where

$$\|f^m\|^2 = h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(f_{ij}^m)^2.$$

Proof. Making the inner product on both hand sides of (2.233) with v^n, respectively, it produces

$$\begin{aligned} &\frac{1}{s}\left((\mathcal{I} + s^2\delta_x^2\delta_y^2)\left(a_0^{(\alpha)}v^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})v^k - a_{n-1}^{(\alpha)}v^0\right), v^n\right) \\ &= (\delta_x^2 v^n, v^n) + (\delta_y^2 v^n, v^n) + (f^n, v^n), \quad 1 \leqslant n \leqslant N. \end{aligned} \qquad (2.237)$$

Because of (2.200)–(2.202), we have by noticing (2.235) that

$$\begin{aligned} ((\mathcal{I} + s^2\delta_x^2\delta_y^2)v^k, v^n) &= (v^k, v^n) + s^2(\delta_x\delta_y v^k, \delta_x\delta_y v^n) \\ &= (v^k, v^n)_s, \quad 0 \leqslant k \leqslant n; \end{aligned} \qquad (2.238)$$

$$(\delta_x^2 v^n, v^n) + (\delta_y^2 v^n, v^n) = -(\delta_x v^n, \delta_x v^n) - (\delta_y v^n, \delta_y v^n) = -\|\nabla_h v^n\|^2. \qquad (2.239)$$

By the Cauchy–Schwarz inequality and Lemma 2.10.1, it follows

$$\begin{aligned}(f^n, v^n) &\leqslant \|f^n\| \cdot \|v^n\| \leqslant 6\Big(\frac{1}{L_1^2} + \frac{1}{L_2^2}\Big)\|v^n\|^2 + \frac{1}{24(\frac{1}{L_1^2} + \frac{1}{L_2^2})}\|f^n\|^2 \\ &\leqslant \|\nabla_h v^n\|^2 + \frac{1}{24(\frac{1}{L_1^2} + \frac{1}{L_2^2})}\|f^n\|^2, \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{2.240}$$

Inserting (2.238)–(2.240) into (2.237) and rearranging the result, again by the Cauchy–Schwarz inequality yield

$$\begin{aligned} a_0^{(\alpha)}(v^n, v^n)_s &\leqslant \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})(v^k, v^n)_s + a_{n-1}^{(\alpha)}(v^0, v^n)_s \\ &\quad + \frac{L_1^2 L_2^2}{24(L_1^2 + L_2^2)} s\|f^n\|^2 \\ &\leqslant \frac{1}{2}\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})[(v^k, v^k)_s + (v^n, v^n)_s] \\ &\quad + \frac{1}{2}a_{n-1}^{(\alpha)}[(v^0, v^0)_s + (v^n, v^n)_s] + \frac{L_1^2 L_2^2}{24(L_1^2 + L_2^2)} s\|f^n\|^2, \end{aligned}$$

which simplifies to give

$$\begin{aligned} a_0^{(\alpha)}(v^n, v^n)_s &\leqslant \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})(v^k, v^k)_s + a_{n-1}^{(\alpha)}(v^0, v^0)_s \\ &\quad + \frac{L_1^2 L_2^2}{12(L_1^2 + L_2^2)} s\|f^n\|^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Noticing (2.51), we have

$$\begin{aligned} a_0^{(\alpha)}(v^n, v^n)_s &\leqslant \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})(v^k, v^k)_s + a_{n-1}^{(\alpha)}\Big[(v^0, v^0)_s \\ &\quad + \frac{L_1^2 L_2^2}{12(L_1^2 + L_2^2)} t_n^\alpha \Gamma(1-\alpha)\|f^n\|^2\Big], \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{2.241}$$

Then the claimed result (2.236) can be achieved by the induction method from (2.241). The proof ends. □

2.11.4 Convergence of the difference scheme

We now consider the convergence of the difference scheme (2.228)–(2.230). At this point, the following theorem is true.

Theorem 2.11.3. *Suppose* $\{U_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *and* $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (2.220)-(2.222) *and the difference scheme* (2.228)–(2.230), *respectively. Let*

$$e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N.$$

Then it holds

$$\|e^n\| \leqslant \kappa_3(\tau^{\min\{2\alpha,2-\alpha\}} + h_1^2 + h_2^2), \quad 1 \leqslant n \leqslant N,$$

with

$$\kappa_3 = \frac{L_1L_2}{6}\sqrt{\frac{3L_1L_2}{L_1^2+L_2^2}T^\alpha\Gamma(1-\alpha)}\ c_{13}.$$

Proof. The substraction of (2.228)–(2.230) from (2.224), (2.226)–(2.227), respectively, produces the system of error equations as follows:

$$\begin{cases} \frac{1}{s}(\mathcal{I} + s^2\delta_x^2\delta_y^2)\left[a_0^{(\alpha)}e_{ij}^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})e_{ij}^k - a_{n-1}^{(\alpha)}e_{ij}^0\right] \\ = \delta_x^2 e_{ij}^n + \delta_y^2 e_{ij}^n + (r_{13})_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ e_{ij}^0 = 0, \quad (i,j) \in \omega, \\ e_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (2.225), Theorem 2.11.2 immediately implies

$$\begin{aligned} \|e^n\|^2 &\leqslant (e^n, e^n)_s \\ &\leqslant \frac{L_1^2L_2^2}{12(L_1^2+L_2^2)}\Gamma(1-\alpha)\max_{1\leqslant m\leqslant n}\{t_m^\alpha\|(r_{13})^m\|^2\} \\ &\leqslant \frac{L_1^2L_2^2}{12(L_1^2+L_2^2)}\Gamma(1-\alpha)T^\alpha L_1L_2\left[c_{13}(\tau^{\min\{2\alpha,2-\alpha\}} + h_1^2 + h_2^2)\right]^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

The desired result will be obvious from the above estimate. The proof is completed. □

2.12 Supplementary remarks and discussions

1. The time-fractional subdiffusion equation mainly consists of two types, one is the Caputo type and the other is the R-L type, which are expressed in the form of

$${}_0^CD_t^\alpha u(x,t) = u_{xx}(x,t) + f(x,t),$$

and

$$u_t(x,t) = {}_0\mathbf{D}_t^{1-\alpha}u_{xx}(x,t) + g(x,t),$$

respectively. Under certain conditions, it is possible to convert between them. In this chapter, only for the Caputo type, the finite difference method was considered. In a similar way, the finite difference method for the R-L type can be discussed. On this topic, readers can refer to literatures, such as by Yuste et al.[108, 109]; Langlands and Henry[42]; Liu et al.[8, 120]; Cui[10, 11] and Zhang et al.[112, 114].

2. The Caputo derivative can be discretized using either the G-L approximation or the interpolation approximation. Sections 2.1, 2.2 and 2.10 reported the first kind of methods based on the G-L formula and the first-order convergent difference schemes in time have been obtained. Indeed, there are also related superconvergent results based on the G-L formula. From Theorem 1.4.1, we know that if the function $f \in \mathscr{C}^{2+\alpha}(\mathcal{R})$,

$$A_{h,p}^{\alpha} f(t) = {}_{-\infty}\mathbf{D}_t^{\alpha} f(t) + \left(p - \frac{\alpha}{2}\right) {}_{-\infty}\mathbf{D}_t^{\alpha+1} f(t) h + O(h^2).$$

Let $p = \frac{\alpha}{2}$, $t = t_{n-\frac{\alpha}{2}}$, with $t_{n-\frac{\alpha}{2}} = (n - \frac{\alpha}{2})h$, then we have

$$A_{h,0}^{\alpha} f(t_n) = A_{h,\frac{\alpha}{2}}^{\alpha} f(t_{n-\frac{\alpha}{2}}) = {}_{-\infty}\mathbf{D}_t^{\alpha} f(t_{n-\frac{\alpha}{2}}) + O(h^2),$$

which means that the second-order accuracy can be achieved using $A_{h,0}^{\alpha} f(t_n)$ to approximate ${}_{-\infty}\mathbf{D}_t^{\alpha} f(t_{n-\frac{\alpha}{2}})$. By the linear interpolation, further it holds that

$$A_{h,0}^{\alpha} f(t_n) = \left(1 - \frac{\alpha}{2}\right) {}_{-\infty}\mathbf{D}_t^{\alpha} f(t_n) + \frac{\alpha}{2} {}_{-\infty}\mathbf{D}_t^{\alpha} f(t_{n-1}) + O(h^2),$$

which says that the second-order accuracy can be achieved using $A_{h,0}^{\alpha} f(t_n)$ to approximate a linear combination of ${}_{-\infty}\mathbf{D}_t^{\alpha} f(t_n)$ and ${}_{-\infty}\mathbf{D}_t^{\alpha} f(t_{n-1})$. Dimitrov[13] and Gao et al.[28] have reported the related research results on the problem of Caputo type and R-L type, respectively. In addition, using the shifted and weighted G-L formula (1.32) to approximate the R-L fractional derivative, Wang and Vong[96] developed the finite difference method for two-term time-fractional subdiffusion equations of R-L type, where the second-order accuracy in time can be achieved. Ji et al.[40] applied the shifted and weighted G-L formula (1.37) in Theorem 1.4.4 to approximate the time-fractional derivative and presented a third-order convergent method in time.

3. For the Caputo time-fractional subdiffusion equations, Sections 2.3, 2.5, 2.8 and 2.11 reported the finite difference methods based on the L1 formula to approximate Caputo fractional derivatives. Indeed, there are also some superconvergent works based on the interpolation approximation. Alikhanov[1] proposed a superconvergent interpolation approximation, also called the L2-1_σ approximation, for the Caputo fractional derivative based on the work in [31] and the second-order numerical method for solving the time-fractional subdiffusion equation was investigated using this approximation. The authors in [19] developed this method and applied it to solve the multiterm fractional subdiffusion equation and obtained a temporal second

order difference scheme. Du et al. investigated the L2-1_σ method for the variable-order fractional subdiffusion equation in [16].

4. This chapter mainly discussed the finite difference method for solving the time-fractional subdiffusion equations with the Dirichlet boundary value conditions. For the problem with the Neumann boundary value condition, the readers can refer to the works by Langlands and Henry[42], Zhao and Sun[115] and Ren et al.[69]. In addition, Gao et al.[22, 30] studied the 1D time-fractional subdiffusion equations on space unbounded domains. The results on the 2D problem can be found in [33].

5. The operator $\mathcal{A}$ defined in Section 2.1 is called an average operator. It is often used to construct the compact difference scheme, so that it is also called a compact operator. For the time-fractional subdiffusion equation, Gao and Sun[21] proposed a higher-order difference scheme by using the L1 approximation for the time-fractional derivative of order $\alpha(0 < \alpha < 1)$ and the compact approximation for the spatial derivative, which achieved the convergence of order $2 - \alpha$ in time and four in space in the maximum norm.

7. The second-order method in space was discussed for 1D multiterm time-fractional subdiffusion equations in Section 2.8 and Section 2.9. Regarding the fourth-order methods in space for the same problem and the corresponding 2D problem, readers can refer to [67].

8. Based on the SOE approximation for the kernel function in the fractional derivative, this chapter illustrated two kinds of fast methods to solve the time-fractional subdiffusion equations[41, 101]. By taking into account of the special structure of resultant difference schemes for the time-fractional diffusion equations, Lu et al.[53] developed a different fast method.

9. Lv and Xu[55] considered a numerical method for the time-fractional subdiffusion equation based on the L1-2 method[31]. The $(3-\alpha)$-order convergence of the scheme has been proved. Zhu and Xu further studied the fast L1-2 difference method for the fractional subdiffusion equations[118].

10. Sections 2.10 and 2.11 introduced two kinds of ADI difference methods for solving 2D time-fractional subdiffusion equations based on the G-L formula and L1 approximation, respectively. The discrete energy method was used to analyze the unique solvability, stability and convergence of the resultant schemes in the L^2 norm. Using the similar techniques, interested readers can try to give the estimates in the H^1 norm, or refer to the work in [111]. Besides, Cui[11, 12] reported the compact ADI difference methods for 2D problems and the Fourier method was used for the theoretical analysis. The compact ADI difference method for 2D problem in the R-L type was also studied in [112] and the discrete energy method was applied for the theoretical analysis. Moreover, another different ADI difference scheme was obtained in [111] by adding a different perturbation term into equation (2.223).

11. Stynes et al.[74] studied the numerical solution to the fractional subdiffusion equation with the initial singularity on the graded mesh. Shen et al.[72] further provided a fast difference scheme for this kind of fractional subdiffusion equations.

Exercises 2

2.1 Using the discrete energy method in Section 2.2, show that the difference scheme in Section 2.1 is uniquely solvable, convergent and unconditionally stable with respect to the initial value and the source function on the right-hand side.

2.2 Using the discrete energy method in Section 2.5, show that the difference scheme in Section 2.3 is uniquely solvable, convergent and unconditionally stable with respect to the initial value and the source function on the right-hand side.

2.3 For the problem (2.154)–(2.156), when $\varphi(x) = 0$, construct the difference scheme

$$\begin{cases} \tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)} u_i^{n-k} + \tau^{-\alpha_1}\sum_{k=0}^{n} g_k^{(\alpha_1)} u_i^{n-k} = \delta_x^2 u_i^n + f_i^n, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\hat{u}(x,t)$ like that in Section 2.1 and suppose $\hat{u}(x,\cdot) \in \mathscr{C}^{1+\alpha}(\mathcal{R})$. For this difference scheme, try to
(1) show the unique solvability;
(2) show the stability with respect to the initial value and function f;
(3) show the convergence.

2.4 Consider the following problem of fourth-order fractional subdiffusion equation:

$$\begin{cases} {}_0^C D_t^{\alpha} u(x,t) + u_{xxxx}(x,t) = f(x,t), \quad x \in (0,L),\ t \in (0,T], & (2.242) \\ u(x,0) = \varphi(x), \quad x \in (0,L), & (2.243) \\ u(0,t) = \mu_0(t), \quad u(L,t) = \nu_0(t), \quad t \in [0,T], & (2.244) \\ u_{xx}(0,t) = \mu_1(t), \quad u_{xx}(L,t) = \nu_1(t), \quad t \in [0,T], & (2.245) \end{cases}$$

where $\alpha \in (0,1)$, the functions f, φ, μ_0, ν_0, μ_1, ν_1 are all given and $\varphi(0) = \mu_0(0)$, $\varphi(L) = \nu_0(0)$, $\varphi_{xx}(0) = \mu_1(0)$, $\varphi_{xx}(L) = \nu_1(0)$.
Let $v(x,t) = u_{xx}(x,t)$, then the fourth-order equation (2.242) can be rewritten as a system of two second-order equations. Construct a difference scheme for (2.242)–(2.245) by using L2-1_σ approximation or fast L2-1_σ approximation.
For this difference scheme, try to
(1) show the unique solvability;
(2) show the stability with respect to the initial value and function f;
(3) show the convergence.

2.5 For the problem (2.197)–(2.199), construct the difference scheme

$$\begin{cases} \tau^{-\alpha}\sum_{k=0}^{n} g_k^{(\alpha)} u_{ij}^{n-k} = \delta_x^2 u_{ij}^n + \delta_y^2 u_{ij}^n + f_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ u_{ij}^0 = 0, \quad (i,j) \in \omega, \\ u_{ij}^n = \mu(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\hat{u}(x, y, t)$ like that in Section 2.10 and suppose $\hat{u}(x, y, \cdot) \in \mathscr{C}^{1+\alpha}(\mathcal{R})$. For this difference scheme, try to

(1) show the unique solvability;

(2) show the stability with respect to the initial value and function f;

(3) show the convergence.

2.6 For the problem (2.220)–(2.222), construct the difference scheme:

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)}u_{ij}^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_{ij}^k - a_{n-1}^{(\alpha)}u_{ij}^0\right] \\ = \delta_x^2 u_{ij}^n + \delta_y^2 u_{ij}^n + f_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, \\ u_{ij}^n = \mu(x_i, y_j, t_n), \quad (i,j) \in \partial\omega, \quad 0 \leqslant n \leqslant N. \end{cases}$$

For this difference scheme, try to

(1) show the unique solvability;

(2) show the stability with respect to the initial value and function f;

(3) show the convergence.

3 Difference methods for the time-fractional wave equations

This chapter will develop the difference methods for solving the time-fractional wave equations. The discussion on 1D problem is given in the former 8 sections. The time-fractional derivative is discretized by L1 approximation or L2-1_σ approximation. The spatial derivative is discretized by the second-order central difference quotient approximation or the compact approximation. The fast L1 approximation and fast L2-1_σ approximation are concerned. The difference methods for the multi-term time-fractional wave equation and the time-fractional mixed diffusion and wave equation are also investigated. The ADI and compact ADI methods for 2D problem are established. The chapter consists of 11 sections.

3.1 The second-order method in space based on L1 approximation for 1D problem

Consider the following problem of the time-fractional wave equations:

$$\begin{cases} {}_0^C D_t^\gamma u(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], & (3.1) \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in (0,L), & (3.2) \\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), \quad t \in [0,T], & (3.3) \end{cases}$$

where $\gamma \in (1,2)$, the functions f, φ, ψ, μ, ν are given and $\varphi(0) = \mu(0)$, $\varphi(L) = \nu(0)$, $\psi(0) = \mu'(0)$, $\psi(L) = \nu'(0)$. Suppose $u \in C^{(4,3)}([0,L] \times [0,T])$.

Take the same mesh partition and notations as those in Section 2.1. For the mesh function $v = \{v_i^k \mid 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant N\}$ defined on $\Omega_h \times \Omega_\tau$, denote

$$v_i^{k-\frac{1}{2}} = \frac{1}{2}(v_i^k + v_i^{k-1}), \quad \delta_t v_i^{k-\frac{1}{2}} = \frac{1}{\tau}(v_i^k - v_i^{k-1}).$$

Define the same mesh function spaces $\mathcal{U}_h$ and $\mathring{\mathcal{U}}_h$ as those in Section 2.1.

Denote

$$U_i^n = u(x_i, t_n), \quad f_i^n = f(x_i, t_n), \quad \psi_i = \psi(x_i), \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N.$$

3.1.1 Derivation of the difference scheme

Considering equation (3.1) at the point (x_i, t_n), we have

$${}_0^C D_t^\gamma u(x_i, t_n) = u_{xx}(x_i, t_n) + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 0 \leqslant n \leqslant N.$$

https://doi.org/10.1515/9783110616064-003

Taking an average on two adjacent time levels gives

$$\begin{aligned}&\frac{1}{2}\big[{}_0^C D_t^\gamma u(x_i,t_n) + {}_0^C D_t^\gamma u(x_i,t_{n-1})\big]\\&= \frac{1}{2}\big[u_{xx}(x_i,t_n) + u_{xx}(x_i,t_{n-1})\big] + f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N,\end{aligned}$$

where $f_i^{n-\frac{1}{2}} = \frac{1}{2}(f_i^n + f_i^{n-1})$.

For the approximation of the time-fractional derivative and spatial derivative in the equality above, the L1 formula (1.69) and the second-order central difference quotient are applied, respectively, and it follows from Theorem 1.6.2 and Lemma 2.1.3 that

$$\begin{aligned}&\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\Big[b_0^{(\gamma)}\delta_t U_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t U_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_i\Big]\\&= \delta_x^2 U_i^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}} + (r_1)_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N,\end{aligned} \tag{3.4}$$

and there is a positive constant c_1 such that

$$\big|(r_1)_i^{n-\frac{1}{2}}\big| \leqslant c_1(\tau^{3-\gamma} + h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{3.5}$$

where $\{b_l^{(\gamma)}\}$ is defined in (1.64).

Noticing the initial-boundary value conditions (3.2)–(3.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (3.6)\\ U_0^n = \mu(t_n), \quad U_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (3.7)\end{cases}$$

Omitting the small term $(r_1)_i^{n-\frac{1}{2}}$ in (3.4) and replacing the exact solution U_i^n with its numerical one u_i^n arrives at a difference scheme for solving (3.1)–(3.3) as follows:

$$\begin{cases} \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\Big[b_0^{(\gamma)}\delta_t u_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t u_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_i\Big] & \\ = \delta_x^2 u_i^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.8)\\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (3.9)\\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (3.10)\end{cases}$$

Denote

$$\eta = \tau^{\gamma-1}\Gamma(3-\gamma).$$

3.1.2 Solvability of the difference scheme

Theorem 3.1.1. *The difference scheme* (3.8)–(3.10) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The value of u^0 is determined by (3.9)–(3.10). Now assume that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n can be obtained from (3.8) and (3.10). To show its unique solvability, it suffices to verify that the corresponding homogeneous one

$$\begin{cases} \dfrac{1}{\eta\tau} u_i^n = \dfrac{1}{2}\delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (3.11)\\ u_0^n = u_M^n = 0 & (3.12) \end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (3.11) with u^n and noticing (3.12), we have

$$\frac{1}{\eta\tau}(u^n, u^n) = \frac{1}{2}(\delta_x^2 u^n, u^n) = -\frac{1}{2}\|\delta_x u^n\|^2 \leqslant 0,$$

thus $\|u^n\| = 0$. It follows $u^n = 0$ from (3.12).

By the principle of induction, the theorem is true. The proof ends. □

3.1.3 Stability of the difference scheme

The stability of the difference scheme (3.8)–(3.10) will be analyzed in this subsection. The following theorem is true.

Theorem 3.1.2. *Suppose* $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \dfrac{1}{\eta}\left[b_0^{(\gamma)}\delta_t v_i^{n-\frac{1}{2}} - \displaystyle\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t v_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_i\right] \\ = \delta_x^2 v_i^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.13)\\ v_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (3.14)\\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N. & (3.15) \end{cases}$$

Then it holds

$$\begin{aligned} \|\delta_x v^n\|^2 &\leqslant \|\delta_x v^0\|^2 + \frac{t_n^{2-\gamma}}{\Gamma(3-\gamma)}\|\psi\|^2 \\ &\quad + \Gamma(2-\gamma)t_n^{\gamma-1} \cdot \tau \sum_{k=1}^{n} \|f^{k-\frac{1}{2}}\|^2, \quad 1 \leqslant n \leqslant N, \end{aligned} \tag{3.16}$$

where

$$\|\psi\|^2 = h\sum_{i=1}^{M-1}\psi_i^2, \quad \|f^{k-\frac{1}{2}}\|^2 = h\sum_{i=1}^{M-1}(f_i^{k-\frac{1}{2}})^2.$$

Proof. Making the inner product on both hand sides of (3.13) with $\eta\delta_t v^{n-\frac{1}{2}}$ produces

$$\begin{aligned} b_0^{(\gamma)}(\delta_t v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) = & \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\delta_t v^{k-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) \\ & + b_{n-1}^{(\gamma)}(\psi, \delta_t v^{n-\frac{1}{2}}) + \eta(\delta_x^2 v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) \\ & + \eta(f^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{3.17}$$

Noticing (3.15), the application of the summation by parts arrives at

$$\begin{aligned} (\delta_x^2 v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) &= -(\delta_x v^{n-\frac{1}{2}}, \delta_x\delta_t v^{n-\frac{1}{2}}) \\ &= -\frac{1}{2\tau}(\|\delta_x v^n\|^2 - \|\delta_x v^{n-1}\|^2). \end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.17), it follows from the Cauchy–Schwarz inequality that

$$\begin{aligned} & b_0^{(\gamma)}\|\delta_t v^{n-\frac{1}{2}}\|^2 + \frac{\eta}{2\tau}(\|\delta_x v^n\|^2 - \|\delta_x v^{n-1}\|^2) \\ = & \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\delta_t v^{k-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) + b_{n-1}^{(\gamma)}(\psi, \delta_t v^{n-\frac{1}{2}}) \\ & + \eta(f^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) \\ \leqslant & \frac{1}{2}\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\|\delta_t v^{k-\frac{1}{2}}\|^2 + \|\delta_t v^{n-\frac{1}{2}}\|^2) \\ & + \frac{1}{2}b_{n-1}^{(\gamma)}(\|\psi\|^2 + \|\delta_t v^{n-\frac{1}{2}}\|^2) + \eta(f^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}), \end{aligned}$$

which can be simplified to

$$\begin{aligned} & b_0^{(\gamma)}\|\delta_t v^{n-\frac{1}{2}}\|^2 + \frac{\eta}{\tau}(\|\delta_x v^n\|^2 - \|\delta_x v^{n-1}\|^2) \\ \leqslant & \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\|\delta_t v^{k-\frac{1}{2}}\|^2 + b_{n-1}^{(\gamma)}\|\psi\|^2 \\ & + 2\eta(f^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N. \end{aligned}$$

The result can further be reformulated as

$$\|\delta_x v^n\|^2 + \frac{\tau}{\eta}\sum_{k=1}^{n} b_{n-k}^{(\gamma)}\|\delta_t v^{k-\frac{1}{2}}\|^2$$

$$\leqslant \|\delta_x v^{n-1}\|^2 + \frac{\tau}{\eta}\sum_{k=1}^{n-1} b_{n-k-1}^{(\gamma)}\|\delta_t v^{k-\frac{1}{2}}\|^2 + \frac{\tau}{\eta} b_{n-1}^{(\gamma)}\|\psi\|^2$$
$$+ 2\tau(f^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N.$$

Let

$$F^0 = \|\delta_x v^0\|^2, \quad F^n = \|\delta_x v^n\|^2 + \frac{\tau}{\eta}\sum_{k=1}^{n} b_{n-k}^{(\gamma)}\|\delta_t v^{k-\frac{1}{2}}\|^2, \quad n \geqslant 1,$$

then

$$F^n \leqslant F^{n-1} + \frac{\tau}{\eta} b_{n-1}^{(\gamma)}\|\psi\|^2 + 2\tau(f^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N.$$

The recursive process will lead to

$$F^n \leqslant F^0 + \frac{\tau}{\eta}\sum_{k=0}^{n-1} b_k^{(\gamma)}\|\psi\|^2 + 2\tau\sum_{k=1}^{n}(f^{k-\frac{1}{2}}, \delta_t v^{k-\frac{1}{2}})$$
$$\leqslant F^0 + \frac{\tau}{\eta}\sum_{k=0}^{n-1} b_k^{(\gamma)}\|\psi\|^2 + \tau\sum_{k=1}^{n}\left(\frac{\eta}{b_{n-k}^{(\gamma)}}\|f^{k-\frac{1}{2}}\|^2 + \frac{b_{n-k}^{(\gamma)}}{\eta}\|\delta_t v^{k-\frac{1}{2}}\|^2\right),$$
$$1 \leqslant n \leqslant N.$$

Thus,

$$\|\delta_x v^n\|^2 \leqslant \|\delta_x v^0\|^2 + \frac{\tau}{\eta}\sum_{k=0}^{n-1} b_k^{(\gamma)}\|\psi\|^2 + \tau\sum_{k=1}^{n}\frac{\eta}{b_{n-k}^{(\gamma)}}\|f^{k-\frac{1}{2}}\|^2, \quad 1 \leqslant n \leqslant N. \tag{3.19}$$

By the definition of $b_k^{(\gamma)}$ and Lemma 1.6.1, it is easy to know that

$$(2-\gamma)(k+1)^{1-\gamma} < b_k^{(\gamma)} < (2-\gamma)k^{1-\gamma},$$

from which, we can get

$$b_{n-k}^{(\gamma)} > (2-\gamma)(n-k+1)^{1-\gamma} \geqslant (2-\gamma)n^{1-\gamma}, \quad 1 \leqslant k \leqslant n.$$

Therefore,

$$\frac{\eta}{b_{n-k}^{(\gamma)}} \leqslant \frac{\tau^{\gamma-1}\Gamma(3-\gamma)}{(2-\gamma)n^{1-\gamma}} = \Gamma(2-\gamma)(n\tau)^{\gamma-1} = \Gamma(2-\gamma)t_n^{\gamma-1}, \tag{3.20}$$

from which it follows

$$\tau\sum_{k=1}^{n}\frac{\eta}{b_{n-k}^{(\gamma)}}\|f^{k-\frac{1}{2}}\|^2 \leqslant \Gamma(2-\gamma)t_n^{\gamma-1} \cdot \tau\sum_{k=1}^{n}\|f^{k-\frac{1}{2}}\|^2. \tag{3.21}$$

In addition, it follows from the definition of $\{b_k^{(\gamma)}\}$ that

$$\begin{aligned}\frac{\tau}{\eta}\sum_{k=0}^{n-1} b_k^{(\gamma)} &= \frac{\tau}{\tau^{\gamma-1}\Gamma(3-\gamma)}\sum_{k=0}^{n-1}\left[(k+1)^{2-\gamma}-k^{2-\gamma}\right]\\ &= \frac{\tau^{2-\gamma}}{\Gamma(3-\gamma)}n^{2-\gamma} = \frac{t_n^{2-\gamma}}{\Gamma(3-\gamma)}.\end{aligned} \tag{3.22}$$

The substitution of (3.21) and (3.22) into (3.19) will get the assertion (3.16). The proof ends. □

3.1.4 Convergence of the difference scheme

Theorem 3.1.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (3.1)–(3.3) *and the difference scheme* (3.8)–(3.10), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then it holds

$$\|e^n\|_\infty \leqslant \frac{c_1 L}{2}\sqrt{T^\gamma\Gamma(2-\gamma)}(\tau^{3-\gamma}+h^2), \quad 1 \leqslant n \leqslant N.$$

Proof. Subtracting (3.8)–(3.10) from (3.4), (3.6)–(3.7), respectively, produces the system of error equations as follows:

$$\begin{cases}\dfrac{1}{\eta}\left[b_0^{(\gamma)}\delta_t e_i^{n-\frac{1}{2}} - \displaystyle\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t e_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\cdot 0\right] \\ = \delta_x^2 e_i^{n-\frac{1}{2}} + (r_1)_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.23)\\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, & (3.24)\\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. & (3.25)\end{cases}$$

Noticing (3.5), an immediate consequence of Theorem 3.1.2 into (3.23)–(3.25) arrives at

$$\begin{aligned}\|\delta_x e^n\|^2 &\leqslant t_n^{\gamma-1}\Gamma(2-\gamma)\tau\sum_{k=1}^{n}\|(r_1)^{k-\frac{1}{2}}\|^2\\ &\leqslant T^\gamma\Gamma(2-\gamma)Lc_1^2(\tau^{3-\gamma}+h^2)^2, \quad 1 \leqslant n \leqslant N.\end{aligned}$$

Taking the square root on both hand sides of the inequality above and noticing Lemma 2.1.1, we have

$$\|e^n\|_\infty \leqslant \frac{\sqrt{L}}{2}\|\delta_x e^n\| \leqslant \frac{c_1 L}{2}\sqrt{T^\gamma\Gamma(2-\gamma)}(\tau^{3-\gamma}+h^2), \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

3.2 The fast difference method based on L1 approximation for 1D problem

In this section, we will propose a fast difference scheme for (3.1)–(3.3) based on the fast L1 approximation.

3.2.1 Derivation of the difference scheme

Denote

$$v(x,t) = u_t(x,t), \quad \alpha = \gamma - 1.$$

Thus, the problem (3.1)–(3.3) is equivalent to the following one:

$$\begin{cases} {}_0^C D_t^\alpha v(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], & (3.26)\\ u_t(x,t) = v(x,t), \quad x \in [0,L],\ t \in (0,T], & (3.27)\\ u(x,0) = \varphi(x), \quad v(x,0) = \psi(x), \quad x \in (0,L), & (3.28)\\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), \quad t \in [0,T]. & (3.29) \end{cases}$$

Denote

$$U_i^n = u(x_i,t_n), \quad V_i^n = v(x_i,t_n), \quad f_i^n = f(x_i,t_n), \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$
$$\varphi_i = \varphi(x_i), \quad \psi_i = \psi(x_i), \quad 0 \leqslant i \leqslant M.$$

Considering equation (3.26) at the point (x_i, t_n), we have

$${}_0^C D_t^\alpha v(x_i,t_n) = u_{xx}(x_i,t_n) + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 0 \leqslant n \leqslant N.$$

Applying the fast L1 approximation and second-order central difference quotient to approximate Caputo and spatial derivatives in the equality above respectively, it follows from Theorem 1.7.1 that

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\left[\displaystyle\sum_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + \hat{a}_0^{(\alpha)}(V_i^n - V_i^{n-1})\right] & \\ = \delta_x^2 U_i^n + f_i^n + (r_2)_i^n, \quad 1 \leqslant i \leqslant M-1, 1 \leqslant n \leqslant N, & (3.30)\\ F_{l,i}^1 = 0, \quad F_{l,i}^n = e^{-s_l\tau} F_{l,i}^{n-1} + B_l(V_i^{n-1} - V_i^{n-2}), & \\ \qquad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N, & (3.31) \end{cases}$$

and there exists a positive constant c_2 such that

$$\left|(r_2)_i^n\right| \leqslant c_2(\tau^{2-\alpha} + h^2 + \epsilon), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{3.32}$$

Substituting (3.31) into (3.30), we obtain

$$\frac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)}V_i^n-\sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)}-\hat{a}_k^{(\alpha)})V_i^{n-k}-\hat{a}_{n-1}^{(\alpha)}V_i^0\right]$$
$$=\delta_x^2U_i^n+f_i^n+(r_2)_i^n,\quad 1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N. \tag{3.33}$$

Considering equation (3.27) at the point $(x_i,t_{n-\frac{1}{2}})$, we have

$$\delta_tU_i^{n-\frac{1}{2}}=V_i^{n-\frac{1}{2}}+(r_3)_i^n,\quad 0\leqslant i\leqslant M,\ 1\leqslant n\leqslant N, \tag{3.34}$$

and there exists a positive constant c_3 such that

$$\left|(r_3)_i^n\right|\leqslant c_3\tau^2,\quad 0\leqslant i\leqslant M,\ 1\leqslant n\leqslant N. \tag{3.35}$$

Noticing the initial-boundary value conditions (3.28)–(3.29), we have

$$\begin{cases} U_i^0=\varphi_i,\quad V_i^0=\psi_i,\quad 0\leqslant i\leqslant M, & (3.36)\\ U_0^n=\mu(t_n),\quad U_M^n=\nu(t_n),\quad 1\leqslant n\leqslant N. & (3.37)\end{cases}$$

Omitting the small terms $(r_2)_i^n$ and $(r_3)_i^n$ in (3.30) and (3.34), and replacing the exact solution $\{U_i^n,V_i^n\}$ with its numerical one $\{u_i^n,v_i^n\}$ arrive at the difference scheme for solving (3.26)–(3.29) as follows:

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\left[\displaystyle\sum_{l=1}^{N_{\exp}}\omega_lF_{l,i}^n+\hat{a}_0^{(\alpha)}(v_i^n-v_i^{n-1})\right] & \\ =\delta_x^2u_i^n+f_i^n,\quad 1\leqslant i\leqslant M-1,\quad 1\leqslant n\leqslant N, & (3.38)\\ F_{l,i}^1=0,\quad F_{l,i}^n=e^{-s_l\tau}F_{l,i}^{n-1}+B_l(v_i^{n-1}-v_i^{n-2}), & \\ \qquad 1\leqslant l\leqslant N_{\exp},\ 1\leqslant i\leqslant M-1,\ 2\leqslant n\leqslant N, & (3.39)\\ \delta_tu_i^{n-\frac{1}{2}}=v_i^{n-\frac{1}{2}},\quad 0\leqslant i\leqslant M,\quad 1\leqslant n\leqslant N, & (3.40)\\ u_i^0=\varphi_i,\quad v_i^0=\psi_i,\quad 0\leqslant i\leqslant M, & (3.41)\\ u_0^n=\mu(t_n),\quad u_M^n=\nu(t_n),\quad 1\leqslant n\leqslant N. & (3.42)\end{cases}$$

Substituting (3.39) into (3.38) yields

$$\frac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)}v_i^n-\sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)}-\hat{a}_k^{(\alpha)})v_i^{n-k}-\hat{a}_{n-1}^{(\alpha)}v_i^0\right]=\delta_x^2u_i^n+f_i^n,$$
$$1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N. \tag{3.43}$$

3.2.2 Solvability of the difference scheme

Theorem 3.2.1. *The difference scheme* (3.38)–(3.42) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \dots, u_M^n), \quad v^n = (v_0^n, v_1^n, \dots, v_M^n).$$

The value of $\{u^0, v^0\}$ is determined by (3.41). Now assume that the value of $\{u^0, v^0, u^1, v^1, \dots, u^{n-1}, v^{n-1}\}$ has been uniquely determined. From (3.40), we have

$$v_i^n = 2v_i^{n-\frac{1}{2}} - v_i^{n-1} = 2\delta_t u_i^{n-\frac{1}{2}} - v_i^{n-1}, \quad 0 \leqslant i \leqslant M,\ 1 \leqslant n \leqslant N. \tag{3.44}$$

Substituting (3.44) into (3.38) and noticing (3.42) give a linear system in u^n as follows:

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\left[\sum\limits_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + \hat{a}_0^{(\alpha)}(2\delta_t u_i^{n-\frac{1}{2}} - 2v_i^{n-1})\right] \\ = \delta_x^2 u_i^n + f_i^n, \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n). \end{cases}$$

To show its unique solvability, it suffices to verify that the corresponding homogeneous one

$$\begin{cases} \dfrac{\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)} \cdot \dfrac{2}{\tau} u_i^n = \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (3.45) \\ u_0^n = 0, \quad u_M^n = 0 & (3.46) \end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (3.45) with u^n and noticing (3.46), we have

$$\frac{\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)} \cdot \frac{2}{\tau}\|u^n\|^2 + \|\delta_x u^n\|^2 = 0,$$

thus $\|u^n\| = 0$. It follows $u^n = 0$.

When u^n is determined, v^n will be obtained from (3.44).

By the principle of induction, the theorem is true. The proof ends. □

3.2.3 Stability of the difference scheme

Theorem 3.2.2. *Suppose* $\{u_i^n, v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\left[\displaystyle\sum_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + \hat{a}_0^{(\alpha)}(v_i^n - v_i^{n-1})\right] = \delta_x^2 u_i^n + p_i^n, \\ \qquad\qquad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, & (3.47) \\ F_{l,i}^1 = 0, \quad F_{l,i}^n = e^{-s_l\tau}F_{l,i}^{n-1} + B_l(v_i^{n-1} - v_i^{n-2}), \\ \qquad\qquad 1 \leqslant l \leqslant N_{\exp}, \quad 1 \leqslant i \leqslant M-1, \quad 2 \leqslant n \leqslant N, & (3.48) \\ \delta_t u_i^{n-\frac{1}{2}} = v_i^{n-\frac{1}{2}} + q_i^n, \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant n \leqslant N, & (3.49) \\ u_i^0 = \varphi_i, \quad v_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, & (3.50) \\ u_0^n = 0, \quad u_M^n = 0, \quad 1 \leqslant n \leqslant N. & (3.51) \end{cases}$$

Then it holds

$$\|\delta_x u^n\|^2 \leqslant \|\delta_x u^0\|^2 + \left[\frac{\frac{4}{3}t_n^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{t_n}{\Gamma(1-\alpha)}\epsilon\right]\|v^0\|^2$$
$$+ 2\Gamma(1-\alpha)t_n^{1+\alpha}\left[\max\left\{\|p^1\|, \max_{2\leqslant k\leqslant n}\|p^{k-\frac{1}{2}}\|\right\} + \frac{2\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)}\max_{1\leqslant k\leqslant n}\|q^k\|\right]^2, \quad 1 \leqslant n \leqslant N,$$

where

$$\|w^n\|^2 = h\sum_{i=1}^{M-1}(w_i^n)^2, \quad w = v, p, q,$$

and

$$p_i^{n-\frac{1}{2}} = \frac{1}{2}(p_i^n + p_i^{n-1}), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.$$

Proof. Denote

$$Q_i^1 = p_i^1 + \frac{2\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)}q_i^1, \quad 1 \leqslant i \leqslant M-1,$$
$$Q_i^n = p_i^{n-\frac{1}{2}} + \frac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)}q_i^n - \sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)} - \hat{a}_k^{(\alpha)})q_i^{n-k}\right],$$
$$1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N.$$

Substituting (3.48) into (3.47), we have

$$\frac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)}v_i^n - \sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)} - \hat{a}_k^{(\alpha)})v_i^{n-k} - \hat{a}_{n-1}^{(\alpha)}v_i^0\right] = \delta_x^2 u_i^n + p_i^n,$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N,$$

which is equivalent to

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}(2v_i^{\frac{1}{2}} - 2v_i^0) = \delta_x^2 u_i^1 + p_i^1, \quad 1 \leqslant i \leqslant M-1, & (3.52) \\ \dfrac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)} v_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)} - \hat{a}_k^{(\alpha)})v_i^{n-k-\frac{1}{2}} - \hat{a}_{n-1}^{(\alpha)} v_i^0\right] & \\ = \delta_x^2 u_i^{n-\frac{1}{2}} + p_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N. & (3.53) \end{cases}$$

Substituting (3.49) into (3.52)–(3.53) yields

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}(2\delta_t u_i^{\frac{1}{2}} - 2v_i^0) = \delta_x^2 u_i^1 + Q_i^1, \quad 1 \leqslant i \leqslant M-1, & (3.54) \\ \dfrac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)} \delta_t u_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)} - \hat{a}_k^{(\alpha)})\delta_t u_i^{n-k-\frac{1}{2}} - \hat{a}_{n-1}^{(\alpha)} v_i^0\right] & \\ = \delta_x^2 u_i^{n-\frac{1}{2}} + Q_i^n, \quad 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N. & (3.55) \end{cases}$$

(I) Making the inner product on both hand sides of (3.54) with $\delta_t u^{\frac{1}{2}}$ produces

$$\begin{aligned} &\frac{2}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}\|\delta_t u^{\frac{1}{2}}\|^2 - (\delta_x^2 u^1, \delta_t u^{\frac{1}{2}}) \\ =& \frac{2}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}(v^0, \delta_t u^{\frac{1}{2}}) + (Q^1, \delta_t u^{\frac{1}{2}}) \\ \leqslant& \frac{1}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}\left(\frac{2}{3}\|v^0\|^2 + \frac{3}{2}\|\delta_t u^{\frac{1}{2}}\|^2\right) + \|Q^1\| \cdot \|\delta_t u^{\frac{1}{2}}\|. \end{aligned}$$

Noticing (3.51), we have

$$-(\delta_x^2 u^1, \delta_t u^{\frac{1}{2}}) = (\delta_x u^1, \delta_t \delta_x u^{\frac{1}{2}}) = \frac{1}{2\tau}(\|\delta_x u^1\|^2 - \|\delta_x u^0\|^2) + \frac{\tau}{2}\|\delta_x \delta_t u^{\frac{1}{2}}\|^2,$$

hence it follows that

$$\begin{aligned} &\frac{2}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}\|\delta_t u^{\frac{1}{2}}\|^2 + \frac{1}{2\tau}(\|\delta_x u^1\|^2 - \|\delta_x u^0\|^2) \\ \leqslant& \frac{1}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}\left(\frac{2}{3}\|v^0\|^2 + \frac{3}{2}\|\delta_t u^{\frac{1}{2}}\|^2\right) + \|Q^1\| \cdot \|\delta_t u^{\frac{1}{2}}\|. \end{aligned}$$

Multiplying both hand sides of the inequality above by 2τ arrives at

$$\begin{aligned} &\frac{\tau}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}\|\delta_t u^{\frac{1}{2}}\|^2 + \|\delta_x u^1\|^2 \\ \leqslant& \|\delta_x u^0\|^2 + \frac{4}{3} \cdot \frac{\tau}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}\|v^0\|^2 + 2\tau\|Q^1\| \cdot \|\delta_t u^{\frac{1}{2}}\|. \end{aligned} \tag{3.56}$$

(II) Making the inner product on both hand sides of (3.55) with $2\delta_t u^{n-\frac{1}{2}}$ produces

$$\frac{2}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}\|\delta_t u^{n-\frac{1}{2}}\|^2 + \frac{1}{\tau}(\|\delta_x u^n\|^2 - \|\delta_x u^{n-1}\|^2)$$

$$
\begin{aligned}
&= \frac{2}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)}-\hat{a}_k^{(\alpha)})(\delta_t u^{n-k-\frac{1}{2}},\delta_t u^{n-\frac{1}{2}})+\hat{a}_{n-1}^{(\alpha)}(v^0,\delta_t u^{n-\frac{1}{2}})\right]\\
&\quad + 2(Q^n,\delta_t u^{n-\frac{1}{2}})\\
&\leqslant \frac{1}{\Gamma(1-\alpha)}\left[\sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)}-\hat{a}_k^{(\alpha)})(\|\delta_t u^{n-k-\frac{1}{2}}\|^2+\|\delta_t u^{n-\frac{1}{2}}\|^2)\right.\\
&\quad \left.+\hat{a}_{n-1}^{(\alpha)}(\|v^0\|^2+\|\delta_t u^{n-\frac{1}{2}}\|^2)\right]+2\|Q^n\|\cdot\|\delta_t u^{n-\frac{1}{2}}\|,\quad 2\leqslant n\leqslant N,
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}\|\delta_t u^{n-\frac{1}{2}}\|^2+\frac{1}{\tau}(\|\delta_x u^n\|^2-\|\delta_x u^{n-1}\|^2)\\
&\leqslant \frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)}-\hat{a}_k^{(\alpha)})\|\delta_t u^{n-k-\frac{1}{2}}\|^2+\frac{1}{\Gamma(1-\alpha)}\hat{a}_{n-1}^{(\alpha)}\|v^0\|^2\\
&\quad +2\|Q^n\|\cdot\|\delta_t u^{n-\frac{1}{2}}\|,\quad 2\leqslant n\leqslant N.
\end{aligned}
$$

The above inequality can be rewritten as

$$
\begin{aligned}
&\frac{\tau}{\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\hat{a}_k^{(\alpha)}\|\delta_t u^{n-k-\frac{1}{2}}\|^2+\|\delta_x u^n\|^2\\
&\leqslant \frac{\tau}{\Gamma(1-\alpha)}\sum_{k=1}^{n-1}\hat{a}_{k-1}^{(\alpha)}\|\delta_t u^{n-k-\frac{1}{2}}\|^2+\|\delta_x u^{n-1}\|^2\\
&\quad +\frac{\tau}{\Gamma(1-\alpha)}\hat{a}_{n-1}^{(\alpha)}\|v^0\|^2+2\tau\|Q^n\|\cdot\|\delta_t u^{n-\frac{1}{2}}\|\\
&= \frac{\tau}{\Gamma(1-\alpha)}\sum_{k=0}^{n-2}\hat{a}_k^{(\alpha)}\|\delta_t u^{n-1-k-\frac{1}{2}}\|^2+\|\delta_x u^{n-1}\|^2\\
&\quad +\frac{\tau}{\Gamma(1-\alpha)}\hat{a}_{n-1}^{(\alpha)}\|v^0\|^2+2\tau\|Q^n\|\cdot\|\delta_t u^{n-\frac{1}{2}}\|,\quad 2\leqslant n\leqslant N.
\end{aligned}
$$

Replacing the superscript n with m and summing up for m from 2 to n on both hand sides of the above inequality lead to

$$
\begin{aligned}
&\frac{\tau}{\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\hat{a}_k^{(\alpha)}\|\delta_t u^{n-k-\frac{1}{2}}\|^2+\|\delta_x u^n\|^2\\
&\leqslant \frac{\tau}{\Gamma(1-\alpha)}\hat{a}_0^{(\alpha)}\|\delta_t u^{\frac{1}{2}}\|^2+\|\delta_x u^1\|^2\\
&\quad +\frac{\tau}{\Gamma(1-\alpha)}\sum_{m=1}^{n-1}\hat{a}_m^{(\alpha)}\|v^0\|^2+2\tau\sum_{m=2}^{n}\|Q^m\|\cdot\|\delta_t u^{m-\frac{1}{2}}\|,\quad 2\leqslant n\leqslant N.
\end{aligned}
\tag{3.57}
$$

Combining (3.56) with (3.57) produces

$$
\frac{\tau}{\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\hat{a}_k^{(\alpha)}\|\delta_t u^{n-k-\frac{1}{2}}\|^2+\|\delta_x u^n\|^2
$$

$$\leqslant \|\delta_x u^0\|^2 + \frac{\tau}{\Gamma(1-\alpha)}\left[\frac{4}{3}\hat{a}_0^{(\alpha)} + \sum_{m=1}^{n-1}\hat{a}_m^{(\alpha)}\right]\|v^0\|^2 + 2\tau\sum_{m=1}^{n}\|Q^m\|\cdot\|\delta_t u^{m-\frac{1}{2}}\|$$
$$= \|\delta_x u^0\|^2 + \frac{\tau}{\Gamma(1-\alpha)}\left[\frac{4}{3}\hat{a}_0^{(\alpha)} + \sum_{m=1}^{n-1}\hat{a}_m^{(\alpha)}\right]\|v^0\|^2 + 2\tau\sum_{k=0}^{n-1}\|Q^{n-k}\|\cdot\|\delta_t u^{n-k-\frac{1}{2}}\|$$
$$\leqslant \|\delta_x u^0\|^2 + \frac{\tau}{\Gamma(1-\alpha)}\left[\frac{4}{3}\hat{a}_0^{(\alpha)} + \sum_{m=1}^{n-1}\hat{a}_m^{(\alpha)}\right]\|v^0\|^2 + \frac{\tau}{\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\hat{a}_k^{(\alpha)}\|\delta_t u^{n-k-\frac{1}{2}}\|^2$$
$$+ \tau\Gamma(1-\alpha)\sum_{k=0}^{n-1}\frac{1}{\hat{a}_k^{(\alpha)}}\|Q^{n-k}\|^2, \quad 1 \leqslant n \leqslant N,$$

that is,

$$\|\delta_x u^n\|^2 \leqslant \|\delta_x u^0\|^2 + \frac{\tau}{\Gamma(1-\alpha)}\left[\frac{4}{3}\hat{a}_0^{(\alpha)} + \sum_{m=1}^{n-1}\hat{a}_m^{(\alpha)}\right]\|v^0\|^2$$
$$+ \tau\Gamma(1-\alpha)\sum_{k=0}^{n-1}\frac{1}{\hat{a}_k^{(\alpha)}}\|Q^{n-k}\|^2, \quad 1 \leqslant n \leqslant N. \tag{3.58}$$

It follows from (1.126) that

$$\frac{\tau}{\Gamma(1-\alpha)}\left[\frac{4}{3}\hat{a}_0^{(\alpha)} + \sum_{m=1}^{n-1}\hat{a}_m^{(\alpha)}\right]$$
$$\leqslant \frac{\tau}{\Gamma(1-\alpha)}\left[\frac{4}{3}\cdot\frac{\tau^{-\alpha}}{1-\alpha} + \sum_{m=1}^{n-1}\left(\frac{\tau^{-\alpha}}{1-\alpha}a_m^{(\alpha)} + \epsilon\right)\right]$$
$$= \frac{\tau}{\Gamma(1-\alpha)}\left[\frac{4}{3}\cdot\frac{\tau^{-\alpha}}{1-\alpha} + \frac{\tau^{-\alpha}}{1-\alpha}(n^{1-\alpha}-1) + (n-1)\epsilon\right]$$
$$\leqslant \frac{\frac{4}{3}t_n^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{t_n}{\Gamma(1-\alpha)}\epsilon \tag{3.59}$$

and it follows from Lemma 1.7.2 that

$$\tau\Gamma(1-\alpha)\sum_{k=0}^{n-1}\frac{1}{\hat{a}_k^{(\alpha)}}\|Q^{n-k}\|^2$$
$$\leqslant \tau\Gamma(1-\alpha)\max_{1\leqslant k\leqslant n}\|Q^k\|^2\sum_{k=0}^{n-1}\frac{1}{\hat{a}_k^{(\alpha)}}$$
$$\leqslant \tau\Gamma(1-\alpha)\max_{1\leqslant k\leqslant n}\|Q^k\|^2\sum_{k=0}^{n-1}2\,t_{k+1}^{\alpha}$$
$$\leqslant 2\Gamma(1-\alpha)t_n^{1+\alpha}\max_{1\leqslant k\leqslant n}\|Q^k\|^2. \tag{3.60}$$

In addition, it follows from the definition of Q^n that

$$\|Q^1\| \leqslant \|p^1\| + \frac{2\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)}\|q^1\|, \tag{3.61}$$

$$\begin{aligned}
\|Q^n\| &\leqslant \|p^{n-\frac{1}{2}}\| + \frac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)}\|q^n\| + \sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)} - \hat{a}_k^{(\alpha)})\|q^{n-k}\|\right] \\
&\leqslant \|p^{n-\frac{1}{2}}\| + \frac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)}\|q^n\| + (\hat{a}_0^{(\alpha)} - \hat{a}_{n-1}^{(\alpha)})\max_{1\leqslant k\leqslant n-1}\|q^k\|\right] \\
&\leqslant \|p^{n-\frac{1}{2}}\| + \frac{2\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)}\max_{1\leqslant k\leqslant n}\|q^k\|, \quad 2 \leqslant n \leqslant N. \qquad (3.62)
\end{aligned}$$

The substitution of (3.59)–(3.62) into (3.58) yields

$$\begin{aligned}
\|\delta_x u^n\|^2 \leqslant \|\delta_x u^0\|^2 &+ \left[\frac{\frac{4}{3}t_n^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{t_n}{\Gamma(1-\alpha)}\epsilon\right]\|v^0\|^2 \\
&+ 2\Gamma(1-\alpha)t_n^{1+\alpha}\left[\max\left\{\|p^1\|, \max_{2\leqslant k\leqslant n}\|p^{k-\frac{1}{2}}\|\right\}\right. \\
&\left.+ \frac{2\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)}\max_{1\leqslant k\leqslant n}\|q^k\|\right]^2, \quad 1 \leqslant n \leqslant N.
\end{aligned}$$

The proof ends. □

3.2.4 Convergence of the difference scheme

Theorem 3.2.3. *Suppose* $\{U_i^n, V_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n, v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (3.26)–(3.29) *and the difference scheme* (3.38)–(3.42), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad z_i^n = V_i^n - v_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then it holds

$$\|\delta_x e^n\| \leqslant \sqrt{2Lt_n^\gamma\Gamma(2-\gamma)}\left(c_2 + \frac{2}{\Gamma(3-\gamma)}c_3\right)(\tau^{3-\gamma} + h^2 + \epsilon), \quad 1 \leqslant n \leqslant N.$$

Proof. Subtracting (3.43), (3.40)–(3.42) from (3.33), (3.34), (3.36)–(3.37), respectively, produces the system of error equations as follows:

$$\begin{cases}
\dfrac{1}{\Gamma(1-\alpha)}\left[\hat{a}_0^{(\alpha)}z_i^n - \displaystyle\sum_{k=1}^{n-1}(\hat{a}_{k-1}^{(\alpha)} - \hat{a}_k^{(\alpha)})z_i^{n-k} - \hat{a}_{n-1}^{(\alpha)}z_i^0\right] \\
= \delta_x^2 e_i^n + (r_2)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\
\delta_t e_i^{n-\frac{1}{2}} = z_i^{n-\frac{1}{2}} + (r_3)_i^n, \quad 0 \leqslant i \leqslant M,\ 1 \leqslant n \leqslant N, \\
e_i^0 = 0, \quad z_i^0 = 0, \quad 0 \leqslant i \leqslant M, \\
e_0^n = 0, \quad e_M^n = 0, \quad 1 \leqslant n \leqslant N.
\end{cases}$$

Noticing (3.32) and (3.35), an immediate consequence of Theorem 3.2.2 reads

$$\begin{aligned}
\|\delta_x e^n\|^2 &\leqslant \|\delta_x e^0\|^2 + \left[\frac{\frac{4}{3}t_n^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{t_n}{\Gamma(1-\alpha)}\epsilon\right]\|z^0\|^2 \\
&\quad + 2\Gamma(1-\alpha)t_n^{1+\alpha}\left[\max\left\{\|(r_2)^1\|, \max_{2\leqslant k\leqslant n}\|(r_2)^{k-\frac{1}{2}}\|\right\}\right. \\
&\quad \left. + \frac{2\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)}\max_{1\leqslant k\leqslant n}\|(r_3)^k\|\right]^2 \\
&= 2\Gamma(1-\alpha)t_n^{1+\alpha}\left[\max\left\{\|(r_2)^1\|, \max_{2\leqslant k\leqslant n}\|(r_2)^{k-\frac{1}{2}}\|\right\}\right. \\
&\quad \left. + \frac{2\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)}\max_{1\leqslant k\leqslant n}\|(r_3)^k\|\right]^2 \\
&\leqslant 2\Gamma(1-\alpha)t_n^{1+\alpha}\left[\sqrt{L}\,c_2(\tau^{2-\alpha}+h^2+\epsilon) + \frac{2\hat{a}_0^{(\alpha)}}{\Gamma(1-\alpha)}\sqrt{L}\,c_3\tau^2\right]^2 \\
&= 2\Gamma(1-\alpha)t_n^{1+\alpha}\left[\sqrt{L}\,c_2(\tau^{2-\alpha}+h^2+\epsilon) + \frac{2}{\Gamma(2-\alpha)}\sqrt{L}\,c_3\tau^{2-\alpha}\right]^2 \\
&\leqslant 2\Gamma(1-\alpha)t_n^{1+\alpha}\left[\left(c_2 + \frac{2}{\Gamma(2-\alpha)}c_3\right)\sqrt{L}\,(\tau^{2-\alpha}+h^2+\epsilon)\right]^2, \quad 1\leqslant n\leqslant N.
\end{aligned}$$

Taking the square root on both hand sides of the inequality above arrives at

$$\begin{aligned}
\|\delta_x e^n\| &\leqslant \sqrt{2Lt_n^{1+\alpha}\Gamma(1-\alpha)}\left(c_2 + \frac{2}{\Gamma(2-\alpha)}c_3\right)(\tau^{2-\alpha}+h^2+\epsilon) \\
&= \sqrt{2Lt_n^{\gamma}\Gamma(2-\gamma)}\left(c_2 + \frac{2}{\Gamma(3-\gamma)}c_3\right)(\tau^{3-\gamma}+h^2+\epsilon), \quad 1\leqslant n\leqslant N.
\end{aligned}$$

The proof ends. □

3.3 The fourth-order method in space based on L1 approximation for 1D problem

In this section, we continue to consider the problem (3.1)–(3.3), but another high order difference scheme in space will be developed. Suppose the exact solution $u \in C^{(6,3)}([0,L]\times[0,T])$.

3.3.1 Derivation of the difference scheme

Considering equation (3.1) at the point (x_i, t_n), we have

$${}_0^C D_t^{\gamma} u(x_i,t_n) = u_{xx}(x_i,t_n) + f_i^n, \quad 0\leqslant i\leqslant M,\ 0\leqslant n\leqslant N.$$

Performing the operator $\mathcal{A}$ to both hand sides of the equality above and taking an average on two adjacent time levels arrive at

$$\mathcal{A}\left\{\frac{1}{2}\left[{}_0^C D_t^\gamma u(x_i,t_n) + {}_0^C D_t^\gamma u(x_i,t_{n-1})\right]\right\}$$
$$= \mathcal{A}\left\{\frac{1}{2}\left[u_{xx}(x_i,t_n) + u_{xx}(x_i,t_{n-1})\right]\right\} + \mathcal{A}f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.$$

Applying the L1 formula (1.69) to approximate the time-fractional derivative in the equality above, it follows from Theorem 1.6.2 and Lemma 2.1.3 that

$$\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\mathcal{A}\left\{b_0^{(\gamma)}\delta_t U_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}\left(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)}\right)\delta_t U_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_i\right\}$$
$$= \delta_x^2 U_i^{n-\frac{1}{2}} + \mathcal{A}f_i^{n-\frac{1}{2}} + (r_4)_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{3.63}$$

and there is a positive constant c_4 such that

$$\left|(r_4)_i^{n-\frac{1}{2}}\right| \leqslant c_4(\tau^{3-\gamma} + h^4), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{3.64}$$

where $\{b_l^{(\gamma)}\}$ is defined in (1.64).

Noticing the initial-boundary value conditions (3.2)–(3.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (3.65)\\ U_0^n = \mu(t_n), \quad U_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (3.66)\end{cases}$$

Omitting the small term $(r_4)_i^{n-\frac{1}{2}}$ in (3.63) and replacing the exact solution U_i^n with its numerical one u_i^n produce a difference scheme for (3.1)–(3.3) as follows:

$$\begin{cases} \dfrac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\mathcal{A}\left\{b_0^{(\gamma)}\delta_t u_i^{n-\frac{1}{2}} - \sum\limits_{k=1}^{n-1}\left(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)}\right)\delta_t u_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_i\right\} & \\ = \delta_x^2 u_i^{n-\frac{1}{2}} + \mathcal{A}f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.67)\\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (3.68)\\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (3.69)\end{cases}$$

Denote

$$\eta = \tau^{\gamma-1}\Gamma(3-\gamma).$$

3.3.2 Solvability of the difference scheme

We now proceed to discuss the unique solvability of the difference scheme (3.67)–(3.69). The following theorem is true.

Theorem 3.3.1. *The difference scheme* (3.67)–(3.69) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The value of u^0 is determined by (3.68)–(3.69). Suppose the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n can be obtained from (3.67) and (3.69). To show its unique solvability, it suffices to prove the corresponding homogeneous one

$$\begin{cases} \dfrac{1}{\eta\tau}\mathcal{A}u_i^n = \dfrac{1}{2}\delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (3.70)\\ u_0^n = u_M^n = 0 & (3.71)\end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (3.70) with u^n yields

$$\frac{1}{\eta\tau}(\mathcal{A}u^n, u^n) = \frac{1}{2}(\delta_x^2 u^n, u^n). \tag{3.72}$$

Noticing (3.71), it follows from the summation by parts and Lemma 2.1.1 that

$$\begin{aligned}(\mathcal{A}u^n, u^n) &= \left(\left(\mathcal{I} + \frac{h^2}{12}\delta_x^2\right)u^n, u^n\right)\\ &= \|u^n\|^2 - \frac{h^2}{12}\|\delta_x u^n\|^2 \geqslant \frac{2}{3}\|u^n\|^2.\end{aligned}$$

Substituting this result into (3.72) arrives at

$$\frac{2}{3}\cdot\frac{1}{\eta\tau}\|u^n\|^2 \leqslant -\frac{1}{2}\|\delta_x u^n\|^2 \leqslant 0,$$

thus $\|u^n\|^2 = 0$. It follows $u^n = 0$ from (3.71).

By the principle of induction, the theorem is true. The proof ends. □

3.3.3 Stability of the difference scheme

The stability analysis on the difference scheme (3.67)–(3.69) will be carried out in this subsection.

Theorem 3.3.2. *Suppose* $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \dfrac{1}{\eta}\mathcal{A}\left\{b_0^{(\gamma)}\delta_t v_i^{n-\frac{1}{2}} - \sum\limits_{k=1}^{n-1}(b_{n-k-1} - b_{n-k}^{(\gamma)})\delta_t v_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_i\right\} & \\ = \delta_x^2 v_i^{n-\frac{1}{2}} + g_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.73)\\ v_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (3.74)\\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N. & (3.75)\end{cases}$$

Then it holds

$$\|\delta_x v^n\|_A^2 \leqslant \|\delta_x v^0\|_A^2 + \frac{t_n^{2-\gamma}}{\Gamma(3-\gamma)}\|\mathcal{A}\psi\|^2 + \Gamma(2-\gamma)t_n^{\gamma-1}\,\tau\sum_{k=1}^{n}\|g^{k-\frac{1}{2}}\|^2,$$
$$1 \leqslant n \leqslant N, \tag{3.76}$$

where

$$\|\mathcal{A}\psi\|^2 = h\sum_{i=1}^{M-1}(\mathcal{A}\psi_i)^2, \quad \|g^{k-\frac{1}{2}}\|^2 = h\sum_{i=1}^{M-1}(g_i^{k-\frac{1}{2}})^2.$$

Proof. Making the inner product on both hand sides of (3.73) with $\eta\mathcal{A}\delta_t v^{n-\frac{1}{2}}$, it follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}
&b_0^{(\gamma)}\|\mathcal{A}\delta_t v^{n-\frac{1}{2}}\|^2 \\
&= \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\mathcal{A}\delta_t v^{k-\frac{1}{2}}, \mathcal{A}\delta_t v^{n-\frac{1}{2}}) \\
&\quad + b_{n-1}^{(\gamma)}(\mathcal{A}\psi, \mathcal{A}\delta_t v^{n-\frac{1}{2}}) + \eta(\delta_x^2 v^{n-\frac{1}{2}}, \mathcal{A}\delta_t v^{n-\frac{1}{2}}) + \eta(g^{n-\frac{1}{2}}, \mathcal{A}\delta_t v^{n-\frac{1}{2}}) \\
&\leqslant \frac{1}{2}\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\|\mathcal{A}\delta_t v^{k-\frac{1}{2}}\|^2 + \|\mathcal{A}\delta_t v^{n-\frac{1}{2}}\|^2) \\
&\quad + \frac{1}{2}b_{n-1}^{(\gamma)}(\|\mathcal{A}\psi\|^2 + \|\mathcal{A}\delta_t v^{n-\frac{1}{2}}\|^2) + \eta(\delta_x^2 v^{n-\frac{1}{2}}, \mathcal{A}\delta_t v^{n-\frac{1}{2}}) \\
&\quad + \eta(g^{n-\frac{1}{2}}, \mathcal{A}\delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N,
\end{aligned}$$

which can be simplified to

$$\begin{aligned}
&b_0^{(\gamma)}\|\mathcal{A}\delta_t v^{n-\frac{1}{2}}\|^2 \\
&\leqslant \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\|\mathcal{A}\delta_t v^{k-\frac{1}{2}}\|^2 + b_{n-1}^{(\gamma)}\|\mathcal{A}\psi\|^2 \\
&\quad + 2\eta(\delta_x^2 v^{n-\frac{1}{2}}, \mathcal{A}\delta_t v^{n-\frac{1}{2}}) + 2\eta(g^{n-\frac{1}{2}}, \mathcal{A}\delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N.
\end{aligned} \tag{3.77}$$

Applying the summation by parts and noticing (3.75), we have

$$\begin{aligned}
-(\delta_x^2 v^{n-\frac{1}{2}}, \mathcal{A}\delta_t v^{n-\frac{1}{2}}) &= (\delta_t v^{n-\frac{1}{2}}, v^{n-\frac{1}{2}})_{1,A} \\
&= \frac{1}{2\tau}[(v^n, v^n)_{1,A} - (v^{n-1}, v^{n-1})_{1,A}] \\
&= \frac{1}{2\tau}(\|\delta_x v^n\|_A^2 - \|\delta_x v^{n-1}\|_A^2).
\end{aligned} \tag{3.78}$$

The substitution of (3.78) into (3.77) yields

$$\begin{aligned}
&\sum_{k=1}^{n} b_{n-k}^{(\gamma)}\|\mathcal{A}\delta_t v^{k-\frac{1}{2}}\|^2 + \frac{\eta}{\tau}(\|\delta_x v^n\|_A^2 - \|\delta_x v^{n-1}\|_A^2) \\
&\leqslant \sum_{k=1}^{n-1} b_{n-k-1}^{(\gamma)}\|\mathcal{A}\delta_t v^{k-\frac{1}{2}}\|^2 + b_{n-1}^{(\gamma)}\|\mathcal{A}\psi\|^2 + 2\eta(g^{n-\frac{1}{2}}, \mathcal{A}\delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N,
\end{aligned}$$

which can be rearranged as

$$\begin{aligned}&\|\delta_x v^n\|_A^2 + \frac{\tau}{\eta}\sum_{k=1}^{n} b_{n-k}^{(\gamma)}\|\mathcal{A}\delta_t v^{k-\frac{1}{2}}\|^2\\ \leqslant\ &\|\delta_x v^{n-1}\|_A^2 + \frac{\tau}{\eta}\sum_{k=1}^{n-1} b_{n-k-1}^{(\gamma)}\|\mathcal{A}\delta_t v^{k-\frac{1}{2}}\|^2\\ &+ \frac{\tau}{\eta} b_{n-1}^{(\gamma)}\|\mathcal{A}\psi\|^2 + 2\tau(g^{n-\frac{1}{2}}, \mathcal{A}\delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{3.79}$$

Let

$$G^0 = \|\delta_x v^0\|_A^2, \quad G^n = \|\delta_x v^n\|_A^2 + \frac{\tau}{\eta}\sum_{k=1}^{n} b_{n-k}^{(\gamma)}\|\mathcal{A}\delta_t v^{k-\frac{1}{2}}\|^2, \quad 1 \leqslant n \leqslant N.$$

It follows from (3.79) that

$$G^n \leqslant G^{n-1} + \frac{\tau}{\eta} b_{n-1}^{(\gamma)}\|\mathcal{A}\psi\|^2 + 2\tau(g^{n-\frac{1}{2}}, \mathcal{A}\delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N.$$

The recursive process will lead to

$$\begin{aligned}G^n &\leqslant G^0 + \frac{\tau}{\eta}\sum_{k=0}^{n-1} b_k^{(\gamma)}\|\mathcal{A}\psi\|^2 + 2\tau\sum_{k=1}^{n}(g^{k-\frac{1}{2}}, \mathcal{A}\delta_t v^{k-\frac{1}{2}})\\ &\leqslant G^0 + \frac{\tau}{\eta}\sum_{k=0}^{n-1} b_k^{(\gamma)}\|\mathcal{A}\psi\|^2\\ &\quad + \tau\sum_{k=1}^{n}\left(\frac{\eta}{b_{n-k}^{(\gamma)}}\|g^{k-\frac{1}{2}}\|^2 + \frac{b_{n-k}^{(\gamma)}}{\eta}\|\mathcal{A}\delta_t v^{k-\frac{1}{2}}\|^2\right), \quad 1 \leqslant n \leqslant N,\end{aligned}$$

that is,

$$\|\delta_x v^n\|_A^2 \leqslant \|\delta_x v^0\|_A^2 + \frac{\tau}{\eta}\sum_{k=0}^{n-1} b_k^{(\gamma)}\|\mathcal{A}\psi\|^2 + \tau\sum_{k=1}^{n}\frac{\eta}{b_{n-k}^{(\gamma)}}\|g^{k-\frac{1}{2}}\|^2, \quad 1 \leqslant n \leqslant N. \tag{3.80}$$

By (3.20) and (3.22), it follows (3.76) from (3.80). The proof ends. □

3.3.4 Convergence of the difference scheme

We now present the error analysis of the difference scheme (3.67)–(3.69).

Theorem 3.3.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (3.1)–(3.3) *and the difference scheme* (3.67)–(3.69), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then it holds

$$\|e^n\|_\infty \leqslant \frac{L}{4}\sqrt{6T^\gamma\Gamma(2-\gamma)}\, c_4(\tau^{3-\gamma}+h^4), \quad 1 \leqslant n \leqslant N.$$

Proof. The subtraction of (3.67)–(3.69) from (3.63), (3.65)–(3.66), respectively, produces the system of error equations as follows:

$$\begin{cases} \frac{1}{\eta}\mathcal{A}\left\{b_0^{(\gamma)}\delta_t e_i^{n-\frac{1}{2}} - \sum\limits_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t e_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\cdot 0\right\} \\ = \delta_x^2 e_i^{n-\frac{1}{2}} + (r_4)_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (3.64), the application of Theorem 3.3.2 yields

$$\begin{aligned} \|\delta_x e^n\|_A^2 &\leqslant t_n^{\gamma-1}\Gamma(2-\gamma)\tau\sum_{k=1}^{n}\|(r_4)^{k-\frac{1}{2}}\|^2 \\ &\leqslant T^\gamma\Gamma(2-\gamma)Lc_4^2(\tau^{3-\gamma}+h^4)^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Taking the square root on both hand sides of the inequality above, it follows from Lemma 2.1.1 and Lemma 2.1.2 that

$$\begin{aligned} \|e^n\|_\infty &\leqslant \frac{\sqrt{L}}{2}\|\delta_x e^n\| \leqslant \frac{\sqrt{L}}{2}\cdot\sqrt{\frac{3}{2}}\|\delta_x e^n\|_A \\ &\leqslant \frac{L}{4}\sqrt{6T^\gamma\Gamma(2-\gamma)}\, c_4(\tau^{3-\gamma}+h^4), \quad 1 \leqslant n \leqslant N. \end{aligned}$$

The proof ends. □

3.4 The difference method based on L2-1_σ approximation for 1D problem

Consider the following problem of the time-fractional wave equation:

$$\begin{cases} {}_0^C D_t^\gamma u(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], & (3.81) \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in [0,L], & (3.82) \\ u(0,t) = 0, \quad u(L,t) = 0, \quad t \in (0,T], & (3.83) \end{cases}$$

where $\gamma \in (1,2)$, the functions f, φ, ψ are all given and $\varphi(0) = 0$, $\varphi(L) = 0$, $\psi(0) = 0$, $\psi(L) = 0$. Suppose the exact solution $u \in C^{(4,4)}([0,L]\times[0,T])$.

In this section, we will use the L2-1_σ approximation to establish a temporal second-order difference scheme.

3.4.1 Derivation of the difference scheme

At first, a useful lemma is listed.

Lemma 3.4.1. [81]

(I) *Suppose* $f \in C^2[t_k, t_{k+1}]$, *then it holds*

$$\frac{1}{2}[f(t_k) + f(t_{k+1})]$$
$$= f(t_{k+\frac{1}{2}}) + \frac{\tau^2}{8}\int_0^1 \left[f_{tt}\left(t_{k+\frac{1}{2}} - \frac{1}{2}\tau s\right) + f_{tt}\left(t_{k+\frac{1}{2}} + \frac{1}{2}\tau s\right)\right](1-s)\mathrm{d}s. \tag{3.84}$$

(II) *Suppose* $f \in C^2[t_k, t_{k+1}]$ *and* $\sigma \in (0,1)$, *thus we have*

$$(1-\sigma)f(t_k) + \sigma f(t_{k+1})$$
$$= f(t_{k+\sigma}) + \tau^2 \int_0^1 [\sigma(1-\sigma)^2 f_{tt}(t_{k+\sigma} + (1-\sigma)\tau s)$$
$$+ (1-\sigma)\sigma^2 f_{tt}(t_{k+\sigma} - \sigma\tau s)](1-s)\mathrm{d}s. \tag{3.85}$$

(III) *Suppose* $f \in C^3[t_k, t_{k+1}]$, *then one has*

$$\frac{1}{\tau}[f(t_{k+1}) - f(t_k)] = f'(t_{k+\frac{1}{2}})$$
$$+ \frac{\tau^2}{16}\int_0^1 \left[f_{ttt}\left(t_{k+\frac{1}{2}} + \frac{1}{2}\tau s\right) + f_{ttt}\left(t_{k+\frac{1}{2}} - \frac{1}{2}\tau s\right)\right](1-s)^2\mathrm{d}s. \tag{3.86}$$

(IV) *Suppose* $f \in C^3[t_k, t_{k+1}]$ *and* $\sigma \in (0,1)$, *then one has*

$$\frac{1}{2\tau}[(2\sigma+1)f(t_{k+1}) - 4\sigma f(t_k) + (2\sigma-1)f(t_{k-1})]$$
$$= f'(t_{k+\sigma}) + \frac{\tau^2}{4}\left[(2\sigma+1)(1-\sigma)^3\int_0^1 f_{ttt}(t_{k+\sigma} + (1-\sigma)\tau s)(1-s)^2\mathrm{d}s\right.$$
$$+ 4\sigma^4\int_0^1 f_{ttt}(t_{k+\sigma} - \sigma\tau s)(1-s)^2\mathrm{d}s$$
$$\left. - (2\sigma-1)(1+\sigma)^3\int_0^1 f_{ttt}(t_{k+\sigma} - (1+\sigma)\tau s)(1-s)^2\mathrm{d}s\right]. \tag{3.87}$$

Proof. (I) Expanding $f(t_k)$ and $f(t_{k+1})$ at $t = t_{k+\frac{1}{2}}$ to the second-order derivative term with the help of Taylor expansion with the integral remainder, and averaging the results yield (3.84).

(II) By the same process as (I), but expanding is performed at the point $t = t_{k+\sigma}$ to the second-order derivative term. Multiplying the results by $1 - \sigma$ and σ, respectively, and then adding the results together will give (3.85).

(III) Expand $f(t_k)$ and $f(t_{k+1})$ at $t = t_{k+\frac{1}{2}}$ with the help of Taylor expansion with the integral remainder to the third-order derivative term. The equality (3.86) can be obtained.

(IV) Expand $f(t_{k+1})$, $f(t_k)$ and $f(t_{k-1})$ at $t = t_{k+\sigma}$ to the third-order derivative term with the help of Taylor expansion with the integral remainder. Then multiplying the obtained results by $2\sigma + 1$, -4σ and $2\sigma - 1$, respectively, and summing up the results will lead to (3.87). □

For any $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ defined on $\Omega_h \times \Omega_\tau$, introduce the following notation:

$$D_{\bar{t}} u_i^n = \frac{1}{2\tau}\left[(2\sigma+1)u_i^n - 4\sigma u_i^{n-1} + (2\sigma-1)u_i^{n-2}\right], \quad n \geqslant 2.$$

Let

$$v(x,t) = u_t(x,t), \quad \alpha = \gamma - 1, \quad \sigma = 1 - \frac{\alpha}{2}, \quad s = \tau^\alpha \Gamma(2-\alpha).$$

Thus, the problem (3.81)–(3.83) is equivalent to

$$\begin{cases} {}_0^C D_t^\alpha v(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], & (3.88)\\ u_t(x,t) = v(x,t), \quad x \in [0,L],\ t \in (0,T], & (3.89)\\ u(x,0) = \varphi(x), \quad v(x,0) = \psi(x), \quad x \in [0,L], & (3.90)\\ u(0,t) = 0, \quad u(L,t) = 0, \quad t \in (0,T]. & (3.91) \end{cases}$$

Denote

$$\begin{cases} U_i^n = u(x_i,t_n), \quad V_i^n = v(x_i,t_n), \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,\\ \varphi_i = \varphi(x_i), \quad \psi_i = \psi(x_i), \quad 0 \leqslant i \leqslant M. \end{cases}$$

Considering (3.88) at the point $(x_i, t_{n-1+\sigma})$, we have

$${}_0^C D_t^\alpha v(x_i,t_{n-1+\sigma}) = u_{xx}(x_i,t_{n-1+\sigma}) + f_i^{n-1+\sigma},$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{3.92}$$

where $f_i^{n-1+\sigma} = f(x_i, t_{n-1+\sigma})$.

Applying the L2-1_σ formula (1.81) to approximate the Caputo derivative gives

$${}_0^C D_t^\alpha v(x_i,t_{n-1+\sigma}) = \frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(V_i^{n-k} - V_i^{n-k-1}) + O(\tau^{3-\alpha}). \tag{3.93}$$

Combining the linear interpolation (3.85) with the second-order central difference quotient (Lemma 2.1.3) for the spatial second-order derivative, we have

$$u_{xx}(x_i,t_{n-1+\sigma}) = \sigma u_{xx}(x_i,t_n) + (1-\sigma)u_{xx}(x_i,t_{n-1}) + O(\tau^2)$$

$$= \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + O(h^2) + O(\tau^2). \tag{3.94}$$

Substituting (3.93) and (3.94) into (3.92) arrives at

$$\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(V_i^{n-k} - V_i^{n-k-1}) = \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + f_i^{n-1+\sigma} + (r_5)_i^n,$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{3.95}$$

and there exists a positive constant c_5 such that

$$|(r_5)_i^n| \leqslant c_5(\tau^2 + h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{3.96}$$

Considering (3.89) at the points $(x_i, t_{\frac{1}{2}})$ and $(x_i, t_{n-1+\sigma})$, respectively, we have

$$\begin{cases} u_t(x_i, t_{\frac{1}{2}}) = v(x_i, t_{\frac{1}{2}}), \quad 0 \leqslant i \leqslant M, \\ u_t(x_i, t_{n-1+\sigma}) = v(x_i, t_{n-1+\sigma}), \quad 0 \leqslant i \leqslant M, \quad 2 \leqslant n \leqslant N. \end{cases}$$

It follows from Lemma 3.4.1 that

$$\begin{cases} \delta_t U_i^{\frac{1}{2}} = V_i^{\frac{1}{2}} + (r_6)_i^1, \quad 0 \leqslant i \leqslant M, & (3.97) \\ D_{\bar{t}} U_i^n = \sigma V_i^n + (1-\sigma)V_i^{n-1} + (r_6)_i^n, \quad 0 \leqslant i \leqslant M, \quad 2 \leqslant n \leqslant N, & (3.98) \end{cases}$$

and there exists a positive constant c_6 such that

$$|\delta_x^2 (r_6)_i^n| \leqslant c_6\tau^2, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{3.99}$$

In addition, from (3.89) and (3.91), we have

$$(r_6)_0^n = 0, \quad (r_6)_M^n = 0, \quad 1 \leqslant n \leqslant N. \tag{3.100}$$

Noticing the initial-boundary value conditions (3.90)–(3.91), we have

$$\begin{cases} U_i^0 = \varphi_i, \quad V_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, & (3.101) \\ U_0^n = 0, \quad U_M^n = 0, \quad 1 \leqslant n \leqslant N. & (3.102) \end{cases}$$

Omitting the small term $(r_5)_i^n$ and $(r_6)_i^n$ in (3.95), (3.97) and (3.98) and replacing the exact solution $\{U_i^n, V_i^n\}$ with its numerical one $\{u_i^n, v_i^n\}$ produce the difference scheme for (3.88)–(3.91) as follows:

$$\begin{cases} \frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(v_i^{n-k} - v_i^{n-k-1}) = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma}, & \\ \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.103) \\ \delta_t u_i^{\frac{1}{2}} = v_i^{\frac{1}{2}}, \quad 0 \leqslant i \leqslant M, & (3.104) \\ D_{\bar{t}} u_i^n = \sigma v_i^n + (1-\sigma)v_i^{n-1}, \quad 0 \leqslant i \leqslant M, \quad 2 \leqslant n \leqslant N, & (3.105) \\ u_i^0 = \varphi_i, \quad v_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, & (3.106) \\ u_0^n = 0, \quad u_M^n = 0, \quad 1 \leqslant n \leqslant N. & (3.107) \end{cases}$$

On each time level, only one tridiagonal system of linear algebraic equations need be solved. See the process of the proof for Theorem 3.4.1.

Remark 3.4.1. By Lemma 3.4.1, it is easy to write $(r_6)_i^n$ in the integral expression, and then obtain (3.99).

Remark 3.4.2. For the nonhomogeneous boundary value problem:

$$\begin{cases} {}_0^C D_t^\gamma u(x,t) = u_{xx}(x,t) + f(x,t), & x \in (0,L),\ t \in (0,T], \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), & x \in [0,L], \\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), & t \in (0,T], \end{cases}$$

the equality (3.100) is in general not valid. In this case, let $v(x,t) = u_t(x,t)$, then the problem above can be written as the following equivalent one:

$$\begin{cases} {}_0^C D_t^\alpha v(x,t) = u_{xx}(x,t) + f(x,t), & x \in (0,L),\ t \in (0,T], \\ u_{txx}(x,t) = v_{xx}(x,t), \quad x \in (0,L), & t \in (0,T], \\ u(x,0) = \varphi(x), \quad v(x,0) = \psi(x), & x \in [0,L], \\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), & t \in (0,T], \\ v(0,t) = \mu'(t), \quad v(L,t) = \nu'(t), & t \in (0,T], \end{cases}$$

for which we construct the following difference scheme:

$$\begin{cases} \dfrac{1}{s}\displaystyle\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(v_i^{n-k} - v_i^{n-k-1}) = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma}, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ \delta_x^2 \delta_t u_i^{\frac{1}{2}} = \delta_x^2 v_i^{\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1, \\ \delta_x^2 D_{\bar t} u_i^n = \delta_x^2(\sigma v_i^n + (1-\sigma)v_i^{n-1}), \quad 1 \leqslant i \leqslant M-1, \quad 2 \leqslant n \leqslant N, \\ u_i^0 = \varphi_i, \quad v_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 1 \leqslant n \leqslant N, \\ v_0^n = \mu'(t_n), \quad v_M^n = \nu'(t_n), \quad 1 \leqslant n \leqslant N. \end{cases}$$

The interested readers can refer to [81].

Another way is firstly to make the boundary value conditions homogeneous and then construct the corresponding difference scheme.

3.4.2 Solvability of the difference scheme

Theorem 3.4.1. *The difference scheme* (3.103)–(3.107) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n), \quad v^n = (v_0^n, v_1^n, \ldots, v_M^n).$$

(I) The value of $\{u^0, v^0\}$ is determined by (3.106).

(II) It follows from (3.104) that

$$v_i^1 = 2\delta_t u_i^{\frac{1}{2}} - v_i^0, \quad 0 \leqslant i \leqslant M. \tag{3.108}$$

Substituting (3.108) into (3.103), and noticing (3.107), we obtain the linear system in u^1 as follows:

$$\begin{cases} \frac{1}{s} c_0^{(1,\alpha)} (2\delta_t u_i^{\frac{1}{2}} - 2v_i^0) = \sigma\delta_x^2 u_i^1 + (1-\sigma)\delta_x^2 u_i^0 + f_i^\sigma, \quad 1 \leqslant i \leqslant M-1, & (3.109) \\ u_0^1 = 0, \quad u_M^1 = 0. & (3.110) \end{cases}$$

Considering its homogeneous one, we have

$$\begin{cases} \frac{2}{s\tau} c_0^{(1,\alpha)} u_i^1 = \sigma\delta_x^2 u_i^1, \quad 1 \leqslant i \leqslant M-1, & (3.111) \\ u_0^1 = 0, \quad u_M^1 = 0. & (3.112) \end{cases}$$

Making the inner product on both hand sides of (3.111) with u^1 and noticing (3.112) produce

$$\frac{2}{s\tau} c_0^{(1,\alpha)} \|u^1\|^2 + \sigma\|\delta_x u^1\|^2 = 0,$$

which implies $u^1 = 0$, hence the system (3.109)–(3.110) has a unique solution. Once u^1 is obtained, v^1 is followed from (3.108).

(III) Suppose the value of $\{u^0, v^0, u^1, v^1, \ldots, u^{n-1}, v^{n-1}\}$ has been uniquely determined, then it follows from (3.105) that

$$v_i^n = \frac{1}{\sigma}(D_{\bar{t}} u_i^n - (1-\sigma)v_i^{n-1}), \quad 0 \leqslant i \leqslant M. \tag{3.113}$$

Substituting (3.113) into (3.103), noticing (3.107), one can obtain the linear system in u^n as follows:

$$\begin{cases} \frac{1}{s}\left\{ c_0^{(n,\alpha)} \left[\frac{1}{\sigma}(D_{\bar{t}} u_i^n - (1-\sigma)v_i^{n-1}) - v_i^{n-1} \right] + \sum_{k=1}^{n-1} c_k^{(n,\alpha)} (v_i^{n-k} - v_i^{n-k-1}) \right\} \\ = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma}, \quad 1 \leqslant i \leqslant M-1, & (3.114) \\ u_0^n = 0, \quad u_M^n = 0. & (3.115) \end{cases}$$

To show its unique solvability, it suffices to prove the corresponding homogeneous one

$$\begin{cases} \frac{1}{s} c_0^{(n,\alpha)} \cdot \frac{1}{\sigma} \cdot \frac{2\sigma+1}{2\tau} u_i^n = \sigma\delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (3.116) \\ u_0^n = 0, \quad u_M^n = 0 & (3.117) \end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (3.116) with u^n yields

$$\frac{2\sigma+1}{2s\tau\sigma}c_0^{(n,\alpha)}\|u^n\|^2+\sigma\|\delta_x u^n\|^2=0,$$

which implies $u^n=0$, and the system (3.114)–(3.115) has a unique solution u^n. Once u^n is obtained, v^n is followed from (3.113).

By the principle of induction, the theorem is true. The proof ends. □

3.4.3 Stability of the difference scheme

Before analyzing the stability of the difference scheme, we give two lemmas.

Lemma 3.4.2. *Suppose $u^0,u^1,\ldots,u^N\in\mathring{\mathcal{U}}_h$, $(\cdot,\cdot)$ is an inner product on $\mathring{\mathcal{U}}_h$ and $\|\cdot\|$ is the induced norm. Denote*

$$E^n\equiv(2\sigma+1)\|u^n\|^2-(2\sigma-1)\|u^{n-1}\|^2+(2\sigma^2+\sigma-1)\|u^n-u^{n-1}\|^2,\quad n\geqslant 1.$$

Then we have

$$(D_{\bar t}u^n,\sigma u^n+(1-\sigma)u^{n-1})\geqslant\frac{1}{4\tau}(E^n-E^{n-1}),\quad n\geqslant 2$$

and

$$E^n\geqslant\frac{1}{\sigma}\|u^n\|^2,\quad n\geqslant 1.$$

Proof. The operator $D_{\bar t}u^n$ can be rewritten as

$$D_{\bar t}u^n=2\sigma\frac{u^n-u^{n-1}}{\tau}-(2\sigma-1)\frac{u^n-u^{n-2}}{2\tau},$$

or

$$D_{\bar t}u^n=\left(\sigma+\frac{1}{2}\right)\frac{u^n-u^{n-1}}{\tau}-\left(\sigma-\frac{1}{2}\right)\frac{u^{n-1}-u^{n-2}}{\tau}.$$

Noticing the identities $(a-b)a=\frac{1}{2}[a^2-b^2+(a-b)^2]$, $(a-b)b=\frac{1}{2}[a^2-b^2-(a-b)^2]$, we have

$$\begin{aligned}&(D_{\bar t}u^n,\sigma u^n+(1-\sigma)u^{n-1})\\ &=\sigma\left[2\sigma\left(\frac{u^n-u^{n-1}}{\tau},u^n\right)-(2\sigma-1)\left(\frac{u^n-u^{n-2}}{2\tau},u^n\right)\right]\\ &\quad+(1-\sigma)\left[\left(\sigma+\frac{1}{2}\right)\left(\frac{u^n-u^{n-1}}{\tau},u^{n-1}\right)-\left(\sigma-\frac{1}{2}\right)\left(\frac{u^{n-1}-u^{n-2}}{\tau},u^{n-1}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \sigma\Big[\frac{\sigma}{\tau}(\|u^n\|^2 - \|u^{n-1}\|^2 + \|u^n - u^{n-1}\|^2) \\
&\quad - \frac{2\sigma-1}{4\tau}(\|u^n\|^2 - \|u^{n-2}\|^2 + \|u^n - u^{n-2}\|^2)\Big] \\
&\quad + (1-\sigma)\Big[\frac{\sigma+\frac{1}{2}}{2\tau}(\|u^n\|^2 - \|u^{n-1}\|^2 - \|u^n - u^{n-1}\|^2) \\
&\quad - \frac{\sigma-\frac{1}{2}}{2\tau}(\|u^{n-1}\|^2 - \|u^{n-2}\|^2 + \|u^{n-1} - u^{n-2}\|^2)\Big] \\
&\geqslant \sigma\Big[\frac{\sigma}{\tau}(\|u^n\|^2 - \|u^{n-1}\|^2 + \|u^n - u^{n-1}\|^2) \\
&\quad - \frac{2\sigma-1}{4\tau}(\|u^n\|^2 - \|u^{n-2}\|^2 + 2\|u^n - u^{n-1}\|^2 + 2\|u^{n-1} - u^{n-2}\|^2)\Big] \\
&\quad + (1-\sigma)\Big[\frac{\sigma+\frac{1}{2}}{2\tau}(\|u^n\|^2 - \|u^{n-1}\|^2 - \|u^n - u^{n-1}\|^2) \\
&\quad - \frac{\sigma-\frac{1}{2}}{2\tau}(\|u^{n-1}\|^2 - \|u^{n-2}\|^2 + \|u^{n-1} - u^{n-2}\|^2)\Big] \\
&= \frac{2\sigma+1}{4\tau}(\|u^n\|^2 - \|u^{n-1}\|^2) - \frac{2\sigma-1}{4\tau}(\|u^{n-1}\|^2 - \|u^{n-2}\|^2) \\
&\quad + \frac{2\sigma^2+\sigma-1}{4\tau}(\|u^n - u^{n-1}\|^2 - \|u^{n-1} - u^{n-2}\|^2) \\
&= \frac{1}{4\tau}(E^n - E^{n-1}), \quad n \geqslant 2.
\end{aligned}
$$

Further, by the Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
E^n &= (2\sigma^2+3\sigma)\|u^n\|^2 + (2\sigma^2-\sigma)\|u^{n-1}\|^2 - 2(2\sigma-1)(\sigma+1)(u^n, u^{n-1}) \\
&\geqslant (2\sigma^2+3\sigma)\|u^n\|^2 + (2\sigma^2-\sigma)\|u^{n-1}\|^2 - \Big[(2\sigma-1)\sigma\|u^{n-1}\|^2 \\
&\quad + \frac{(2\sigma-1)(\sigma+1)^2}{\sigma}\|u^n\|^2\Big] \\
&= \frac{1}{\sigma}\|u^n\|^2.
\end{aligned}
$$

The proof ends. □

Lemma 3.4.3. *The following two inequalities are valid:*

$$
\sum_{m=2}^{n}(c_{m-2}^{(m,\alpha)} - c_{m-2}^{(m-1,\alpha)}) \leqslant \frac{1-\alpha}{12}\left(\frac{\alpha}{\sigma}+1\right)\sigma^{-\alpha},
$$

$$
\sum_{m=2}^{n} c_{m-1}^{(m,\alpha)} \leqslant \frac{3}{2}(n-1+\sigma)^{1-\alpha}.
$$

Proof. (I) Denote

$$A_n \equiv \sum_{m=2}^{n} \left(c_{m-2}^{(m,\alpha)} - c_{m-2}^{(m-1,\alpha)}\right).$$

When $n = 2$:

$$A_n = c_0^{(2,\alpha)} - c_0^{(1,\alpha)} = \frac{1}{2-\alpha}\left[(1+\sigma)^{2-\alpha} - \sigma^{2-\alpha}\right] - \frac{1}{2}\left[(1+\sigma)^{1-\sigma} + \sigma^{1-\sigma}\right].$$

When $n \geqslant 3$:

$$\begin{aligned} A_n = & \sum_{m=2}^{n} \left[\frac{1}{2-\alpha}\left((m-1+\sigma)^{2-\alpha} - 2(m-2+\sigma)^{2-\alpha} + (m-3+\sigma)^{2-\alpha}\right) \right. \\ & - \frac{1}{2}\left((m-1+\sigma)^{1-\alpha} - 2(m-2+\sigma)^{1-\alpha} + (m-3+\sigma)^{1-\alpha}\right) \\ & + \frac{1}{2-\alpha}\left((m-2+\sigma)^{2-\alpha} - (m-3+\sigma)^{2-\alpha}\right) \\ & \left. - \frac{1}{2}\left(3(m-2+\sigma)^{1-\alpha} - (m-3+\sigma)^{1-\alpha}\right) \right] \\ = & \sum_{m=2}^{n} \left[\frac{1}{2-\alpha}\left((m-1+\sigma)^{2-\alpha} - (m-2+\sigma)^{2-\alpha}\right) \right. \\ & \left. - \frac{1}{2}\left((m-1+\sigma)^{1-\alpha} + (m-2+\sigma)^{1-\alpha}\right) \right]. \end{aligned}$$

It is easy to see that the equality above is also true for $n = 2$.
Let

$$f(x) = (x+\sigma)^{1-\alpha},$$

then we have

$$\begin{aligned} A_n &= \sum_{m=2}^{n} \left\{ \int_{m-2}^{m-1} f(x)\mathrm{d}x - \frac{1}{2}\left[f(m-1) + f(m-2)\right] \right\} \\ &= \sum_{m=2}^{n} \left(-\frac{1}{12} f''(\xi_m) \right), \quad \xi_m \in (m-2, m-1). \end{aligned}$$

A direct calculation yields

$$-f''(x) = (1-\alpha)\alpha(x+\sigma)^{-\alpha-1}.$$

Then it follows that

$$A_n = \frac{1}{12}\left[-f''(\xi_2) + \sum_{m=3}^{n} \left(-f''(\xi_m)\right) \right]$$

$$
\begin{aligned}
&= \frac{1}{12}(1-\alpha)\alpha\left[(\xi_2+\sigma)^{-\alpha-1} + \sum_{m=3}^{n}(\xi_m+\sigma)^{-\alpha-1}\right]\\
&\leqslant \frac{1}{12}(1-\alpha)\alpha\left[\sigma^{-\alpha-1} + \sum_{m=3}^{n}(m-2+\sigma)^{-\alpha-1}\right]\\
&\leqslant \frac{1}{12}(1-\alpha)\alpha\left[\sigma^{-\alpha-1} + \sum_{m=3}^{n}\int_{m-3}^{m-2}(x+\sigma)^{-\alpha-1}\mathrm{d}x\right]\\
&= \frac{1}{12}(1-\alpha)\alpha\left[\sigma^{-\alpha-1} + \int_{0}^{n-2}(x+\sigma)^{-\alpha-1}\mathrm{d}x\right]\\
&= \frac{1}{12}(1-\alpha)\alpha\left[\sigma^{-\alpha-1} + \frac{\sigma^{-\alpha}-(n-2+\sigma)^{-\alpha}}{\alpha}\right]\\
&\leqslant \frac{1}{12}(1-\alpha)\alpha\left[\sigma^{-\alpha-1} + \frac{\sigma^{-\alpha}}{\alpha}\right]\\
&= \frac{1-\alpha}{12}\sigma^{-\alpha}\left(\frac{\alpha}{\sigma}+1\right).
\end{aligned}
$$

(II) Denote

$$
B_n \equiv \sum_{m=2}^{n} c_{m-1}^{(m,\alpha)}.
$$

Then

$$
\begin{aligned}
B_n = &\sum_{m=2}^{n}\left[\frac{1}{2}\Big(3(m-1+\sigma)^{1-\alpha}-(m-2+\sigma)^{1-\alpha}\Big)\right.\\
&\left.-\frac{1}{2-\alpha}\Big((m-1+\sigma)^{2-\alpha}-(m-2+\sigma)^{2-\alpha}\Big)\right]\\
=&\sum_{m=2}^{n}(m-1+\sigma)^{1-\alpha}+\frac{1}{2}[(n-1+\sigma)^{1-\alpha}-\sigma^{1-\alpha}]\\
&-\frac{1}{2-\alpha}[(n-1+\sigma)^{2-\alpha}-\sigma^{2-\alpha}]\\
\leqslant&\sum_{m=2}^{n-1}\int_{m-1}^{m}(x+\sigma)^{1-\alpha}\mathrm{d}x+(n-1+\sigma)^{1-\alpha}\\
&+\frac{1}{2}[(n-1+\sigma)^{1-\alpha}-\sigma^{1-\alpha}]-\frac{1}{2-\alpha}[(n-1+\sigma)^{2-\alpha}-\sigma^{2-\alpha}]\\
=&\frac{(n-1+\sigma)^{2-\alpha}-(1+\sigma)^{2-\alpha}}{2-\alpha}+\frac{1}{2}[3(n-1+\sigma)^{1-\alpha}-\sigma^{1-\alpha}]\\
&-\frac{1}{2-\alpha}[(n-1+\sigma)^{2-\alpha}-\sigma^{2-\alpha}]\\
\leqslant&\frac{1}{2}[3(n-1+\sigma)^{1-\alpha}-\sigma^{1-\alpha}]
\end{aligned}
$$

$$\leqslant \frac{3}{2}(n-1+\sigma)^{1-\alpha}.$$

The proof ends. □

Theorem 3.4.2. *Suppose* $\{u^n, v^n \mid 0 \leqslant n \leqslant N\}$ *satisfies*

$$\begin{cases} \frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(v_i^{n-k} - v_i^{n-k-1}) = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + p_i^n, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.118) \\ \delta_t u_i^{\frac{1}{2}} = v_i^{\frac{1}{2}} + q_i^1, \quad 0 \leqslant i \leqslant M, & (3.119) \\ D_{\bar{t}} u_i^n = \sigma v_i^n + (1-\sigma)v_i^{n-1} + q_i^n, \quad 0 \leqslant i \leqslant M,\ 2 \leqslant n \leqslant N, & (3.120) \\ u_i^0 = \varphi_i, \quad v_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, & (3.121) \\ u_0^n = 0, \quad u_M^n = 0, \quad 1 \leqslant n \leqslant N, & (3.122) \end{cases}$$

where $q_0^n = q_M^n = 0\,(1 \leqslant n \leqslant N)$ *and* $\varphi_0 = \varphi_M = \psi_0 = \psi_M = 0$. *Then there exists a constant C such that*

$$\tau\sum_{k=1}^{n}\|v^k\|^2 + \|\delta_x u^n\|^2$$
$$\leqslant C\left[\tau^{1-\alpha}\|v^0\|^2 + \|\delta_x u^0\|^2 + \tau\sum_{m=1}^{n}\|p^m\|^2 + \tau\sum_{m=1}^{n}\|\delta_x^2 q^m\|^2\right], \quad 1 \leqslant n \leqslant N, \tag{3.123}$$

where

$$\|v^k\|^2 = h\sum_{i=1}^{M-1}(v_i^k)^2, \quad \|p^k\|^2 = h\sum_{i=1}^{M-1}(p_i^k)^2, \quad \|\delta_x^2 q^k\|^2 = h\sum_{i=1}^{M-1}(\delta_x^2 q_i^k)^2.$$

Proof. The combination of (3.119)–(3.122) with $q_0^n = q_M^n = 0\,(1 \leqslant n \leqslant N)$ and $\varphi_0 = \varphi_M = \psi_0 = \psi_M = 0$ leads to

$$v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N.$$

Consequently, we have

$$\begin{cases} v_0^{\frac{1}{2}} = 0, \quad v_M^{\frac{1}{2}} = 0, & (3.124) \\ \sigma v_0^n + (1-\sigma)v_0^{n-1} = 0, \quad \sigma v_M^n + (1-\sigma)v_M^{n-1} = 0, \quad 2 \leqslant n \leqslant N. & (3.125) \end{cases}$$

(I) When $n = 1$, (3.118) reads

$$\frac{1}{s}c_0^{(1,\alpha)}(v_i^1 - v_i^0) = \sigma\delta_x^2 u_i^1 + (1-\sigma)\delta_x^2 u_i^0 + p_i^1, \quad 1 \leqslant i \leqslant M-1. \tag{3.126}$$

Making the inner product on both hand sides of (3.126) with $v^{\frac{1}{2}}$ and noticing (3.124), we obtain

$$\frac{1}{2s}c_0^{(1,\alpha)}(\|v^1\|^2 - \|v^0\|^2) = -(\delta_x(\sigma u^1 + (1-\sigma)u^0), \delta_x v^{\frac{1}{2}}) + (p^1, v^{\frac{1}{2}}). \tag{3.127}$$

It follows from (3.119) that

$$\delta_t\delta_x^2 u_i^{\frac{1}{2}} = \delta_x^2 v_i^{\frac{1}{2}} + \delta_x^2 q_i^1, \quad 1 \leqslant i \leqslant M-1. \tag{3.128}$$

Making the inner product on both hand sides of (3.128) with $-(\sigma u^1 + (1-\sigma)u^0)$ and noticing

$$\sigma u_0^1 + (1-\sigma)u_0^0 = 0, \quad \sigma u_M^1 + (1-\sigma)u_M^0 = 0,$$

we get

$$\begin{aligned}&\frac{1}{\tau}(\delta_x u^1 - \delta_x u^0, \delta_x(\sigma u^1 + (1-\sigma)u^0)) \\ &= (\delta_x v^{\frac{1}{2}}, \delta_x(\sigma u^1 + (1-\sigma)u^0)) - (\delta_x^2 q^1, \sigma u^1 + (1-\sigma)u^0).\end{aligned} \tag{3.129}$$

Adding (3.127) and (3.129) yields

$$\begin{aligned}&\frac{1}{2s}c_0^{(1,\alpha)}(\|v^1\|^2 - \|v^0\|^2) + \frac{1}{\tau}(\delta_x u^1 - \delta_x u^0, \delta_x(\sigma u^1 + (1-\sigma)u^0)) \\ &= (p^1, v^{\frac{1}{2}}) - (\delta_x^2 q^1, \sigma u^1 + (1-\sigma)u^0).\end{aligned}$$

Noticing the fact that

$$\begin{aligned}&(\delta_x u^1 - \delta_x u^0, \delta_x(\sigma u^1 + (1-\sigma)u^0)) \\ &= \sigma(\delta_x u^1, \delta_x u^1) - (2\sigma-1)(\delta_x u^1, \delta_x u^0) - (1-\sigma)(\delta_x u^0, \delta_x u^0) \\ &= \frac{\sigma}{4}\|\delta_x u^1\|^2 + (2\sigma-1)\left\|\frac{1}{2}\sqrt{\frac{3\sigma}{2\sigma-1}}\delta_x u^1 - \sqrt{\frac{2\sigma-1}{3\sigma}}\delta_x u^0\right\|^2 \\ &\quad - \frac{\sigma^2-\sigma+1}{3\sigma}\|\delta_x u^0\|^2\end{aligned}$$

and

$$c_0^{(1,\alpha)} = \sigma^{1-\alpha},$$

we arrive at

$$\begin{aligned}&\frac{\sigma^{1-\alpha}\tau^{-\alpha}}{2\Gamma(2-\alpha)}(\|v^1\|^2 - \|v^0\|^2) + \frac{1}{\tau}\left(\frac{\sigma}{4}\|\delta_x u^1\|^2 - \frac{\sigma^2-\sigma+1}{3\sigma}\|\delta_x u^0\|^2\right) \\ &\leqslant (p^1, v^{\frac{1}{2}}) - (\delta_x^2 q^1, \sigma u^1 + (1-\sigma)u^0),\end{aligned}$$

which can be rewritten as

$$\frac{\sigma^{1-\alpha}\tau^{1-\alpha}}{2\Gamma(2-\alpha)}(\|v^1\|^2 - \|v^0\|^2) + \frac{\sigma}{4}\|\delta_x u^1\|^2 - \frac{\sigma^2-\sigma+1}{3\sigma}\|\delta_x u^0\|^2$$

$$
\begin{aligned}
&\leqslant \tau(p^1, v^{\frac{1}{2}}) - \tau(\delta_x^2 q^1, \sigma u^1 + (1-\sigma)u^0)\\
&\leqslant \tau\|p^1\| \cdot \|v^{\frac{1}{2}}\| + \tau\|\delta_x^2 q^1\| \cdot \|\sigma u^1 + (1-\sigma)u^0\|\\
&\leqslant \tau\left[\frac{1}{2\varepsilon_0}\|v^{\frac{1}{2}}\|^2 + \frac{\varepsilon_0}{2}\|p^1\|^2\right] + \tau\left[\frac{1}{2}\|\sigma u^1 + (1-\sigma)u^0\|^2 + \frac{1}{2}\|\delta_x^2 q^1\|^2\right]\\
&\leqslant \tau\left[\frac{1}{4\varepsilon_0}(\|v^1\|^2 + \|v^0\|^2) + \frac{\varepsilon_0}{2}\|p^1\|^2\right] + \tau\left[\frac{1}{2}(\|u^1\|^2 + \|u^0\|^2) + \frac{1}{2}\|\delta_x^2 q^1\|^2\right].
\end{aligned}
$$

Letting $\varepsilon_0 = \Gamma(2-\alpha)\sigma^{\alpha-1}\tau^{\alpha}$, there exists a positive constant C_1 such that

$$
\begin{aligned}
&\tau^{1-\alpha}\|v^1\|^2 + \|\delta_x u^1\|^2\\
&\leqslant C_1(\tau^{1-\alpha}\|v^0\|^2 + \|\delta_x u^0\|^2 + \tau\|u^0\|^2 + \tau\|u^1\|^2 + \tau^{1+\alpha}\|p^1\|^2 + \tau\|\delta_x^2 q^1\|^2). \qquad (3.130)
\end{aligned}
$$

(II) Making the inner product on both hand sides of (3.118) with $\sigma v^n + (1-\sigma)v^{n-1}$ and noticing (3.125), we have

$$
\begin{aligned}
&\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(v^{n-k} - v^{n-k-1}, \sigma v^n + (1-\sigma)v^{n-1})\\
&= (\delta_x^2(\sigma u^n + (1-\sigma)u^{n-1}), \sigma v^n + (1-\sigma)v^{n-1}) + (p^n, \sigma v^n + (1-\sigma)v^{n-1})\\
&= -(\delta_x(\sigma u^n + (1-\sigma)u^{n-1}), \delta_x(\sigma v^n + (1-\sigma)v^{n-1}))\\
&\quad + (p^n, \sigma v^n + (1-\sigma)v^{n-1}), \quad 2 \leqslant n \leqslant N. \qquad (3.131)
\end{aligned}
$$

For the left-hand side of (3.131), by Lemma 2.6.1, we can obtain

$$
\begin{aligned}
&\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(v^{n-k} - v^{n-k-1}, \sigma v^n + (1-\sigma)v^{n-1})\\
&\geqslant \frac{1}{2}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|v^{n-k}\|^2 - \|v^{n-k-1}\|^2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}\cdot\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|v^{n-k}\|^2 - \|v^{n-k-1}\|^2)\\
&\leqslant -(\delta_x(\sigma u^n + (1-\sigma)u^{n-1}), \delta_x(\sigma v^n + (1-\sigma)v^{n-1}))\\
&\quad + (p^n, \sigma v^n + (1-\sigma)v^{n-1}), \quad 2 \leqslant n \leqslant N. \qquad (3.132)
\end{aligned}
$$

It follows from (3.120) that

$$
D_{\bar{t}}\delta_x^2 u_i^n = \delta_x^2(\sigma v_i^n + (1-\sigma)v_i^{n-1}) + \delta_x^2 q_i^n, \quad 1 \leqslant i \leqslant M-1, \quad 2 \leqslant n \leqslant N.
$$

Making the inner product on both hand sides of the equation above with $-(\sigma u^n + (1-\sigma)u^{n-1})$ and noticing

$$
\sigma u_0^n + (1-\sigma)u_0^{n-1} = 0, \quad \sigma u_M^n + (1-\sigma)u_M^{n-1} = 0, \quad 2 \leqslant n \leqslant N,
$$

we obtain

$$\begin{aligned}&(D_{\bar t}\delta_x u^n, \delta_x(\sigma u^n + (1-\sigma)u^{n-1}))\\ &= (\delta_x(\sigma v^n + (1-\sigma)v^{n-1}), \delta_x(\sigma u^n + (1-\sigma)u^{n-1}))\\ &\quad - (\delta_x^2 q^n, \sigma u^n + (1-\sigma)u^{n-1}), \quad 2 \leqslant n \leqslant N. \end{aligned} \tag{3.133}$$

Denote

$$\begin{aligned}F^n &= (2\sigma+1)\|\delta_x u^n\|^2 - (2\sigma-1)\|\delta_x u^{n-1}\|^2\\ &\quad + (2\sigma^2+\sigma-1)\|\delta_x(u^n - u^{n-1})\|^2, \quad n \geqslant 1.\end{aligned}$$

With the help of Lemma 3.4.2, we have

$$F^n \geqslant \frac{1}{\sigma}\|\delta_x u^n\|^2, \quad n \geqslant 1 \tag{3.134}$$

and

$$(D_{\bar t}\delta_x u^n, \delta_x(\sigma u^n + (1-\sigma)u^{n-1})) \geqslant \frac{1}{4\tau}(F^n - F^{n-1}), \quad n \geqslant 2. \tag{3.135}$$

Combining (3.133) with (3.135), we get

$$\begin{aligned}\frac{1}{4\tau}(F^n - F^{n-1}) &\leqslant (\delta_x(\sigma v^n + (1-\sigma)v^{n-1}), \delta_x(\sigma u^n + (1-\sigma)u^{n-1}))\\ &\quad - (\delta_x^2 q^n, \sigma u^n + (1-\sigma)u^{n-1}), \quad 2 \leqslant n \leqslant N.\end{aligned} \tag{3.136}$$

Adding (3.132) and (3.136) arrives at

$$\begin{aligned}&\frac{1}{2}\cdot\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|v^{n-k}\|^2 - \|v^{n-k-1}\|^2) + \frac{1}{4\tau}(F^n - F^{n-1})\\ &\leqslant (p^n, \sigma v^n + (1-\sigma)v^{n-1}) - (\delta_x^2 q^n, \sigma u^n + (1-\sigma)u^{n-1})\\ &\leqslant \|p^n\|\cdot\|\sigma v^n + (1-\sigma)v^{n-1}\| + \|\delta_x^2 q^n\|\cdot\|\sigma u^n + (1-\sigma)u^{n-1}\|, \quad 2 \leqslant n \leqslant N.\end{aligned}$$

Noticing the fact that

$$\begin{aligned}&\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|v^{n-k}\|^2 - \|v^{n-k-1}\|^2)\\ &= \sum_{k=0}^{n-1} c_k^{(n,\alpha)}\|v^{n-k}\|^2 - \sum_{k=0}^{n-2} c_k^{(n-1,\alpha)}\|v^{n-1-k}\|^2\\ &\quad - \sum_{k=0}^{n-2}(c_k^{(n,\alpha)} - c_k^{(n-1,\alpha)})\|v^{n-1-k}\|^2 - c_{n-1}^{(n,\alpha)}\|v^0\|^2\\ &= \sum_{k=0}^{n-1} c_k^{(n,\alpha)}\|v^{n-k}\|^2 - \sum_{k=0}^{n-2} c_k^{(n-1,\alpha)}\|v^{n-1-k}\|^2\end{aligned}$$

$$-\,(c_{n-2}^{(n,\alpha)}-c_{n-2}^{(n-1,\alpha)})\|v^1\|^2-c_{n-1}^{(n,\alpha)}\|v^0\|^2,$$

we have

$$\begin{aligned}&\frac{1}{2s}\left(\sum_{k=0}^{n-1}c_k^{(n,\alpha)}\|v^{n-k}\|^2-\sum_{k=0}^{n-2}c_k^{(n-1,\alpha)}\|v^{n-1-k}\|^2\right)+\frac{1}{4\tau}(F^n-F^{n-1})\\ &\leqslant\frac{1}{2s}[(c_{n-2}^{(n,\alpha)}-c_{n-2}^{(n-1,\alpha)})\|v^1\|^2+c_{n-1}^{(n,\alpha)}\|v^0\|^2]\\ &\quad+\|p^n\|\cdot\|\sigma v^n+(1-\sigma)v^{n-1}\|+\|\delta_x^2q^n\|\cdot\|\sigma u^n+(1-\sigma)u^{n-1}\|,\quad 2\leqslant n\leqslant N.\end{aligned}$$

Replacing the superscript n with m and summing up for m from 2 to n on both hand sides of the inequality above yields

$$\begin{aligned}&\frac{1}{2s}\left(\sum_{k=0}^{n-1}c_k^{(n,\alpha)}\|v^{n-k}\|^2-c_0^{(1,\alpha)}\|v^1\|^2\right)+\frac{1}{4\tau}(F^n-F^1)\\ &\leqslant\frac{1}{2s}\left[\sum_{m=2}^{n}(c_{m-2}^{(m,\alpha)}-c_{m-2}^{(m-1,\alpha)})\|v^1\|^2+\sum_{m=2}^{n}c_{m-1}^{(m,\alpha)}\|v^0\|^2\right]\\ &\quad+\sum_{m=2}^{n}\|p^m\|\cdot\|\sigma v^m+(1-\sigma)v^{m-1}\|\\ &\quad+\sum_{m=2}^{n}\|\delta_x^2q^m\|\cdot\|\sigma u^m+(1-\sigma)u^{m-1}\|,\quad 2\leqslant n\leqslant N.\end{aligned}$$

Using Lemma 3.4.3, we have

$$\begin{aligned}&\frac{1}{2s}\left(\sum_{k=0}^{n-1}c_k^{(n,\alpha)}\|v^{n-k}\|^2-c_0^{(1,\alpha)}\|v^1\|^2\right)+\frac{1}{4\tau}(F^n-F^1)\\ &\leqslant\frac{1}{2s}\left[\frac{1-\alpha}{12}\left(\frac{\alpha}{\sigma}+1\right)\sigma^{-\alpha}\|v^1\|^2+\frac{3}{2}(n-1+\sigma)^{1-\alpha}\|v^0\|^2\right]\\ &\quad+\sum_{m=2}^{n}\|p^m\|\cdot\|\sigma v^m+(1-\sigma)v^{m-1}\|\\ &\quad+\sum_{m=2}^{n}\|\delta_x^2q^m\|\cdot\|\sigma u^m+(1-\sigma)u^{m-1}\|,\quad 2\leqslant n\leqslant N.\end{aligned}\tag{3.137}$$

Using Lemma 1.6.3, we have

$$c_0^{(1,\alpha)}=\sigma^{1-\alpha}$$

and

$$c_0^{(n,\alpha)}>c_1^{(n,\alpha)}>c_2^{(n,\alpha)}>\cdots>c_{n-1}^{(n,\alpha)}>(1-\alpha)n^{-\alpha},\quad n\geqslant 2.$$

It follows from (3.137) that

$$\begin{aligned}
&\frac{1-\alpha}{2s}n^{-\alpha}\sum_{k=0}^{n-1}\|v^{n-k}\|^2+\frac{1}{4\tau}(F^n-F^1)\\
&\leqslant \frac{1}{2s}\sigma^{1-\alpha}\|v^1\|^2+\frac{1}{2s}\left[\frac{1-\alpha}{12}\left(\frac{\alpha}{\sigma}+1\right)\sigma^{-\alpha}\|v^1\|^2+\frac{3}{2}(n-1+\sigma)^{1-\alpha}\|v^0\|^2\right]\\
&\quad+\sum_{m=2}^{n}\|p^m\|\cdot\|\sigma v^m+(1-\sigma)v^{m-1}\|\\
&\quad+\sum_{m=2}^{n}\|\delta_x^2 q^m\|\cdot\|\sigma u^m+(1-\sigma)u^{m-1}\|,\quad 2\leqslant n\leqslant N.
\end{aligned}$$

Noticing

$$\frac{1-\alpha}{2s}n^{-\alpha}=\frac{1-\alpha}{2}\cdot\frac{n^{-\alpha}}{\Gamma(2-\alpha)\tau^\alpha}\geqslant\frac{1}{2T^\alpha\Gamma(1-\alpha)},$$

it yields that

$$\begin{aligned}
&\frac{1}{2T^\alpha\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\|v^{n-k}\|^2+\frac{1}{4\tau}(F^n-F^1)\\
&\leqslant\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\sigma^{1-\alpha}\|v^1\|^2\\
&\quad+\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12}\left(\frac{\alpha}{\sigma}+1\right)\sigma^{-\alpha}\|v^1\|^2+\frac{3}{2}(n-1+\sigma)^{1-\alpha}\|v^0\|^2\right]\\
&\quad+\sum_{m=2}^{n}\|p^m\|\cdot\|\sigma v^m+(1-\sigma)v^{m-1}\|+\sum_{m=2}^{n}\|\delta_x^2 q^m\|\cdot\|\sigma u^m+(1-\sigma)u^{m-1}\|,\quad 2\leqslant n\leqslant N.
\end{aligned}$$

Multiplying both hand sides of the inequality above by 4τ, we get

$$\begin{aligned}
&\frac{2}{T^\alpha\Gamma(1-\alpha)}\tau\sum_{k=0}^{n-1}\|v^{n-k}\|^2+F^n\\
&\leqslant F^1+\frac{2\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sigma^{1-\alpha}\|v^1\|^2\\
&\quad+\frac{2\tau^{1-\alpha}}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12}\left(\frac{\alpha}{\sigma}+1\right)\sigma^{-\alpha}\|v^1\|^2+\frac{3}{2}(n-1+\sigma)^{1-\alpha}\|v^0\|^2\right]\\
&\quad+4\tau\sum_{m=2}^{n}\|p^m\|\cdot\|\sigma v^m+(1-\sigma)v^{m-1}\|\\
&\quad+4\tau\sum_{m=2}^{n}\|\delta_x^2 q^m\|\cdot\|\sigma u^m+(1-\sigma)u^{m-1}\|\\
&\leqslant F^1+\frac{2\tau^{1-\alpha}}{\Gamma(2-\alpha)}\left[\sigma^{1-\alpha}+\frac{1-\alpha}{12}\left(\frac{\alpha}{\sigma}+1\right)\sigma^{-\alpha}\right]\|v^1\|^2+\frac{3}{\Gamma(2-\alpha)}T^{1-\alpha}\|v^0\|^2
\end{aligned}$$

$$\begin{aligned}
&+ 2\tau \sum_{m=2}^{n} \left[\frac{1}{\varepsilon_0} \|\sigma v^m + (1-\sigma) v^{m-1}\|^2 + \varepsilon_0 \|p^m\|^2 \right] \\
&+ 2\tau \sum_{m=2}^{n} [\|\sigma u^m + (1-\sigma) u^{m-1}\|^2 + \|\delta_x^2 q^m\|^2] \\
\leqslant{}& F^1 + \frac{2\tau^{1-\alpha}}{\Gamma(2-\alpha)} \left[\sigma^{1-\alpha} + \frac{1-\alpha}{12} \left(\frac{\alpha}{\sigma} + 1 \right) \sigma^{-\alpha} \right] \|v^1\|^2 + \frac{3}{\Gamma(2-\alpha)} T^{1-\alpha} \|v^0\|^2 \\
&+ \frac{2\tau}{\varepsilon_0} \sum_{m=2}^{n} (\|v^m\|^2 + \|v^{m-1}\|^2) + 2\tau\varepsilon_0 \sum_{m=2}^{n} \|p^m\|^2 \\
&+ 2\tau \sum_{m=2}^{n} (\|u^m\|^2 + \|u^{m-1}\|^2 + \|\delta_x^2 q^m\|^2), \quad 2 \leqslant n \leqslant N.
\end{aligned}$$

Taking $\varepsilon_0 = 4T^\alpha \Gamma(1-\alpha)$ in the inequality above, noticing (3.130), (3.134) and

$$\begin{aligned}
F^1 &= (2\sigma+1)\|\delta_x u^1\|^2 - (2\sigma-1)\|\delta_x u^0\|^2 + (2\sigma^2+\sigma-1)\|\delta_x(u^1-u^0)\|^2 \\
&\leqslant (2\sigma+1)\|\delta_x u^1\|^2 - (2\sigma-1)\|\delta_x u^0\|^2 \\
&\quad + 2(2\sigma^2+\sigma-1)(\|\delta_x u^1\|^2 + \|\delta_x u^0\|^2) \\
&= (4\sigma^2+4\sigma-1)\|\delta_x u^1\|^2 + (4\sigma^2-1)\|\delta_x u^0\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\tau \sum_{k=0}^{n-1} \|v^{n-k}\|^2 + \|\delta_x u^n\|^2 \\
\leqslant{}& C_2 \Bigg(\tau^{1-\alpha} \|v^0\|^2 + \|\delta_x u^0\|^2 + \tau \|u^0\|^2 \\
&\quad + \tau \sum_{m=1}^{n} \|u^m\|^2 + \tau \sum_{m=1}^{n} \|p^m\|^2 + \tau \sum_{m=1}^{n} \|\delta_x^2 q^m\|^2 \Bigg) \\
\leqslant{}& C_2 \Bigg(\tau^{1-\alpha} \|v^0\|^2 + \|\delta_x u^0\|^2 + \tau \|u^0\|^2 \\
&\quad + \tau \sum_{m=1}^{n} \frac{L^2}{6} \|\delta_x u^m\|^2 + \tau \sum_{m=1}^{n} \|p^m\|^2 + \tau \sum_{m=1}^{n} \|\delta_x^2 q^m\|^2 \Bigg), \quad 1 \leqslant n \leqslant N,
\end{aligned}$$

with C_2 a constant. An application of the Gronwall inequality yields (3.123). The proof ends. □

3.4.4 Convergence of the difference scheme

Theorem 3.4.3. *Suppose* $\{U_i^n, V_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n, v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (3.88)–(3.91) *and the difference scheme* (3.103)–

(3.107), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad z_i^n = V_i^n - v_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then it holds

$$\tau \sum_{k=1}^{n} \|z^k\|^2 + \|\delta_x e^n\|^2 \leqslant C(c_5^2 + c_6^2) LT(\tau^2 + h^2)^2, \quad 1 \leqslant n \leqslant N,$$

where C is defined in Theorem 3.4.2.

Proof. The subtraction of (3.103)–(3.107) from (3.95), (3.97)–(3.98), (3.101)–(3.102), respectively, produces the system of error equations as follows:

$$\begin{cases} \dfrac{1}{s}\displaystyle\sum_{k=0}^{n-1} c_k^{(n,\alpha)} (z_i^{n-k} - z_i^{n-k-1}) = \sigma\delta_x^2 e_i^n + (1-\sigma)\delta_x^2 e_i^{n-1} + (r_5)_i^n, \\ \qquad\qquad\qquad\qquad\qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ \delta_t e_i^{\frac{1}{2}} = z_i^{\frac{1}{2}} + (r_6)_i^1, \quad 0 \leqslant i \leqslant M, \\ D_{\bar{t}} e_i^n = \sigma z_i^n + (1-\sigma) z_i^{n-1} + (r_6)_i^n, \quad 0 \leqslant i \leqslant M,\ 2 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad z_i^0 = 0, \quad 0 \leqslant i \leqslant M, \\ e_0^n = 0, \quad e_M^n = 0, \quad 1 \leqslant n \leqslant N. \end{cases}$$

Noticing (3.96), (3.99)–(3.100), the application of Theorem 3.4.2 yields

$$\begin{aligned} &\tau \sum_{k=1}^{n} \|z^k\|^2 + \|\delta_x e^n\|^2 \\ &\leqslant C\left[\tau^{1-\alpha}\|z^0\|^2 + \|\delta_x e^0\|^2 + \tau \sum_{m=1}^{n} \|(r_5)^m\|^2 + \tau \sum_{m=1}^{n} \|\delta_x^2 (r_6)^m\|^2\right] \\ &\leqslant C\left\{\tau \sum_{m=1}^{n} L[c_5(\tau^2 + h^2)]^2 + \tau \sum_{m=1}^{n} L(c_6\tau^2)^2\right\} \\ &\leqslant C(c_5^2 + c_6^2) LT(\tau^2 + h^2)^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

The proof ends. □

3.5 The fast difference method based on L2-1_σ approximation for 1D problem

In this section, we continue to consider the problem (3.81)–(3.83) and develop a fast temporal second-order difference scheme.

3.5.1 Derivation of the difference scheme

For the mesh function $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ defined on $\Omega_h \times \Omega_\tau$, as what we did in last section, introduce the following notation:

$$D_{\bar{t}} u_i^n = \frac{1}{2\tau}\left[(2\sigma+1)u_i^n - 4\sigma u_i^{n-1} + (2\sigma-1)u_i^{n-2}\right].$$

Let

$$v(x,t) = u_t(x,t), \quad \alpha = \gamma - 1, \quad \sigma = 1 - \frac{\alpha}{2}, \quad s = \tau^\alpha \Gamma(2-\alpha).$$

Thus, the considered problem (3.81)–(3.83) is equivalent to the following one:

$$\begin{cases} {}_0^C D_t^\alpha v(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], & (3.138)\\ u_t(x,t) = v(x,t), \quad x \in [0,L],\ t \in (0,T], & (3.139)\\ u(x,0) = \varphi(x), \quad v(x,0) = \psi(x), \quad x \in [0,L], & (3.140)\\ u(0,t) = 0, \quad u(L,t) = 0, \quad t \in (0,T]. & (3.141) \end{cases}$$

Denote

$$\begin{cases} U_i^n = u(x_i,t_n), \quad V_i^n = v(x_i,t_n), \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,\\ \varphi_i = \varphi(x_i), \quad \psi_i = \psi(x_i), \quad 0 \leqslant i \leqslant M. \end{cases}$$

Considering (3.138) at the point $(x_i, t_{n-1+\sigma})$, we have

$${}_0^C D_t^\alpha v(x_i,t_{n-1+\sigma}) = u_{xx}(x_i,t_{n-1+\sigma}) + f_i^{n-1+\sigma},$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \qquad (3.142)$$

Applying the theory in Subsection 1.7.2 to approximate Caputo derivative gives

$$\begin{cases} {}_0^C D_t^\alpha v(x_i,t_{n-1+\sigma}) = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\sum_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + d_0^{(1,\alpha)}(V_i^n - V_i^{n-1})\\ \qquad + O(\tau^{3-\alpha} + \epsilon), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.143)\\ F_{l,i}^1 = 0, \quad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1, & (3.144)\\ F_{l,i}^n = e^{-s_l\tau} F_{l,i}^{n-1} + A_l(V_i^{n-1} - V_i^{n-2}) + B_l(V_i^n - V_i^{n-1}),\\ \qquad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N. & (3.145) \end{cases}$$

For the spatial second-order derivative, by (3.85) and Lemma 2.1.3, we have

$$\begin{aligned} u_{xx}(x_i,t_{n-1+\sigma}) &= \sigma u_{xx}(x_i,t_n) + (1-\sigma)u_{xx}(x_i,t_{n-1}) + O(\tau^2)\\ &= \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + O(h^2+\tau^2). \end{aligned} \qquad (3.146)$$

Substituting (3.143) and (3.146) into (3.142) arrives at

$$\begin{aligned}&\frac{1}{\Gamma(1-\alpha)}\sum_{l=1}^{N_{\exp}}\omega_l F_{l,i}^n + d_0^{(1,\alpha)}(V_i^n - V_i^{n-1})\\&= \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + f_i^{n-1+\sigma} + (r_7)_i^n,\\&\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N \end{aligned} \tag{3.147}$$

and there exists a positive constant c_7 such that

$$\left|(r_7)_i^n\right| \leqslant c_7(\tau^2 + h^2 + \epsilon), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{3.148}$$

Substituting (3.144)–(3.145) into (3.147) and eliminating the intermediate variable $\{F_{l,i}^n\}$ yield

$$\begin{aligned}\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(V_i^{n-k} - V_i^{n-k-1}) &= \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + f_i^{n-1+\sigma} + (r_7)_i^n,\\&\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.\end{aligned}$$

Considering (3.139) at the points $(x_i, t_{\frac{1}{2}})$ and $(x_i, t_{n-1+\sigma})$, respectively, we have

$$\begin{cases} \delta_t U_i^{\frac{1}{2}} = V_i^{\frac{1}{2}} + (r_8)_i^1,\ 0 \leqslant i \leqslant M, & (3.149)\\ D_{\bar{t}} U_i^n = \sigma V_i^n + (1-\sigma)V_i^{n-1} + (r_8)_i^n, \quad 0 \leqslant i \leqslant M,\ 2 \leqslant n \leqslant N, & (3.150)\end{cases}$$

and there exists a positive constant c_8 such that

$$\left|\delta_x^2(r_8)_i^n\right| \leqslant c_8\tau^2, \quad 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N. \tag{3.151}$$

In addition, from homogeneous boundary value conditions (3.141), we can obtain

$$(r_8)_0^n = 0, \quad (r_8)_M^n = 0, \quad 1 \leqslant n \leqslant N. \tag{3.152}$$

Noticing the initial-boundary value conditions (3.140)–(3.141), we have

$$\begin{cases} U_i^0 = \varphi_i, \quad V_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, & (3.153)\\ U_0^n = 0, \quad U_M^n = 0, \quad 1 \leqslant n \leqslant N. & (3.154)\end{cases}$$

Omitting the small term $(r_7)_i^n$ and $(r_8)_i^n$ in (3.147), (3.149)–(3.150) and replacing the exact solution $\{U_i^n, V_i^n\}$ with its numerical one $\{u_i^n, v_i^n\}$ produce the difference scheme for

(3.138)–(3.141) as follows:

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \sum\limits_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + d_0^{(1,\alpha)}(v_i^n - v_i^{n-1}) = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma}, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.155) \\ F_{l,i}^1 = 0, \quad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1, & (3.156) \\ F_{l,i}^n = e^{-s_l\tau} F_{l,i}^{n-1} + A_l(v_i^{n-1} - v_i^{n-2}) + B_l(v_i^n - v_i^{n-1}), \\ \qquad\qquad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N, & (3.157) \\ \delta_t u_i^{\frac{1}{2}} = v_i^{\frac{1}{2}}, \quad 0 \leqslant i \leqslant M, & (3.158) \\ D_{\bar{t}} u_i^n = \sigma v_i^n + (1-\sigma) v_i^{n-1}, \quad 0 \leqslant i \leqslant M, \quad 2 \leqslant n \leqslant N, & (3.159) \\ u_i^0 = \varphi_i, \quad v_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, & (3.160) \\ u_0^n = 0, \quad u_M^n = 0, \quad 1 \leqslant n \leqslant N. & (3.161) \end{cases}$$

Substituting (3.156)–(3.157) into (3.155) and eliminating the intermediate variable $\{F_{l,i}^n\}$ yield

$$\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(v_i^{n-k} - v_i^{n-k-1}) = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma},$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{3.162}$$

3.5.2 Solvability of the difference scheme

Theorem 3.5.1. *The difference scheme* (3.155)–(3.161) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n), \quad v^n = (v_0^n, v_1^n, \ldots, v_M^n).$$

(I) The value of $\{u^0, v^0\}$ is determined by (3.160).
(II) It follows from (3.158) that

$$v_i^1 = 2\delta_t u_i^{\frac{1}{2}} - v_i^0, \quad 0 \leqslant i \leqslant M. \tag{3.163}$$

Substituting (3.163) into (3.162), and noticing (3.161), we obtain the linear system in u^1:

$$\begin{cases} d_0^{(1,\alpha)}(2\delta_t u_i^{\frac{1}{2}} - 2v_i^0) = \sigma\delta_x^2 u_i^1 + (1-\sigma)\delta_x^2 u_i^0 + f_i^\sigma, \\ \qquad\qquad 1 \leqslant i \leqslant M-1, & (3.164) \\ u_0^1 = 0, \quad u_M^1 = 0. & (3.165) \end{cases}$$

Considering its homogeneous one, we have

$$\begin{cases} d_0^{(1,\alpha)} \dfrac{2}{\tau} u_i^1 = \sigma \delta_x^2 u_i^1, \quad 1 \leqslant i \leqslant M-1, & (3.166) \\ u_0^1 = 0, \quad u_M^1 = 0. & (3.167) \end{cases}$$

Making the inner product on both hand sides of (3.166) with u^1 produces

$$d_0^{(1,\alpha)} \frac{2}{\tau} \|u^1\|^2 + \sigma \|\delta_x u^1\|^2 = 0,$$

which implies $u^1 = 0$, hence the system (3.164)–(3.165) has a unique solution. Once u^1 is obtained, v^1 is followed from (3.163).

(III) Suppose the value of $\{u^0, v^0, u^1, v^1, \ldots, u^{n-1}, v^{n-1}\}$ has been uniquely determined, then it follows from (3.159) that

$$v_i^n = \frac{1}{\sigma}(D_{\bar{t}} u_i^n - (1-\sigma) v_i^{n-1}), \quad 0 \leqslant i \leqslant M. \tag{3.168}$$

Substituting (3.168) into (3.162), noticing (3.161), one can obtain the linear system in u^n as follows:

$$\begin{cases} d_0^{(n,\alpha)} \left[\dfrac{1}{\sigma}(D_{\bar{t}} u_i^n - (1-\sigma) v_i^{n-1}) - v_i^{n-1} \right] + \displaystyle\sum_{k=1}^{n-1} d_k^{(n,\alpha)} (v_i^{n-k} - v_i^{n-k-1}) \\ = \sigma \delta_x^2 u_i^n + (1-\sigma) \delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma}, \quad 1 \leqslant i \leqslant M-1, & (3.169) \\ u_0^n = 0, \quad u_M^n = 0. & (3.170) \end{cases}$$

To show its unique solvability, it suffices to prove the corresponding homogeneous one

$$\begin{cases} d_0^{(n,\alpha)} \cdot \dfrac{1}{\sigma} \cdot \dfrac{2\sigma+1}{2\tau} u_i^n = \sigma \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (3.171) \\ u_0^n = 0, \quad u_M^n = 0 & (3.172) \end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (3.171) with u^n yields

$$d_0^{(n,\alpha)} \cdot \frac{1}{\sigma} \cdot \frac{2\sigma+1}{2\tau} \|u^n\|^2 + \sigma \|\delta_x u^n\|^2 = 0,$$

which implies $u^n = 0$, and the system (3.169)–(3.170) has a unique solution u^n. Once u^n is obtained, v^n is followed from (3.168).

By the principle of induction, the theorem is true. The proof ends. □

3.5.3 Stability of the difference scheme

Theorem 3.5.2. *Suppose* $\{u^n, v^n \mid 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\sum_{l=1}^{N_{\exp}} \omega_l F_{l,i}^n + d_0^{(1,\alpha)}(v_i^n - v_i^{n-1}) \\ = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + p_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.173) \\ F_{l,i}^1 = 0, \quad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1, & (3.174) \\ F_{l,i}^n = e^{-s_l\tau} F_{l,i}^{n-1} + A_l(v_i^{n-1} - v_i^{n-2}) + B_l(v_i^n - v_i^{n-1}), \\ \qquad\qquad 1 \leqslant l \leqslant N_{\exp},\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N, & (3.175) \\ \delta_t u_i^{\frac{1}{2}} = v_i^{\frac{1}{2}} + q_i^1, \quad 0 \leqslant i \leqslant M, & (3.176) \\ D_{\bar{t}} u_i^n = \sigma v_i^n + (1-\sigma)v_i^{n-1} + q_i^n, \quad 0 \leqslant i \leqslant M,\ 2 \leqslant n \leqslant N, & (3.177) \\ u_i^0 = \varphi_i, \quad v_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, & (3.178) \\ u_0^n = 0, \quad u_M^n = 0, \quad 1 \leqslant n \leqslant N, & (3.179) \end{cases}$$

where $q_0^n = q_M^n = 0 \quad (1 \leqslant n \leqslant N)$ *and* $\varphi_0 = \varphi_M = \psi_0 = \psi_M = 0$*. Then there exists a positive constant C such that*

$$\tau\sum_{k=1}^{n}\|v^k\|^2 + \|\delta_x u^n\|^2$$
$$\leqslant C\left[\tau^{1-\alpha}\|v^0\|^2 + \epsilon\tau^{\alpha-1}\|\delta_x u^0\|^2 + \tau\sum_{m=1}^{n}\|p^m\|^2 + \tau\sum_{m=1}^{n}\|\delta_x^2 q^m\|^2\right], \quad 1 \leqslant n \leqslant N, \tag{3.180}$$

in which

$$\|v^k\|^2 = h\sum_{i=1}^{M-1}(v_i^k)^2, \quad \|p^k\|^2 = h\sum_{i=1}^{M-1}(p_i^k)^2, \quad \|\delta_x^2 q^k\|^2 = h\sum_{i=1}^{M-1}(\delta_x^2 q_i^k)^2.$$

Proof. It follows from (3.176)–(3.179), $q_0^n = q_M^n = 0\,(1 \leqslant n \leqslant N)$ and $\varphi_0 = \varphi_M = \psi_0 = \psi_M = 0$ that

$$v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N.$$

It is easy to obtain

$$\begin{cases} v_0^{\frac{1}{2}} = 0, \quad v_M^{\frac{1}{2}} = 0, & (3.181) \\ \sigma v_0^n + (1-\sigma)v_0^{n-1} = 0, \quad \sigma v_M^n + (1-\sigma)v_M^{n-1} = 0, \quad 2 \leqslant n \leqslant N. & (3.182) \end{cases}$$

Substituting (3.174)–(3.175) into (3.173) and getting rid of the intermediate variable $\{F_{l,i}^n\}$, we have

$$\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(v_i^{n-k} - v_i^{n-k-1}) = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + p_i^n,$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{3.183}$$

(I) When $n = 1$, equation (3.183) reads

$$d_0^{(1,\alpha)}(v_i^1 - v_i^0) = \sigma\delta_x^2 u_i^1 + (1-\sigma)\delta_x^2 u_i^0 + p_i^1, \quad 1 \leqslant i \leqslant M-1. \tag{3.184}$$

It is easy to know from (3.176) that

$$\delta_t \delta_x^2 u_i^{\frac{1}{2}} = \delta_x^2 v_i^{\frac{1}{2}} + \delta_x^2 q_i^1, \quad 1 \leqslant i \leqslant M-1. \tag{3.185}$$

Making the inner product on both hand sides of (3.184) with $v^{\frac{1}{2}}$ and noticing (3.181) yield that

$$\frac{1}{2} d_0^{(1,\alpha)}(\|v^1\|^2 - \|v^0\|^2) = -(\delta_x(\sigma u^1 + (1-\sigma)u^0), \delta_x v^{\frac{1}{2}}) + (p^1, v^{\frac{1}{2}}). \tag{3.186}$$

Making the inner product on both hand sides of (3.185) with $-(\sigma u^1 + (1-\sigma)u^0)$ and noticing

$$\sigma u_0^1 + (1-\sigma)u_0^0 = 0, \quad \sigma u_M^1 + (1-\sigma)u_M^0 = 0,$$

arrive at

$$\begin{aligned}&\frac{1}{\tau}(\delta_x u^1 - \delta_x u^0, \delta_x(\sigma u^1 + (1-\sigma)u^0))\\ &= (\delta_x v^{\frac{1}{2}}, \delta_x(\sigma u^1 + (1-\sigma)u^0)) - (\delta_x^2 q^1, \sigma u^1 + (1-\sigma)u^0).\end{aligned} \tag{3.187}$$

Adding (3.186) and (3.187) gives

$$\begin{aligned}&\frac{1}{2} d_0^{(1,\alpha)}(\|v^1\|^2 - \|v^0\|^2) + \frac{1}{\tau}(\delta_x u^1 - \delta_x u^0, \delta_x(\sigma u^1 + (1-\sigma)u^0))\\ &= (p^1, v^{\frac{1}{2}}) - (\delta_x^2 q^1, \sigma u^1 + (1-\sigma)u^0).\end{aligned}$$

Noticing the fact that

$$\begin{aligned}&(\delta_x u^1 - \delta_x u^0, \delta_x(\sigma u^1 + (1-\sigma)u^0))\\ &= \frac{\sigma}{4}\|\delta_x u^1\|^2 + (2\sigma - 1)\left\|\frac{1}{2}\sqrt{\frac{3\sigma}{2\sigma-1}}\delta_x u^1 - \sqrt{\frac{2\sigma-1}{3\sigma}}\delta_x u^0\right\|^2\\ &\quad - \frac{\sigma^2 - \sigma + 1}{3\sigma}\|\delta_x u^0\|^2\end{aligned}$$

and

$$d_0^{(1,\alpha)} = \frac{\sigma^{1-\alpha}\tau^{-\alpha}}{\Gamma(2-\alpha)},$$

we have

$$\begin{aligned}&\frac{\sigma^{1-\alpha}\tau^{-\alpha}}{2\Gamma(2-\alpha)}(\|v^1\|^2 - \|v^0\|^2) + \frac{1}{\tau}\left(\frac{\sigma}{4}\|\delta_x u^1\|^2 - \frac{\sigma^2-\sigma+1}{3\sigma}\|\delta_x u^0\|^2\right)\\ &\leqslant (p^1, v^{\frac{1}{2}}) - (\delta_x^2 q^1, \sigma u^1 + (1-\sigma)u^0),\end{aligned}$$

which can be reduced to

$$\begin{aligned}
&\frac{\sigma^{1-\alpha}\tau^{1-\alpha}}{2\Gamma(2-\alpha)}(\|v^1\|^2-\|v^0\|^2)+\frac{\sigma}{4}\|\delta_x u^1\|^2-\frac{\sigma^2-\sigma+1}{3\sigma}\|\delta_x u^0\|^2\\
&\leqslant \tau(p^1,v^{\frac{1}{2}})-\tau(\delta_x^2 q^1,\sigma u^1+(1-\sigma)u^0)\\
&\leqslant \tau\|p^1\|\cdot\|v^{\frac{1}{2}}\|+\tau\|\delta_x^2 q^1\|\cdot\|\sigma u^1+(1-\sigma)u^0\|\\
&\leqslant \tau\left[\frac{1}{2\varepsilon_0}\|v^{\frac{1}{2}}\|^2+\frac{\varepsilon_0}{2}\|p^1\|^2\right]+\tau\left[\frac{1}{2}\|\sigma u^1+(1-\sigma)u^0\|^2+\frac{1}{2}\|\delta_x^2 q^1\|^2\right]\\
&\leqslant \tau\left[\frac{1}{4\varepsilon_0}(\|v^1\|^2+\|v^0\|^2)+\frac{\varepsilon_0}{2}\|p^1\|^2\right]+\tau\left[\frac{1}{2}(\|u^1\|^2+\|u^0\|^2)+\frac{1}{2}\|\delta_x^2 q^1\|^2\right].
\end{aligned}$$

Taking $\varepsilon_0=\Gamma(2-\alpha)\sigma^{\alpha-1}\tau^{\alpha}$, there exists a positive constant C_1 such that

$$\begin{aligned}
&\tau^{1-\alpha}\|v^1\|^2+\|\delta_x u^1\|^2\\
&\leqslant C_1(\tau^{1-\alpha}\|v^0\|^2+\|\delta_x u^0\|^2+\tau\|u^0\|^2+\tau\|u^1\|^2+\tau^{1+\alpha}\|p^1\|^2+\tau\|\delta_x^2 q^1\|^2).
\end{aligned}\tag{3.188}$$

(II) Making the inner product on both hand sides of (3.183) with $\sigma v^n+(1-\sigma)v^{n-1}$ and noticing (3.182) yield

$$\begin{aligned}
&\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(v^{n-k}-v^{n-k-1},\sigma v^n+(1-\sigma)v^{n-1})\\
&=(\delta_x^2(\sigma u^n+(1-\sigma)u^{n-1}),\sigma v^n+(1-\sigma)v^{n-1})+(p^n,\sigma v^n+(1-\sigma)v^{n-1})\\
&=-(\delta_x(\sigma u^n+(1-\sigma)u^{n-1}),\delta_x(\sigma v^n+(1-\sigma)v^{n-1}))\\
&\quad+(p^n,\sigma v^n+(1-\sigma)v^{n-1}),\quad 2\leqslant n\leqslant N.
\end{aligned}\tag{3.189}$$

For the term on the left-hand side of (3.189), by Lemma 1.7.3 and Lemma 2.6.1, we have

$$\begin{aligned}
&\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(v^{n-k}-v^{n-k-1},\sigma v^n+(1-\sigma)v^{n-1})\\
&\geqslant \frac{1}{2}\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(\|v^{n-k}\|^2-\|v^{n-k-1}\|^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(\|v^{n-k}\|^2-\|v^{n-k-1}\|^2)\\
&\leqslant -(\delta_x(\sigma u^n+(1-\sigma)u^{n-1}),\delta_x(\sigma v^n+(1-\sigma)v^{n-1}))\\
&\quad+(p^n,\sigma v^n+(1-\sigma)v^{n-1}),\quad 2\leqslant n\leqslant N.
\end{aligned}\tag{3.190}$$

It follows from (3.177) that

$$D_{\bar{t}}\delta_x^2 u_i^n=\delta_x^2(\sigma v_i^n+(1-\sigma)v_i^{n-1})+\delta_x^2 q_i^n,\quad 1\leqslant i\leqslant M-1,\ 2\leqslant n\leqslant N.\tag{3.191}$$

Taking the inner product on both hand sides of (3.191) with $-(\sigma u^n + (1-\sigma)u^{n-1})$ and noticing

$$\sigma u_0^n + (1-\sigma)u_0^{n-1} = 0, \quad \sigma u_M^n + (1-\sigma)u_M^{n-1} = 0, \quad 2 \leqslant n \leqslant N$$

lead to

$$\begin{aligned}&(D_{\bar{t}}\delta_x u^n, \delta_x(\sigma u^n + (1-\sigma)u^{n-1}))\\ &= (\delta_x(\sigma v^n + (1-\sigma)v^{n-1}), \delta_x(\sigma u^n + (1-\sigma)u^{n-1}))\\ &\quad - (\delta_x^2 q^n, \sigma u^n + (1-\sigma)u^{n-1}), \quad 2 \leqslant n \leqslant N.\end{aligned} \tag{3.192}$$

Denote

$$\begin{aligned}F^n &= (2\sigma+1)\|\delta_x u^n\|^2 - (2\sigma-1)\|\delta_x u^{n-1}\|^2\\ &\quad + (2\sigma^2+\sigma-1)\|\delta_x(u^n - u^{n-1})\|^2, \quad n \geqslant 1.\end{aligned}$$

Using Lemma 3.4.2, we have

$$F^n \geqslant \frac{1}{\sigma}\|\delta_x u^n\|^2, \quad n \geqslant 1 \tag{3.193}$$

and

$$(D_{\bar{t}}\delta_x u^n, \delta_x(\sigma u^n + (1-\sigma)u^{n-1})) \geqslant \frac{1}{4\tau}(F^n - F^{n-1}), \quad n \geqslant 2. \tag{3.194}$$

Combining (3.192) with (3.194), we obtain

$$\begin{aligned}\frac{1}{4\tau}(F^n - F^{n-1}) &\leqslant (\delta_x(\sigma v^n + (1-\sigma)v^{n-1}), \delta_x(\sigma u^n + (1-\sigma)u^{n-1}))\\ &\quad - (\delta_x^2 q^n, \sigma u^n + (1-\sigma)u^{n-1}), \quad 2 \leqslant n \leqslant N.\end{aligned} \tag{3.195}$$

Adding (3.190) and (3.195) yields

$$\begin{aligned}&\frac{1}{2}\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(\|v^{n-k}\|^2 - \|v^{n-k-1}\|^2) + \frac{1}{4\tau}(F^n - F^{n-1})\\ &\leqslant (p^n, \sigma v^n + (1-\sigma)v^{n-1}) - (\delta_x^2 q^n, \sigma u^n + (1-\sigma)u^{n-1})\\ &\leqslant \|p^n\| \cdot \|\sigma v^n + (1-\sigma)v^{n-1}\| + \|\delta_x^2 q^n\| \cdot \|\sigma u^n + (1-\sigma)u^{n-1}\|, \quad 2 \leqslant n \leqslant N.\end{aligned}$$

Noticing the fact that

$$\begin{aligned}&\sum_{k=0}^{n-1} d_k^{(n,\alpha)}(\|v^{n-k}\|^2 - \|v^{n-k-1}\|^2)\\ &= \sum_{k=0}^{n-1} d_k^{(n,\alpha)}\|v^{n-k}\|^2 - \sum_{k=0}^{n-2} d_k^{(n-1,\alpha)}\|v^{n-1-k}\|^2\end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=0}^{n-2} (d_k^{(n,\alpha)} - d_k^{(n-1,\alpha)}) \|v^{n-1-k}\|^2 - d_{n-1}^{(n,\alpha)} \|v^0\|^2 \\
= &\sum_{k=0}^{n-1} d_k^{(n,\alpha)} \|v^{n-k}\|^2 - \sum_{k=0}^{n-2} d_k^{(n-1,\alpha)} \|v^{n-1-k}\|^2 \\
&- (d_{n-2}^{(n,\alpha)} - d_{n-2}^{(n-1,\alpha)}) \|v^1\|^2 - d_{n-1}^{(n,\alpha)} \|v^0\|^2,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{1}{2}\left(\sum_{k=0}^{n-1} d_k^{(n,\alpha)} \|v^{n-k}\|^2 - \sum_{k=0}^{n-2} d_k^{(n-1,\alpha)} \|v^{n-1-k}\|^2 \right) + \frac{1}{4\tau}(F^n - F^{n-1}) \\
\leqslant &\frac{1}{2}[(d_{n-2}^{(n,\alpha)} - d_{n-2}^{(n-1,\alpha)}) \|v^1\|^2 + d_{n-1}^{(n,\alpha)} \|v^0\|^2] \\
&+ \|p^n\| \cdot \|\sigma v^n + (1-\sigma) v^{n-1}\| + \|\delta_x^2 q^n\| \cdot \|\sigma u^n + (1-\sigma) u^{n-1}\|, \quad 2 \leqslant n \leqslant N. \qquad (3.196)
\end{aligned}
$$

Replacing the superscript n with m, and summing up for m from 2 to n on both hand sides of (3.196), we obtain

$$
\begin{aligned}
&\frac{1}{2}\left(\sum_{k=0}^{n-1} d_k^{(n,\alpha)} \|v^{n-k}\|^2 - d_0^{(1,\alpha)} \|v^1\|^2 \right) + \frac{1}{4\tau}(F^n - F^1) \\
\leqslant &\frac{1}{2}\left[\sum_{m=2}^{n} (d_{m-2}^{(m,\alpha)} - d_{m-2}^{(m-1,\alpha)}) \|v^1\|^2 + \sum_{m=2}^{n} d_{m-1}^{(m,\alpha)} \|v^0\|^2 \right] \\
&+ \sum_{m=2}^{n} \|p^m\| \cdot \|\sigma v^m + (1-\sigma) v^{m-1}\| \\
&+ \sum_{m=2}^{n} \|\delta_x^2 q^m\| \cdot \|\sigma u^m + (1-\sigma) u^{m-1}\|, \quad 2 \leqslant n \leqslant N.
\end{aligned}
$$

Applying Lemma 3.4.3 and (1.147), it follows that

$$
\begin{aligned}
&\sum_{m=2}^{n} (d_{m-2}^{(m,\alpha)} - d_{m-2}^{(m-1,\alpha)}) \\
\leqslant &\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{m=2}^{n} (c_{m-2}^{(m,\alpha)} - c_{m-2}^{(m-1,\alpha)}) + \frac{1}{\Gamma(1-\alpha)} \sum_{m=2}^{n} \frac{9}{4}\epsilon \\
\leqslant &\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \frac{1-\alpha}{12} \left(\frac{\alpha}{\sigma} + 1 \right) \sigma^{-\alpha} + \frac{1}{\Gamma(1-\alpha)} \frac{9}{4}(n-1)\epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{m=2}^{n} d_{m-1}^{(m,\alpha)} &\leqslant \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{m=2}^{n} c_{m-1}^{(m,\alpha)} + \sum_{m=2}^{n} \frac{\epsilon}{\Gamma(1-\alpha)} \\
&\leqslant \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \frac{3}{2}(n-1+\sigma)^{1-\alpha} + (n-1)\frac{\epsilon}{\Gamma(1-\alpha)}.
\end{aligned}
$$

Hence, we arrive at

$$\frac{1}{2}\left(\sum_{k=0}^{n-1} d_k^{(n,\alpha)}\|v^{n-k}\|^2 - d_0^{(1,\alpha)}\|v^1\|^2\right) + \frac{1}{4\tau}(F^n - F^1)$$
$$\leqslant \frac{1}{2}\left[\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \cdot \frac{1-\alpha}{12}\left(\frac{\alpha}{\sigma}+1\right)\sigma^{-\alpha} + \frac{1}{\Gamma(1-\alpha)}\frac{9}{4}(n-1)\epsilon\right)\|v^1\|^2\right.$$
$$\left. + \left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\frac{3}{2}(n-1+\sigma)^{1-\alpha} + (n-1)\frac{\epsilon}{\Gamma(1-\alpha)}\right)\|v^0\|^2\right]$$
$$+ \sum_{m=2}^{n}\|p^m\| \cdot \|\sigma v^m + (1-\sigma)v^{m-1}\|$$
$$+ \sum_{m=2}^{n}\|\delta_x^2 q^m\| \cdot \|\sigma u^m + (1-\sigma)u^{m-1}\|, \quad 2 \leqslant n \leqslant N. \tag{3.197}$$

By Lemma 1.7.3, we get

$$d_0^{(1,\alpha)} = \frac{\sigma^{1-\alpha}\tau^{-\alpha}}{\Gamma(2-\alpha)}$$

and

$$d_0^{(n,\alpha)} > d_1^{(n,\alpha)} > d_2^{(n,\alpha)} > d_3^{(n,\alpha)} > \cdots > d_{n-1}^{(n,\alpha)} > \frac{1}{2t_n^\alpha\Gamma(1-\alpha)}.$$

Thus, the inequality (3.197) can be reduced to

$$\frac{1}{2}\left(\frac{1}{2t_n^\alpha\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\|v^{n-k}\|^2 - \frac{\sigma^{1-\alpha}\tau^{-\alpha}}{\Gamma(2-\alpha)}\|v^1\|^2\right) + \frac{1}{4\tau}(F^n - F^1)$$
$$\leqslant \frac{1}{2}\left[\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\frac{1-\alpha}{12}\left(\frac{\alpha}{\sigma}+1\right)\sigma^{-\alpha} + \frac{1}{\Gamma(1-\alpha)}\frac{9}{4}(n-1)\epsilon\right)\|v^1\|^2\right.$$
$$\left. + \left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\frac{3}{2}(n-1+\sigma)^{1-\alpha} + (n-1)\frac{\epsilon}{\Gamma(1-\alpha)}\right)\|v^0\|^2\right]$$
$$+ \sum_{m=2}^{n}\|p^m\| \cdot \|\sigma v^m + (1-\sigma)v^{m-1}\|$$
$$+ \sum_{m=2}^{n}\|\delta_x^2 q^m\| \cdot \|\sigma u^m + (1-\sigma)u^{m-1}\|, \quad 2 \leqslant n \leqslant N. \tag{3.198}$$

Multiplying both hand sides of (3.198) by 4τ yields

$$\frac{1}{T^\alpha\Gamma(1-\alpha)}\tau\sum_{k=0}^{n-1}\|v^{n-k}\|^2 + F^n$$
$$\leqslant F^1 + \left[\frac{2\sigma^{1-\alpha}\tau^{1-\alpha}}{\Gamma(2-\alpha)} + \left(\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\frac{1-\alpha}{6}\left(\frac{\alpha}{\sigma}+1\right)\sigma^{-\alpha} + \frac{1}{\Gamma(1-\alpha)}\frac{9}{2}T\epsilon\right)\right]\|v^1\|^2$$

$$
\begin{aligned}
&+\left(\frac{3T^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{2\epsilon T}{\Gamma(1-\alpha)}\right)\|v^0\|^2\\
&+4\tau\sum_{m=2}^{n}\|p^m\|\cdot\|\sigma v^m+(1-\sigma)v^{m-1}\|\\
&+4\tau\sum_{m=2}^{n}\|\delta_x^2 q^m\|\cdot\|\sigma u^m+(1-\sigma)u^{m-1}\|\\
\leqslant{}& F^1+\left[\frac{2\sigma^{1-\alpha}\tau^{1-\alpha}}{\Gamma(2-\alpha)}+\left(\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\frac{1-\alpha}{6}\left(\frac{\alpha}{\sigma}+1\right)\sigma^{-\alpha}+\frac{1}{\Gamma(1-\alpha)}\frac{9}{2}T\epsilon\right)\right]\|v^1\|^2\\
&+\left(\frac{3T^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{2\epsilon T}{\Gamma(1-\alpha)}\right)\|v^0\|^2\\
&+2\tau\sum_{m=2}^{n}\left[\frac{1}{\varepsilon_0}\|\sigma v^m+(1-\sigma)v^{m-1}\|^2+\varepsilon_0\|p^m\|^2\right]\\
&+2\tau\sum_{m=2}^{n}[\|\sigma u^m+(1-\sigma)u^{m-1}\|^2+\|\delta_x^2 q^m\|^2]\\
\leqslant{}& F^1+\left[\frac{2\sigma^{1-\alpha}\tau^{1-\alpha}}{\Gamma(1-\alpha)}+\left(\frac{\tau^{1-\alpha}}{\Gamma(1-\alpha)}\frac{1-\alpha}{6}\left(\frac{\alpha}{\sigma}+1\right)\sigma^{-\alpha}+\frac{1}{\Gamma(1-\alpha)}\frac{9}{2}T\epsilon\right)\right]\|v^1\|^2\\
&+\left(\frac{3T^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{2\epsilon T}{\Gamma(1-\alpha)}\right)\|v^0\|^2\\
&+\frac{2\tau}{\varepsilon_0}\sum_{m=2}^{n}(\|v^m\|^2+\|v^{m-1}\|^2)+2\tau\varepsilon_0\sum_{m=2}^{n}\|p^m\|^2\\
&+2\tau\sum_{m=2}^{n}(\|u^m\|^2+\|u^{m-1}\|^2+\|\delta_x^2 q^m\|^2),\quad 2\leqslant n\leqslant N.
\end{aligned}
$$

Taking $\varepsilon_0=8T^{\alpha}\Gamma(1-\alpha)$, noticing (3.188), (3.193) and

$$
\begin{aligned}
F^1&=(2\sigma+1)\|\delta_x u^1\|^2-(2\sigma-1)\|\delta_x u^0\|^2+(2\sigma^2+\sigma-1)\|\delta_x(u^1-u^0)\|^2\\
&\leqslant(2\sigma+1)\|\delta_x u^1\|^2-(2\sigma-1)\|\delta_x u^0\|^2+2(2\sigma^2+\sigma-1)(\|\delta_x u^1\|^2+\|\delta_x u^0\|^2)\\
&=(4\sigma^2+4\sigma-1)\|\delta_x u^1\|^2+(4\sigma^2-1)\|\delta_x u^0\|^2,
\end{aligned}
$$

there exists a positive constant C_2 such that

$$
\begin{aligned}
&\tau\sum_{k=0}^{n-1}\|v^{n-k}\|^2+\|\delta_x u^n\|^2\\
\leqslant{}& C_2\Bigg(\tau^{1-\alpha}\|v^0\|^2+\epsilon\tau^{\alpha-1}\|\delta_x u^0\|^2+\|u^0\|^2\\
&\quad+\tau\sum_{m=1}^{n}\|u^m\|^2+\tau\sum_{m=1}^{n}\|p^m\|^2+\tau\sum_{m=1}^{n}\|\delta_x^2 q^m\|^2\Bigg)\\
\leqslant{}& C_2\Bigg(\tau^{1-\alpha}\|v^0\|^2+\epsilon\tau^{\alpha-1}\|\delta_x u^0\|^2+\|u^0\|^2
\end{aligned}
$$

$$+\tau\sum_{m=1}^{n}\frac{L^2}{6}\|\delta_x u^m\|^2+\tau\sum_{m=1}^{n}\|p^m\|^2+\tau\sum_{m=1}^{n}\|\delta_x^2 q^m\|^2\Big),\quad 1\leqslant n\leqslant N.$$

The inequality (3.180) can be obtained with the help of the Gronwall inequality. The proof ends. □

Remark 3.5.1. In general, ϵ is taken such that $\epsilon\tau^{\alpha-1}\leqslant 1$.

3.5.4 Convergence of the difference scheme

Theorem 3.5.3. *Suppose* $\{U_i^n, V_i^n \mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *and* $\{u_i^n, v_i^n \mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *are solutions of the problem* (3.138)–(3.141) *and the difference scheme* (3.155)–(3.161), *respectively. Let*

$$e_i^n=U_i^n-u_i^n,\quad z_i^n=V_i^n-v_i^n,\quad 0\leqslant i\leqslant M,\ 0\leqslant n\leqslant N,$$

then it holds that

$$\tau\sum_{k=1}^{n}\|z^k\|^2+\|\delta_x e^n\|^2\leqslant C(c_7^2+c_8^2)LT(\tau^2+h^2+\epsilon)^2,\quad 1\leqslant n\leqslant N,$$

where C is defined in Theorem 3.5.2.

Proof. The subtraction of (3.155)–(3.161) from (3.147), (3.144)–(3.145), (3.149)–(3.150) and (3.153)–(3.154), respectively, produces the system of error equations as follows:

$$\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\displaystyle\sum_{l=1}^{N_{\exp}}\omega_l F_{l,i}^n+d_0^{(1,\alpha)}(z_i^n-z_i^{n-1})\\ =\sigma\delta_x^2 e_i^n+(1-\sigma)\delta_x^2 e_i^{n-1}+(r_7)_i^n,\quad 1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N,\\ F_{l,i}^1=0,\quad 1\leqslant l\leqslant N_{\exp},\ 1\leqslant i\leqslant M-1,\\ F_{l,i}^n=e^{-s_l\tau}F_{l,i}^{n-1}+A_l(z_i^{n-1}-z_i^{n-2})+B_l(z_i^n-z_i^{n-1})\\ \qquad\qquad 1\leqslant l\leqslant N_{\exp},\ 1\leqslant i\leqslant M-1,\ 2\leqslant n\leqslant N,\\ \delta_t e_i^{\frac12}=z_i^{\frac12}+(r_8)_i^1,\quad 0\leqslant i\leqslant M,\\ D_{\bar t}e_i^n=\sigma z_i^n+(1-\sigma)z_i^{n-1}+(r_8)_i^n,\quad 0\leqslant i\leqslant M,\ 2\leqslant n\leqslant N,\\ e_i^0=0,\quad z_i^0=0,\quad 0\leqslant i\leqslant M,\\ e_0^n=0,\quad e_M^n=0,\quad 1\leqslant n\leqslant N.\end{cases}$$

Noticing (3.148) and (3.151)–(3.152), the application of Theorem 3.5.2 will lead to

$$\tau\sum_{k=1}^{n}\|z^k\|^2+\|\delta_x e^n\|^2$$
$$\leqslant C\left[\tau^{1-\alpha}\|z^0\|^2+\epsilon\tau^{\alpha-1}\|\delta_x e^0\|^2+\tau\sum_{m=1}^{n}\|(r_7)^m\|^2+\tau\sum_{m=1}^{n}\|\delta_x^2(r_8)^m\|^2\right]$$

$$\leqslant C\left\{\tau \sum_{m=1}^{n} L[c_7(\tau^2 + h^2 + \epsilon)]^2 + \tau \sum_{m=1}^{n} L(c_8\tau^2)^2\right\}$$
$$\leqslant C(c_7^2 + c_8^2)LT(\tau^2 + h^2 + \epsilon)^2, \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

3.6 The difference method based on L1 approximation for the MTTFW equations

In this section, the finite difference method for solving a class of multiterm time-fractional wave (MTTFW) equations will be introduced. For simplicity, take the two-term case with the constant coefficients as an example.

Consider the following problem of the two-term time-fractional wave equations:

$$\begin{cases} {}_0^C D_t^{\gamma_1} u(x,t) + {}_0^C D_t^{\gamma} u(x,t) = u_{xx}(x,t) + f(x,t), \\ \qquad x \in (0,L), t \in (0,T], & (3.199) \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in (0,L), & (3.200) \\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), \quad t \in [0,T], & (3.201) \end{cases}$$

where $1 < \gamma_1 < \gamma < 2$, the functions $f, \varphi, \psi, \mu, \nu$ are given, and $\varphi(0) = \mu(0)$, $\varphi(L) = \nu(0)$, $\psi(0) = \mu'(0)$, $\psi(L) = \nu'(0)$. Suppose $u \in C^{(4,3)}([0,L] \times [0,T])$.

Take the same mesh partition and notations as those in Section 3.1.

3.6.1 Derivation of the difference scheme

Considering equation (3.199) at the point (x_i, t_n), we have

$${}_0^C D_t^{\gamma_1} u(x_i,t_n) + {}_0^C D_t^{\gamma} u(x_i,t_n) = u_{xx}(x_i,t_n) + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 0 \leqslant n \leqslant N.$$

Taking an average on two adjacent time levels arrives at

$$\frac{1}{2}\left[{}_0^C D_t^{\gamma_1} u(x_i,t_n) + {}_0^C D_t^{\gamma_1} u(x_i,t_{n-1})\right] + \frac{1}{2}\left[{}_0^C D_t^{\gamma} u(x_i,t_n) + {}_0^C D_t^{\gamma} u(x_i,t_{n-1})\right]$$
$$= \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}(x_i,t_n) + \frac{\partial^2 u}{\partial x^2}(x_i,t_{n-1})\right] + f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N,$$

where $f_i^{n-\frac{1}{2}} = \frac{1}{2}(f_i^n + f_i^{n-1})$. It follows from Theorem 1.6.2 and Lemma 2.1.3 that

$$\frac{\tau^{1-\gamma_1}}{\Gamma(3-\gamma_1)}\left[b_0^{(\gamma_1)}\delta_t U_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma_1)} - b_{n-k}^{(\gamma_1)})\delta_t U_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma_1)}\psi_i\right]$$

$$+\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\Big[b_0^{(\gamma)}\delta_t U_i^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\delta_t U_i^{k-\frac{1}{2}}-b_{n-1}^{(\gamma)}\psi_i\Big]$$
$$=\delta_x^2 U_i^{n-\frac{1}{2}}+f_i^{n-\frac{1}{2}}+(r_9)_i^{n-\frac{1}{2}},\quad 1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N, \tag{3.202}$$

and there is a positive constant c_9 such that

$$\big|(r_9)_i^{n-\frac{1}{2}}\big|\leqslant c_9(\tau^{3-\gamma}+h^2),\quad 1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N. \tag{3.203}$$

Noticing the initial-boundary value conditions (3.200)–(3.201), we have

$$\begin{cases} U_i^0=\varphi(x_i),\quad 1\leqslant i\leqslant M-1, & (3.204)\\ U_0^n=\mu(t_n),\quad U_M^n=\nu(t_n),\quad 0\leqslant n\leqslant N. & (3.205)\end{cases}$$

Omitting the small term $(r_9)_i^{n-\frac{1}{2}}$ in (3.202) and replacing the exact solution U_i^n with its numerical one u_i^n produce a difference scheme for solving (3.199)–(3.201) as follows:

$$\begin{cases} \dfrac{\tau^{1-\gamma_1}}{\Gamma(3-\gamma_1)}\Big[b_0^{(\gamma_1)}\delta_t u_i^{n-\frac{1}{2}}-\sum\limits_{k=1}^{n-1}(b_{n-k-1}^{(\gamma_1)}-b_{n-k}^{(\gamma_1)})\delta_t u_i^{k-\frac{1}{2}}-b_{n-1}^{(\gamma_1)}\psi_i\Big] & \\ \quad+\dfrac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\Big[b_0^{(\gamma)}\delta_t u_i^{n-\frac{1}{2}}-\sum\limits_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\delta_t u_i^{k-\frac{1}{2}}-b_{n-1}^{(\gamma)}\psi_i\Big] & \\ \quad=\delta_x^2 u_i^{n-\frac{1}{2}}+f_i^{n-\frac{1}{2}},\quad 1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N, & (3.206)\\ u_i^0=\varphi(x_i),\quad 1\leqslant i\leqslant M-1, & (3.207)\\ u_0^n=\mu(t_n),\quad u_M^n=\nu(t_n),\quad 0\leqslant n\leqslant N. & (3.208)\end{cases}$$

Hereinafter, denote

$$\eta=\tau^{\gamma-1}\Gamma(3-\gamma),\quad \eta_1=\tau^{\gamma_1-1}\Gamma(3-\gamma_1).$$

3.6.2 Solvability of the difference scheme

Theorem 3.6.1. *The difference scheme* (3.206)–(3.208) *is uniquely solvable.*

Proof. Let

$$u^n=(u_0^n,u_1^n,\ldots,u_M^n).$$

The value of u^0 is determined by (3.207)–(3.208). Suppose that the values of $u^0,u^1,\ldots,$ u^{n-1} have been uniquely determined. From (3.206) and (3.208), we can obtain the linear system in u^n. To prove its unique solvability, it suffices to show the corresponding homogeneous one

$$\begin{cases} \dfrac{1}{\tau}\Big[\dfrac{1}{\eta_1}b_0^{(\gamma_1)}+\dfrac{1}{\eta}b_0^{(\gamma)}\Big]u_i^n=\dfrac{1}{2}\delta_x^2 u_i^n,\quad 1\leqslant i\leqslant M-1, & (3.209)\\ u_0^n=u_M^n=0 & (3.210)\end{cases}$$

has only the trivial solution.

Making an inner product on both hand sides of (3.209) with u^n and noticing (3.210), we have

$$\frac{1}{\tau}\left[\frac{1}{\eta_1}b_0^{(\gamma_1)}+\frac{1}{\eta}b_0^{(\gamma)}\right]\|u^n\|^2=\frac{1}{2}(\delta_x^2u^n,u^n)=-\frac{1}{2}\|\delta_xu^n\|^2\leqslant 0,$$

thus $\|u^n\|=0$. The combination with (3.210) will arrive at $u^n=0$.

By the principle of induction, the difference scheme (3.206)–(3.208) is uniquely solvable. The proof ends. □

3.6.3 Stability of difference scheme

Theorem 3.6.2. *Suppose* $\{v_i^n \mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases}\dfrac{1}{\eta_1}\left[b_0^{(\gamma_1)}\delta_tv_i^{n-\frac{1}{2}}-\displaystyle\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma_1)}-b_{n-k}^{(\gamma_1)})\delta_tv_i^{k-\frac{1}{2}}-b_{n-1}^{(\gamma_1)}\psi_i\right]\\ \quad+\dfrac{1}{\eta}\left[b_0^{(\gamma)}\delta_tv_i^{n-\frac{1}{2}}-\displaystyle\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\delta_tv_i^{k-\frac{1}{2}}-b_{n-1}^{(\gamma)}\psi_i\right]\\ \quad=\delta_x^2v_i^{n-\frac{1}{2}}+f_i^{n-\frac{1}{2}},\quad 1\leqslant i\leqslant M-1,\ 1\leqslant n\leqslant N, & (3.211)\\ v_i^0=\varphi(x_i),\quad 1\leqslant i\leqslant M-1, & (3.212)\\ v_0^n=0,\quad v_M^n=0,\quad 0\leqslant n\leqslant N. & (3.213)\end{cases}$$

Then it holds

$$\|\delta_xv^n\|^2\leqslant\|\delta_xv^0\|^2+\left[\frac{t_n^{2-\gamma_1}}{\Gamma(3-\gamma_1)}+\frac{t_n^{2-\gamma}}{\Gamma(3-\gamma)}\right]\|\psi\|^2+\frac{1}{4}[t_n^{\gamma_1-1}\Gamma(2-\gamma_1)$$
$$+t_n^{\gamma-1}\Gamma(2-\gamma)]\,\tau\sum_{k=1}^{n}\|f^{k-\frac{1}{2}}\|^2,\quad 1\leqslant n\leqslant N, \tag{3.214}$$

where the definitions of $\|\psi\|$ *and* $\|f^{k-\frac{1}{2}}\|$ *are the same as those in Theorem* 3.1.2.

Proof. Taking an inner product on both hand sides of (3.211) with $\delta_tv^{n-\frac{1}{2}}$, it follows from the Cauchy–Schwarz inequality that

$$\left[\frac{1}{\eta_1}b_0^{(\gamma_1)}+\frac{1}{\eta}b_0^{(\gamma)}\right]\|\delta_tv^{n-\frac{1}{2}}\|^2-(\delta_x^2v^{n-\frac{1}{2}},\delta_tv^{n-\frac{1}{2}})$$
$$=\frac{1}{\eta_1}\left[\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma_1)}-b_{n-k}^{(\gamma_1)})(\delta_tv^{k-\frac{1}{2}},\delta_tv^{n-\frac{1}{2}})+b_{n-1}^{(\gamma_1)}(\psi,\delta_tv^{n-\frac{1}{2}})\right]$$

$$
\begin{aligned}
&+\frac{1}{\eta}\left[\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})(\delta_t v^{k-\frac{1}{2}},\delta_t v^{n-\frac{1}{2}})+b_{n-1}^{(\gamma)}(\psi,\delta_t v^{n-\frac{1}{2}})\right]\\
&+(f^{n-\frac{1}{2}},\delta_t v^{n-\frac{1}{2}})\\
\leqslant{}&\frac{1}{\eta_1}\left[\frac{1}{2}\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma_1)}-b_{n-k}^{(\gamma_1)})(\|\delta_t v^{k-\frac{1}{2}}\|^2+\|\delta_t v^{n-\frac{1}{2}}\|^2)\right.\\
&\left.+\frac{1}{2}b_{n-1}^{(\gamma_1)}(\|\psi\|^2+\|\delta_t v^{n-\frac{1}{2}}\|^2)\right]\\
&+\frac{1}{\eta}\left[\frac{1}{2}\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})(\|\delta_t v^{k-\frac{1}{2}}\|^2+\|\delta_t v^{n-\frac{1}{2}}\|^2)\right.\\
&\left.+\frac{1}{2}b_{n-1}^{(\gamma)}(\|\psi\|^2+\|\delta_t v^{n-\frac{1}{2}}\|^2)\right]\\
&+(f^{n-\frac{1}{2}},\delta_t v^{n-\frac{1}{2}}),\quad 1\leqslant n\leqslant N,
\end{aligned}
$$

which can be rearranged as

$$
\begin{aligned}
&\left[\frac{\tau}{\eta_1}b_0^{(\gamma_1)}+\frac{\tau}{\eta}b_0^{(\gamma)}\right]\|\delta_t v^{n-\frac{1}{2}}\|^2+\|\delta_x v^n\|^2-\|\delta_x v^{n-1}\|^2\\
\leqslant{}&\frac{\tau}{\eta_1}\left[\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma_1)}-b_{n-k}^{(\gamma_1)})\|\delta_t v^{k-\frac{1}{2}}\|^2+b_{n-1}^{(\gamma_1)}\|\psi\|^2\right]\\
&+\frac{\tau}{\eta}\left[\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\|\delta_t v^{k-\frac{1}{2}}\|^2+b_{n-1}^{(\gamma)}\|\psi\|^2\right]\\
&+2\tau(f^{n-\frac{1}{2}},\delta_t v^{n-\frac{1}{2}}),\quad 1\leqslant n\leqslant N
\end{aligned}
\tag{3.215}
$$

by noticing that

$$
-(\delta_x^2 v^{n-\frac{1}{2}},\delta_t v^{n-\frac{1}{2}})=\frac{1}{2\tau}(\|\delta_x v^n\|^2-\|\delta_x v^{n-1}\|^2).
$$

Let

$$
\begin{aligned}
Q^0&=\|\delta_x v^0\|^2,\\
Q^n&=\|\delta_x v^n\|^2+\tau\sum_{k=1}^{n}\left(\frac{b_{n-k}^{(\gamma_1)}}{\eta_1}+\frac{b_{n-k}^{(\gamma)}}{\eta}\right)\|\delta_t v^{k-\frac{1}{2}}\|^2,\quad 1\leqslant n\leqslant N.
\end{aligned}
$$

Then (3.215) reads

$$
Q^n\leqslant Q^{n-1}+\tau\left(\frac{b_{n-1}^{(\gamma_1)}}{\eta_1}+\frac{b_{n-1}^{(\gamma)}}{\eta}\right)\|\psi\|^2+2\tau(f^{n-\frac{1}{2}},\delta_t v^{n-\frac{1}{2}}),\quad 1\leqslant n\leqslant N.
$$

The application of the recursive process produces

$$\begin{aligned} Q^n &\leqslant Q^0 + \tau \sum_{k=0}^{n-1} \left(\frac{b_k^{(\gamma_1)}}{\eta_1} + \frac{b_k^{(\gamma)}}{\eta} \right) \|\psi\|^2 + 2\tau \sum_{k=1}^{n} (f^{k-\frac{1}{2}}, \delta_t v^{k-\frac{1}{2}}) \\ &\leqslant \|\delta_x v^0\|^2 + \tau \sum_{k=0}^{n-1} \left(\frac{b_k^{(\gamma_1)}}{\eta_1} + \frac{b_k^{(\gamma)}}{\eta} \right) \|\psi\|^2 \\ &\quad + \tau \sum_{k=1}^{n} \left[\frac{b_{n-k}^{(\gamma_1)}}{\eta_1} \|\delta_t v^{k-\frac{1}{2}}\|^2 + \frac{\eta_1}{4 b_{n-k}^{(\gamma_1)}} \|f^{k-\frac{1}{2}}\|^2 \right] \\ &\quad + \tau \sum_{k=1}^{n} \left[\frac{b_{n-k}^{(\gamma)}}{\eta} \|\delta_t v^{k-\frac{1}{2}}\|^2 + \frac{\eta}{4 b_{n-k}^{(\gamma)}} \|f^{k-\frac{1}{2}}\|^2 \right], \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\delta_x v^n\|^2 &\leqslant \|\delta_x v^0\|^2 + \tau \sum_{k=0}^{n-1} \left(\frac{b_k^{(\gamma_1)}}{\eta_1} + \frac{b_k^{(\gamma)}}{\eta} \right) \|\psi\|^2 + \tau \sum_{k=1}^{n} \frac{\eta_1}{4 b_{n-k}^{(\gamma_1)}} \|f^{k-\frac{1}{2}}\|^2 \\ &\quad + \tau \sum_{k=1}^{n} \frac{\eta}{4 b_{n-k}^{(\gamma)}} \|f^{k-\frac{1}{2}}\|^2, \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{3.216}$$

By (3.20) and (3.22), the desired result (3.214) is attained from (3.216). The proof ends. □

3.6.4 Convergence of the difference scheme

Theorem 3.6.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (3.199)–(3.201) *and the difference scheme* (3.206)–(3.208), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then it holds

$$\|e^n\|_\infty \leqslant \kappa(\tau^{3-\gamma} + h^2), \quad 1 \leqslant n \leqslant N,$$

where

$$\kappa = \frac{L}{4} \sqrt{T^{\gamma_1}\Gamma(2-\gamma_1) + T^{\gamma}\Gamma(2-\gamma)}\, c_9.$$

Proof. The subtraction of (3.206)–(3.208) from (3.202), (3.204)–(3.205), respectively, produces the system of error equations as follows:

$$\begin{cases} \frac{1}{\eta_1}\left[b_0^{(\gamma_1)}\delta_t e_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma_1)} - b_{n-k}^{(\gamma_1)})\delta_t e_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma_1)}\cdot 0\right] \\ + \frac{1}{\eta}\left[b_0^{(\gamma)}\delta_t e_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t e_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\cdot 0\right] \\ = \delta_x^2 e_i^{n-\frac{1}{2}} + (r_9)_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (3.203) and applying Theorem 3.6.2 immediately arrive at

$$\begin{aligned} \|\delta_x e^n\|^2 &\leqslant \frac{1}{4}[t_n^{\gamma_1-1}\Gamma(2-\gamma_1) + t_n^{\gamma-1}\Gamma(2-\gamma)]\tau\sum_{k=1}^{n}\|(r_9)^{k-\frac{1}{2}}\|^2 \\ &\leqslant \frac{1}{4}[T^{\gamma_1}\Gamma(2-\gamma_1) + T^{\gamma}\Gamma(2-\gamma)]Lc_9^2(\tau^{3-\gamma} + h^2)^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Taking the square root on both hand sides of the inequality above and combining with Lemma 2.1.1 will get the desired conclusion. The proof ends. □

3.7 The difference method based on L2-1_σ approximation for the MTTFW equations

Consider the following problem of the MTTFW equations:

$$\sum_{r=0}^{m}\lambda_r\, {}_0^C D_t^{\gamma_r} u(x,t) = u_{xx}(x,t) + f(x,t),\ x \in (0,L),\ t \in (0,T], \tag{3.217}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in [0,L], \tag{3.218}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \in (0,T], \tag{3.219}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_m$ are some positive constants, $1 \leqslant \gamma_m < \gamma_{m-1} < \cdots < \gamma_0 \leqslant 2$ and at least one γ_r belongs to $(1,2)$, and $\varphi(0) = 0$, $\varphi(L) = 0$, $\psi(0) = 0$, $\psi(L) = 0$. Suppose the exact solution $u \in C^{(4,4)}([0,L] \times [0,T])$.

3.7.1 Derivation of the difference scheme

For any grid function $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ defined on $\Omega_h \times \Omega_\tau$, introduce the following notation:

$$D_{\hat{t}} u_i^n = \frac{1}{2\tau}[(2\sigma+1)u_i^n - 4\sigma u_i^{n-1} + (2\sigma-1)u_i^{n-2}], \quad 0 \leqslant i \leqslant M,\ 2 \leqslant n \leqslant N.$$

Let

$$v(x,t) = u_t(x,t), \quad \alpha_r = \gamma_r - 1.$$

Thus, the problem (3.217)–(3.219) is equivalent to the following one:

$$\begin{cases} \sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} v(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], & (3.220) \\ u_t(x,t) = v(x,t), \quad x \in [0,L],\ t \in (0,T], & (3.221) \\ u(x,0) = \varphi(x), \quad v(x,0) = \psi(x), \quad x \in [0,L], & (3.222) \\ u(0,t) = 0, \quad u(L,t) = 0, \quad t \in (0,T]. & (3.223) \end{cases}$$

Denote

$$U_i^n = u(x_i,t_n), \quad V_i^n = v(x_i,t_n), \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$
$$\varphi_i = \varphi(x_i), \quad \psi_i = \psi(x_i), \quad 0 \leqslant i \leqslant M.$$

As that in Subsection 1.6.4, let $t_{n-1+\sigma} = (n-1+\sigma)\tau$, $f_i^{n-1+\sigma} = f(x_i, t_{n-1+\sigma})$ and σ is the root of $F(\sigma) = 0$.

Considering (3.220) at the point $(x_i, t_{n-1+\sigma})$, we have

$$\sum_{r=0}^{m} \lambda_r {}_0^C D_t^{\alpha_r} v(x_i, t_{n-1+\sigma}) = u_{xx}(x_i, t_{n-1+\sigma}) + f_i^{n-1+\sigma},$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{3.224}$$

Using the theory of Subsection 1.6.4 to approximate the Caputo time-fractional derivative, it follows that

$$\sum_{r=0}^{m} \lambda_r {}_0^C D_t^{\alpha_r} v(x_i, t_{n-1+\sigma}) = \sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} (V_i^{n-k} - V_i^{n-k-1}) + O(\tau^{3-\alpha_0}), \tag{3.225}$$

where

$$\hat{c}_k^{(n,\alpha)} = \sum_{r=0}^{m} \lambda_r \cdot \frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)} c_k^{(n,\alpha_r)}, \quad 0 \leqslant k \leqslant n-1$$

is defined by (1.93).

For the spatial second-order derivative on the right-hand side of (3.224), using (3.85) and Lemma 2.1.3, we have

$$\begin{aligned} u_{xx}(x_i, t_{n-1+\sigma}) &= \sigma u_{xx}(x_i, t_n) + (1-\sigma) u_{xx}(x_i, t_{n-1}) + O(\tau^2) \\ &= \sigma \delta_x^2 U_i^n + (1-\sigma) \delta_x^2 U_i^{n-1} + O(h^2) + O(\tau^2). \end{aligned} \tag{3.226}$$

Substituting (3.225) and (3.226) into (3.224), we obtain

$$\sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)}(V_i^{n-k} - V_i^{n-k-1}) = \sigma\delta_x^2 U_i^n + (1-\sigma)\delta_x^2 U_i^{n-1} + f_i^{n-1+\sigma} + (r_{10})_i^n,$$

$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{3.227}$$

and there exists a positive constant c_{10} such that

$$|(r_{10})_i^n| \leqslant c_{10}(\tau^2 + h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{3.228}$$

Considering (3.221) at the points $(x_i, t_{\frac{1}{2}})$ and $(x_i, t_{n-1+\sigma})$, respectively, we get

$$\begin{cases} \delta_t U_i^{\frac{1}{2}} = V_i^{\frac{1}{2}} + (r_{11})_i^1, \quad 0 \leqslant i \leqslant M, & (3.229) \\ D_{\bar{t}} U_i^n = \sigma V_i^n + (1-\sigma)V_i^{n-1} + (r_{11})_i^n, \quad 0 \leqslant i \leqslant M,\ 2 \leqslant n \leqslant N, & (3.230) \end{cases}$$

and there exists a positive constant c_{11} such that

$$|\delta_x^2 (r_{11})_i^n| \leqslant c_{11}\tau^2, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{3.231}$$

In addition, it is known from homogeneous boundary value condition (3.223) that

$$(r_{11})_0^n = 0, \quad (r_{11})_M^n = 0, \quad 1 \leqslant n \leqslant N. \tag{3.232}$$

Noticing the initial-boundary value conditions (3.222)–(3.223), we have

$$\begin{cases} U_i^0 = \varphi_i, \quad V_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, & (3.233) \\ U_0^n = 0, \quad U_M^n = 0, \quad 1 \leqslant n \leqslant N. & (3.234) \end{cases}$$

Omitting the small term $(r_{10})_i^n$ and $(r_{11})_i^n$ in (3.227), (3.229)–(3.230) and replacing the exact solution $\{U_i^n, V_i^n\}$ with its numerical one $\{u_i^n, v_i^n\}$ produce the difference scheme for (3.220)–(3.223) as follows:

$$\begin{cases} \sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)}(v_i^{n-k} - v_i^{n-k-1}) = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma}, & \\ \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.235) \\ \delta_t u_i^{\frac{1}{2}} = v_i^{\frac{1}{2}}, \quad 0 \leqslant i \leqslant M, & (3.236) \\ D_{\bar{t}} u_i^n = \sigma v_i^n + (1-\sigma)v_i^{n-1}, \quad 0 \leqslant i \leqslant M,\ 2 \leqslant n \leqslant N, & (3.237) \\ u_i^0 = \varphi_i, \quad v_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, & (3.238) \\ u_0^n = 0, \quad u_M^n = 0, \quad 1 \leqslant n \leqslant N. & (3.239) \end{cases}$$

Remark 3.7.1. By Lemma 3.4.1, it is easy to get an integral expression of $(r_{11})_i^n$, from which one can obtain (3.231).

Remark 3.7.2. For the nonhomogeneous boundary value problem

$$\begin{cases} \sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\gamma_r} u(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L), \ t \in (0,T], \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in [0,L], \\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), \quad t \in (0,T], \end{cases}$$

let

$$v(x,t) = u_t(x,t),$$

then one has an equivalent problem

$$\begin{cases} \sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} v(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L), \ t \in (0,T], \\ u_{xxt}(x,t) = v_{xx}(x,t), \quad x \in (0,L), \ t \in (0,T], \\ u(x,0) = \varphi(x), \quad v(x,0) = \psi(x), \quad x \in [0,L], \\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), \quad t \in (0,T], \\ v(0,t) = \mu'(t), \quad v(L,t) = \nu'(t), \quad t \in (0,T]. \end{cases}$$

The following difference scheme can be obtained:

$$\begin{cases} \sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} (v_i^{n-k} - v_i^{n-k-1}) = \sigma \delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma}, \\ \qquad\qquad\qquad\qquad 1 \leqslant i \leqslant M-1, \ 1 \leqslant n \leqslant N, \\ \delta_x^2 \delta_t u_i^{\frac{1}{2}} = \delta_x^2 v_i^{\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1, \\ \delta_x^2 D_{\bar{t}} u_i^n = \delta_x^2 (\sigma v_i^n + (1-\sigma) v_i^{n-1}), \quad 1 \leqslant i \leqslant M-1, \ 2 \leqslant n \leqslant N, \\ u_i^0 = \varphi_i, \quad v_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 1 \leqslant n \leqslant N, \\ v_0^n = \mu'(t_n), \quad v_M^n = \nu'(t_n), \quad 1 \leqslant n \leqslant N. \end{cases}$$

Interested readers are recommended to refer to [83].

Another way is firstly to make the boundary conditions homogeneous and then consider the problem along the line in this subsection.

3.7.2 Solvability of the difference scheme

Theorem 3.7.1. *The difference scheme* (3.235)–(3.239) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n), \quad v^n = (v_0^n, v_1^n, \ldots, v_M^n).$$

(I) The value of $\{u^0, v^0\}$ is determined by (3.238).

(II) It follows from (3.236) that

$$v_i^1 = 2v_i^{\frac{1}{2}} - v_i^0 = 2\delta_t u_i^{\frac{1}{2}} - v_i^0, \quad 0 \leqslant i \leqslant M. \tag{3.240}$$

Substituting (3.240) into (3.235), and noticing (3.239), we can obtain the linear system in u^1

$$\begin{cases} \hat{c}_0^{(1,\alpha)}(2\delta_t u_i^{\frac{1}{2}} - 2v_i^0) = \sigma\delta_x^2 u_i^1 + (1-\sigma)\delta_x^2 u_i^0 + f_i^\sigma, \quad 1 \leqslant i \leqslant M-1, & (3.241) \\ u_0^1 = 0, \quad u_M^1 = 0. & (3.242) \end{cases}$$

Consider its homogeneous system:

$$\begin{cases} \frac{2}{\tau}\hat{c}_0^{(1,\alpha)} u_i^1 = \sigma\delta_x^2 u_i^1, \quad 1 \leqslant i \leqslant M-1, & (3.243) \\ u_0^1 = 0, \quad u_M^1 = 0. & (3.244) \end{cases}$$

Making the inner product on both hand sides of (3.243) with u^1 and using (3.244), we have

$$\frac{2}{\tau}\hat{c}_0^{1,\alpha}\|u^1\|^2 + \sigma\|\delta_x u^1\|^2 = 0,$$

which implies $u^1 = 0$. Then the system (3.241)–(3.242) has a unique solution u^1. Once u^1 is obtained, v^1 can be got from (3.240).

(III) Now assume that the value of $\{u^0, v^0, u^1, v^1, \ldots, u^{n-1}, v^{n-1}\}$ has been uniquely determined, then it follows from (3.237) that

$$v_i^n = \frac{1}{\sigma}(D_{\bar{t}} u_i^n - (1-\sigma)v_i^{n-1}), \quad 0 \leqslant i \leqslant M. \tag{3.245}$$

Substituting (3.245) into (3.235), and noticing (3.239), the linear system in u^n can be obtained as

$$\begin{cases} \hat{c}_0^{(n,\alpha)}\left[\frac{1}{\sigma}(D_{\bar{t}} u_i^n - (1-\sigma)v_i^{n-1}) - v_i^{n-1}\right] + \sum_{k=1}^{n-1}\hat{c}_k^{(n,\alpha)}(v_i^{n-k} - v_i^{n-k-1}) \\ = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + f_i^{n-1+\sigma}, \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = 0, \quad u_M^n = 0. \end{cases}$$

To show its unique solvability, it suffices to verify that the corresponding homogeneous one

$$\begin{cases} \hat{c}_0^{(n,\alpha)} \cdot \frac{1}{\sigma} \cdot \frac{2\sigma+1}{2\tau} u_i^n = \sigma\delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (3.246) \\ u_0^n = 0, \quad u_M^n = 0 & (3.247) \end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (3.246) with u^n and noticing (3.247), we have

$$\frac{2\sigma+1}{2\sigma\tau}\hat{c}_0^{(n,\alpha)}\|u^n\|^2 + \sigma\|\delta_x u^n\|^2 = 0,$$

thus $u^n = 0$. The system has a unique solution. Once u^n is obtained, v^n is followed from (3.245).

By the principle of induction, the theorem is true. The proof ends. □

3.7.3 Stability of the difference scheme

Theorem 3.7.2. *Suppose $\{u^n, v^n \mid 0 \leqslant n \leqslant N\}$ is the solution of the difference scheme*

$$\begin{cases} \sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)}(v_i^{n-k} - v_i^{n-k-1}) = \sigma\delta_x^2 u_i^n + (1-\sigma)\delta_x^2 u_i^{n-1} + p_i^n, \\ \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.248) \\ \delta_t u_i^{\frac{1}{2}} = v_i^{\frac{1}{2}} + q_i^1, \quad 0 \leqslant i \leqslant M, & (3.249) \\ D_{\bar{t}} u_i^n = \sigma v_i^n + (1-\sigma)v_i^{n-1} + q_i^n, \quad 0 \leqslant i \leqslant M,\ 2 \leqslant n \leqslant N, & (3.250) \\ u_i^0 = \varphi_i, \quad v_i^0 = \psi_i, \quad 0 \leqslant i \leqslant M, & (3.251) \\ u_0^n = 0, \quad u_M^n = 0, \quad 1 \leqslant n \leqslant N, & (3.252) \end{cases}$$

where $q_0^n = q_M^n = 0\,(1 \leqslant n \leqslant N)$ and $\varphi_0 = \varphi_M = \psi_0 = \psi_M = 0$. Then there exists a constant C such that

$$\begin{aligned} &\tau\sum_{k=1}^{n}\|v^k\|^2 + \|\delta_x u^n\|^2 \\ &\leqslant C\left[\tau\hat{c}_0^{(1,\alpha)}\|v^0\|^2 + \|\delta_x u^0\|^2 + \tau\sum_{l=1}^{n}\|p^l\|^2 + \tau\sum_{l=1}^{n}\|\delta_x^2 q^l\|^2\right], \quad 1 \leqslant n \leqslant N, \end{aligned} \tag{3.253}$$

where

$$\|v^k\|^2 = h\sum_{i=1}^{M-1}(v_i^k)^2, \quad \|p^k\|^2 = h\sum_{i=1}^{M-1}(p_i^k)^2, \quad \|\delta_x^2 q^k\|^2 = h\sum_{i=1}^{M-1}(\delta_x^2 q_i^k)^2,$$

$$\hat{c}_0^{(1,\alpha)} = \sum_{r=0}^{m}\lambda_r \cdot \frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sigma^{1-\alpha_r}.$$

Proof. It follows from (3.249)–(3.252) and $q_0^n = q_M^n = 0\,(1 \leqslant n \leqslant N)$ together with $\varphi_0 = \varphi_M = \psi_0 = \psi_M = 0$ that

$$v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N.$$

Consequently,

$$\begin{cases} v_0^{\frac{1}{2}} = 0, \ v_M^{\frac{1}{2}} = 0, & (3.254) \\ \sigma v_0^n + (1-\sigma)v_0^{n-1} = 0, \ \sigma v_M^n + (1-\sigma)v_M^{n-1} = 0, \quad 2 \leqslant n \leqslant N. & (3.255) \end{cases}$$

In addition, it follows from (3.251)–(3.252) with $\varphi_0 = \varphi_M = 0$ that

$$\begin{cases} \sigma u_0^1 + (1-\sigma)u_0^0 = 0, \ \sigma u_M^1 + (1-\sigma)u_M^0 = 0, & (3.256) \\ \sigma u_0^n + (1-\sigma)u_0^{n-1} = 0, \ \sigma u_M^n + (1-\sigma)u_M^{n-1} = 0, \quad 2 \leqslant n \leqslant N. & (3.257) \end{cases}$$

(I) When $n = 1$, equation (3.248) reads

$$\hat{c}_0^{(1,\alpha)}(v_i^1 - v_i^0) = \sigma\delta_x^2 u_i^1 + (1-\sigma)\delta_x^2 u_i^0 + p_i^1, \quad 1 \leqslant i \leqslant M-1. \tag{3.258}$$

Making the inner product on both hand sides of (3.258) with $v^{\frac{1}{2}}$ and noticing (3.254), we arrive at

$$\frac{1}{2}\hat{c}_0^{(1,\alpha)}(\|v^1\|^2 - \|v^0\|^2) = -(\delta_x(\sigma u^1 + (1-\sigma)u^0), \delta_x v^{\frac{1}{2}}) + (p^1, v^{\frac{1}{2}}). \tag{3.259}$$

It follows from (3.249) that

$$\delta_t\delta_x^2 u_i^{\frac{1}{2}} = \delta_x^2 v_i^{\frac{1}{2}} + \delta_x^2 q_i^1, \quad 1 \leqslant i \leqslant M-1. \tag{3.260}$$

Making the inner product on both hand sides of (3.260) with $-(\sigma u^1 + (1-\sigma)u^0)$ and noticing (3.256), we get

$$\begin{aligned} &\frac{1}{\tau}(\delta_x u^1 - \delta_x u^0, \delta_x(\sigma u^1 + (1-\sigma)u^0)) \\ &= (\delta_x v^{\frac{1}{2}}, \delta_x(\sigma u^1 + (1-\sigma)u^0)) - (\delta_x^2 q^1, \sigma u^1 + (1-\sigma)u^0). \end{aligned} \tag{3.261}$$

Adding (3.259) and (3.261) yields that

$$\begin{aligned} &\frac{1}{2}\hat{c}_0^{(1,\alpha)}(\|v^1\|^2 - \|v^0\|^2) + \frac{1}{\tau}(\delta_x u^1 - \delta_x u^0, \delta_x(\sigma u^1 + (1-\sigma)u^0)) \\ &= (p^1, v^{\frac{1}{2}}) - (\delta_x^2 q^1, \sigma u^1 + (1-\sigma)u^0). \end{aligned}$$

Noticing the fact

$$\begin{aligned} &(\delta_x u^1 - \delta_x u^0, \delta_x(\sigma u^1 + (1-\sigma)u^0)) \\ &= \frac{\sigma}{4}\|\delta_x u^1\|^2 + (2\sigma-1)\left\|\frac{1}{2}\sqrt{\frac{3\sigma}{2\sigma-1}}\delta_x u^1 - \sqrt{\frac{2\sigma-1}{3\sigma}}\delta_x u^0\right\|^2 \\ &\quad - \frac{\sigma^2-\sigma+1}{3\sigma}\|\delta_x u^0\|^2, \end{aligned}$$

we obtain

$$\begin{aligned}&\frac{1}{2}\hat{c}_0^{(1,\alpha)}(\|v^1\|^2-\|v^0\|^2)+\frac{1}{\tau}\left(\frac{\sigma}{4}\|\delta_x u^1\|^2-\frac{\sigma^2-\sigma+1}{3\sigma}\|\delta_x u^0\|^2\right)\\ &\leqslant (p^1,v^{\frac{1}{2}})-(\delta_x^2 q^1,\sigma u^1+(1-\sigma)u^0),\end{aligned}$$

which can be reduced to

$$\begin{aligned}&\frac{\tau}{2}\hat{c}_0^{(1,\alpha)}(\|v^1\|^2-\|v^0\|^2)+\frac{\sigma}{4}\|\delta_x u^1\|^2-\frac{\sigma^2-\sigma+1}{3\sigma}\|\delta_x u^0\|^2\\ &\leqslant \tau(p^1,v^{\frac{1}{2}})-\tau(\delta_x^2 q^1,\sigma u^1+(1-\sigma)u^0)\\ &\leqslant \tau\|p^1\|\cdot\|v^{\frac{1}{2}}\|+\tau\|\delta_x^2 q^1\|\cdot\|\sigma u^1+(1-\sigma)u^0\|\\ &\leqslant \tau\left[\frac{\varepsilon}{2}\|v^{\frac{1}{2}}\|^2+\frac{1}{2\varepsilon}\|p^1\|^2\right]+\tau\left[\frac{1}{2}\|\sigma u^1+(1-\sigma)u^0\|^2+\frac{1}{2}\|\delta_x^2 q^1\|^2\right]\\ &\leqslant \tau\left[\frac{\varepsilon}{4}(\|v^1\|^2+\|v^0\|^2)+\frac{1}{2\varepsilon}\|p^1\|^2\right]+\tau\left[\frac{1}{2}(\|u^1\|^2+\|u^0\|^2)+\frac{1}{2}\|\delta_x^2 q^1\|^2\right].\end{aligned}$$

Noticing

$$\hat{c}_0^{(1,\alpha)}=\sum_{r=0}^{m}\lambda_r\cdot\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}c_0^{(1,\alpha_r)}=\sum_{r=0}^{m}\lambda_r\cdot\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sigma^{1-\alpha_r}=O(\tau^{-\alpha_0})$$

and then taking $\varepsilon=\hat{c}_0^{(1,\alpha)}$, there exists a constant C_1 such that

$$\begin{aligned}&\tau\hat{c}_0^{(1,\alpha)}\|v^1\|^2+\|\delta_x u^1\|^2\\ &\leqslant C_1(\tau\hat{c}_0^{(1,\alpha)}\|v^0\|^2+\|\delta_x u^0\|^2+\tau\|u^0\|^2\\ &\quad+\tau\|u^1\|^2+\tau^{1+\alpha_0}\|p^1\|^2+\tau\|\delta_x^2 q^1\|^2).\end{aligned}\tag{3.262}$$

(II) Making the inner product on both hand sides of (3.248) with $\sigma v^n+(1-\sigma)v^{n-1}$ and noticing (3.255), we have

$$\begin{aligned}&\sum_{k=0}^{n-1}\hat{c}_k^{(n,\alpha)}(v^{n-k}-v^{n-k-1},\sigma v^n+(1-\sigma)v^{n-1})\\ &=(\delta_x^2(\sigma u^n+(1-\sigma)u^{n-1}),\sigma v^n+(1-\sigma)v^{n-1})+(p^n,\sigma v^n+(1-\sigma)v^{n-1})\\ &=-(\delta_x(\sigma u^n+(1-\sigma)u^{n-1}),\delta_x(\sigma v^n+(1-\sigma)v^{n-1}))\\ &\quad+(p^n,\sigma v^n+(1-\sigma)v^{n-1}),\quad 1\leqslant n\leqslant N.\end{aligned}\tag{3.263}$$

By Lemma 2.6.1, the term on the left-hand side of (3.263) can be estimated as

$$\sum_{k=0}^{n-1}\hat{c}_k^{(n,\alpha)}(v^{n-k}-v^{n-k-1},\sigma v^n+(1-\sigma)v^{n-1})$$

$$\geqslant \frac{1}{2}\sum_{k=0}^{n-1}\hat{c}_k^{(n,\alpha)}(\|v^{n-k}\|^2-\|v^{n-k-1}\|^2).$$

Thus,

$$\begin{aligned}&\frac{1}{2}\sum_{k=0}^{n-1}\hat{c}_k^{(n,\alpha)}(\|v^{n-k}\|^2-\|v^{n-k-1}\|^2)\\ \leqslant& -(\delta_x(\sigma u^n+(1-\sigma)u^{n-1}),\delta_x(\sigma v^n+(1-\sigma)v^{n-1}))\\ &+(p^n,\sigma v^n+(1-\sigma)v^{n-1}),\quad 2\leqslant n\leqslant N.\end{aligned}\tag{3.264}$$

It follows from (3.250) that

$$D_{\bar{t}}\delta_x^2u_i^n=\delta_x^2(\sigma v_i^n+(1-\sigma)v_i^{n-1})+\delta_x^2q_i^n,\quad 1\leqslant i\leqslant M-1,\ 2\leqslant n\leqslant N.$$

Making the inner product on both hand sides of the equality above with $-(\sigma u^n+(1-\sigma)u^{n-1})$ and noticing (3.257) lead to

$$\begin{aligned}&(D_{\bar{t}}\delta_xu^n,\delta_x(\sigma u^n+(1-\sigma)u^{n-1}))\\ =&(\delta_x(\sigma v^n+(1-\sigma)v^{n-1}),\delta_x(\sigma u^n+(1-\sigma)u^{n-1}))\\ &-(\delta_x^2q^n,\sigma u^n+(1-\sigma)u^{n-1}),\quad 2\leqslant n\leqslant N.\end{aligned}\tag{3.265}$$

Denote

$$\begin{aligned}F^n=&(2\sigma+1)\|\delta_xu^n\|^2-(2\sigma-1)\|\delta_xu^{n-1}\|^2\\ &+(2\sigma^2+\sigma-1)\|\delta_x(u^n-u^{n-1})\|^2,\quad n\geqslant 1.\end{aligned}$$

It follows from Lemma 3.4.2 that

$$F^n\geqslant\frac{1}{\sigma}\|\delta_xu^n\|^2,\quad n\geqslant 1\tag{3.266}$$

and

$$(D_{\bar{t}}\delta_xu^n,\delta_x(\sigma u^n+(1-\sigma)u^{n-1}))\geqslant\frac{1}{4\tau}(F^n-F^{n-1}),\quad n\geqslant 2.\tag{3.267}$$

Combining (3.265) with (3.267) gets

$$\begin{aligned}\frac{1}{4\tau}(F^n-F^{n-1})\leqslant&(\delta_x(\sigma v^n+(1-\sigma)v^{n-1}),\delta_x(\sigma u^n+(1-\sigma)u^{n-1}))\\ &-(\delta_x^2q^n,\sigma u^n+(1-\sigma)u^{n-1}),\quad 2\leqslant n\leqslant N.\end{aligned}\tag{3.268}$$

Adding (3.264) and (3.268) yields

$$\frac{1}{2}\sum_{k=0}^{n-1}\hat{c}_k^{(n,\alpha)}(\|v^{n-k}\|^2-\|v^{n-k-1}\|^2)+\frac{1}{4\tau}(F^n-F^{n-1})$$

$$\leqslant (p^n, \sigma v^n + (1-\sigma)v^{n-1}) - (\delta_x^2 q^n, \sigma u^n + (1-\sigma)u^{n-1})$$
$$\leqslant \|p^n\| \cdot \|\sigma v^n + (1-\sigma)v^{n-1}\| + \|\delta_x^2 q^n\| \cdot \|\sigma u^n + (1-\sigma)u^{n-1}\|, \quad 2 \leqslant n \leqslant N.$$

Noticing the fact that

$$\begin{aligned}
&\sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} (\|v^{n-k}\|^2 - \|v^{n-k-1}\|^2) \\
&= \sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} \|v^{n-k}\|^2 - \sum_{k=0}^{n-2} \hat{c}_k^{(n-1,\alpha)} \|v^{n-1-k}\|^2 \\
&\quad - \sum_{k=0}^{n-2} (\hat{c}_k^{(n,\alpha)} - \hat{c}_k^{(n-1,\alpha)}) \|v^{n-1-k}\|^2 - \hat{c}_{n-1}^{(n,\alpha)} \|v^0\|^2 \\
&= \sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} \|v^{n-k}\|^2 - \sum_{k=0}^{n-2} \hat{c}_k^{(n-1,\alpha)} \|v^{n-1-k}\|^2 \\
&\quad - (\hat{c}_{n-2}^{(n,\alpha)} - \hat{c}_{n-2}^{(n-1,\alpha)}) \|v^1\|^2 - \hat{c}_{n-1}^{(n,\alpha)} \|v^0\|^2,
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{2}\left(\sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} \|v^{n-k}\|^2 - \sum_{k=0}^{n-2} \hat{c}_k^{(n-1,\alpha)} \|v^{n-1-k}\|^2 \right) + \frac{1}{4\tau}(F^n - F^{n-1}) \\
&\leqslant \frac{1}{2}[(\hat{c}_{n-2}^{(n,\alpha)} - \hat{c}_{n-2}^{(n-1,\alpha)}) \|v^1\|^2 + \hat{c}_{n-1}^{(n,\alpha)} \|v^0\|^2] + \|p^n\| \cdot \|\sigma v^n + (1-\sigma)v^{n-1}\| \\
&\quad + \|\delta_x^2 q^n\| \cdot \|\sigma u^n + (1-\sigma)u^{n-1}\|, \quad 2 \leqslant n \leqslant N. \qquad (3.269)
\end{aligned}$$

Replacing the superscript n with l, and summing up for l from 2 to n on both hand sides of (3.269), we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} \|v^{n-k}\|^2 - \hat{c}_0^{(1,\alpha)} \|v^1\|^2 \right) + \frac{1}{4\tau}(F^n - F^1) \\
&\leqslant \frac{1}{2}\left(\sum_{l=2}^{n} (\hat{c}_{l-2}^{(l,\alpha)} - \hat{c}_{l-2}^{(l-1,\alpha)}) \|v^1\|^2 + \sum_{l=2}^{n} \hat{c}_{l-1}^{(l,\alpha)} \|v^0\|^2 \right) \\
&\quad + \sum_{l=2}^{n} \|p^l\| \cdot \|\sigma v^l + (1-\sigma)v^{l-1}\| + \sum_{l=2}^{n} \|\delta_x^2 q^l\| \cdot \|\sigma u^l + (1-\sigma)u^{l-1}\|, \quad 2 \leqslant n \leqslant N. \qquad (3.270)
\end{aligned}$$

Multiplying both hand sides of (3.270) by 4τ and arranging the inequality, we have

$$\begin{aligned}
&2\tau \sum_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} \|v^{n-k}\|^2 + F^n \\
&\leqslant F^1 + 2\tau \hat{c}_0^{(1,\alpha)} \|v^1\|^2 + 2\tau \sum_{l=2}^{n} (\hat{c}_{l-2}^{(l,\alpha)} - \hat{c}_{l-2}^{(l-1,\alpha)}) \|v^1\|^2 + 2\tau \sum_{l=2}^{n} \hat{c}_{l-1}^{(l,\alpha)} \|v^0\|^2
\end{aligned}$$

$$+ 4\tau\sum_{l=2}^{n}\|p^l\|\cdot\|\sigma v^l + (1-\sigma)v^{l-1}\| + 4\tau\sum_{l=2}^{n}\|\delta_x^2 q^l\|\cdot\|\sigma u^l + (1-\sigma)u^{l-1}\|, \quad 2 \leqslant n \leqslant N. \tag{3.271}$$

It follows from Lemma 1.6.6 that

$$\hat{c}_1^{(n,\alpha)} > \hat{c}_2^{(n,\alpha)} > \cdots > \hat{c}_{n-2}^{(n,\alpha)} > \hat{c}_{n-1}^{(n,\alpha)} > \sum_{r=0}^{m}\lambda_r\frac{T^{-\alpha_r}}{\Gamma(1-\alpha_r)}. \tag{3.272}$$

By Lemma 3.4.3, we get

$$\begin{aligned}\tau\sum_{l=2}^{n}(\hat{c}_{l-2}^{(l,\alpha)} - \hat{c}_{l-2}^{(l-1,\alpha)}) &= \tau\sum_{l=2}^{n}\sum_{r=0}^{m}\lambda_r\cdot\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}(c_{l-2}^{(l,\alpha_r)} - c_{l-2}^{(l-1,\alpha_r)})\\ &= \tau\sum_{r=0}^{m}\lambda_r\cdot\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{l=2}^{n}(c_{l-2}^{(l,\alpha_r)} - c_{l-2}^{(l-1,\alpha_r)})\\ &\leqslant \sum_{r=0}^{m}\lambda_r\cdot\frac{\tau^{1-\alpha_r}}{\Gamma(2-\alpha_r)}\left[\frac{1-\alpha_r}{12}\left(\frac{\alpha_r}{\sigma}+1\right)\sigma^{-\alpha_r}\right]\\ &= \sum_{r=0}^{m}\frac{\lambda_r}{12}\cdot\frac{\tau^{1-\alpha_r}}{\Gamma(1-\alpha_r)}\left(\frac{\alpha_r}{\sigma}+1\right)\sigma^{-\alpha_r}\end{aligned} \tag{3.273}$$

and

$$\begin{aligned}\tau\sum_{l=2}^{n}\hat{c}_{l-1}^{(l,\alpha)} &= \tau\sum_{l=2}^{n}\sum_{r=0}^{m}\lambda_r\cdot\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}c_{l-1}^{(l,\alpha_r)}\\ &= \sum_{r=0}^{m}\lambda_r\cdot\frac{\tau^{1-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{l=2}^{n}c_{l-1}^{(l,\alpha_r)}\\ &= \sum_{r=0}^{m}\lambda_r\cdot\frac{\tau^{1-\alpha_r}}{\Gamma(2-\alpha_r)}\frac{3}{2}(n-1+\sigma)^{1-\alpha_r}\\ &\leqslant \frac{3}{2}\sum_{r=0}^{m}\lambda_r\cdot\frac{T^{1-\alpha_r}}{\Gamma(2-\alpha_r)}.\end{aligned} \tag{3.274}$$

The application of (3.272)–(3.274) in (3.271) leads to

$$\begin{aligned}&2\sum_{r=0}^{m}\lambda_r\frac{T^{-\alpha_r}}{\Gamma(1-\alpha_r)}\cdot\tau\sum_{k=0}^{n-1}\|v^{n-k}\|^2 + F^n\\ &\leqslant F^1 + 2\tau\hat{c}_0^{(1,\alpha)}\|v^1\|^2 + 2\sum_{r=0}^{m}\frac{\lambda_r}{12}\cdot\frac{\tau^{1-\alpha_r}}{\Gamma(1-\alpha_r)}\left(\frac{\alpha_r}{\sigma}+1\right)\sigma^{-\alpha_r}\|v^1\|^2\\ &\quad + 3\sum_{r=0}^{m}\lambda_r\cdot\frac{T^{1-\alpha_r}}{\Gamma(2-\alpha_r)}\|v^0\|^2 + 4\tau\sum_{l=2}^{n}\|p^l\|\cdot\|\sigma v^l + (1-\sigma)v^{l-1}\|\\ &\quad + 4\tau\sum_{l=2}^{n}\|\delta_x^2 q^l\|\cdot\|\sigma u^l + (1-\sigma)u^{l-1}\|\end{aligned}$$

$$\begin{aligned}
&\leqslant F^1 + 2\tau\hat{c}_0^{(1,\alpha)}\|v^1\|^2 + \sum_{r=0}^{m}\frac{\lambda_r}{6}\cdot\frac{\tau^{1-\alpha_r}}{\Gamma(1-\alpha_r)}\left(\frac{\alpha_r}{\sigma}+1\right)\sigma^{-\alpha_r}\|v^1\|^2\\
&\quad + 3\sum_{r=0}^{m}\lambda_r\cdot\frac{T^{1-\alpha_r}}{\Gamma(2-\alpha_r)}\|v^0\|^2\\
&\quad + 2\tau\sum_{l=2}^{n}\left[\varepsilon_0\,\|\sigma v^l+(1-\sigma)v^{l-1}\|^2+\frac{1}{\varepsilon_0}\,\|p^l\|^2\right]\\
&\quad + 2\tau\sum_{l=2}^{n}[\|\sigma u^l+(1-\sigma)u^{l-1}\|^2+\|\delta_x^2 q^l\|^2]\\
&\leqslant F^1 + 2\tau\hat{c}_0^{(1,\alpha)}\|v^1\|^2 + \sum_{r=0}^{m}\frac{\lambda_r}{6}\cdot\frac{\tau^{1-\alpha_r}}{\Gamma(1-\alpha_r)}\left(\frac{\alpha_r}{\sigma}+1\right)\sigma^{-\alpha_r}\|v^1\|^2\\
&\quad + 3\sum_{r=0}^{m}\lambda_r\cdot\frac{T^{1-\alpha_r}}{\Gamma(2-\alpha_r)}\|v^0\|^2 + 2\tau\varepsilon_0\sum_{l=2}^{n}(\|v^l\|^2+\|v^{l-1}\|^2)\\
&\quad + \frac{2\tau}{\varepsilon_0}\sum_{l=2}^{n}\|p^l\|^2 + 2\tau\sum_{l=2}^{n}(\|u^l\|^2+\|u^{l-1}\|^2+\|\delta_x^2 q^l\|^2),\quad 2\leqslant n\leqslant N.
\end{aligned}$$

Taking $\varepsilon_0 = \frac{1}{4}\sum_{r=0}^{m}\lambda_r\frac{T^{-\alpha_r}}{\Gamma(1-\alpha_r)}$, noticing (3.262), (3.266) and

$$\begin{aligned}
F^1 &= (2\sigma+1)\|\delta_x u^1\|^2 - (2\sigma-1)\|\delta_x u^0\|^2\\
&\quad + (2\sigma^2+\sigma-1)\|\delta_x(u^1-u^0)\|^2\\
&\leqslant (2\sigma+1)\|\delta_x u^1\|^2 - (2\sigma-1)\|\delta_x u^0\|^2\\
&\quad + 2(2\sigma^2+\sigma-1)(\|\delta_x u^1\|^2+\|\delta_x u^0\|^2)\\
&= (4\sigma^2+4\sigma-1)\|\delta_x u^1\|^2 + (4\sigma^2-1)\|\delta_x u^0\|^2,
\end{aligned}$$

it is easy to know that there exists a positive constant C_2 such that

$$\begin{aligned}
&\tau\sum_{k=0}^{n-1}\|v^{n-k}\|^2 + \|\delta_x u^n\|^2\\
&\leqslant C_2\left(\tau\hat{c}_0^{(1,\alpha)}\|v^0\|^2 + \|\delta_x u^0\|^2 + \tau\sum_{l=1}^{n}\|u^l\|^2 + \tau\sum_{l=1}^{n}\|p^l\|^2 + \tau\sum_{l=1}^{n}\|\delta_x^2 q^l\|^2\right)\\
&\leqslant C_2\left(\tau\hat{c}_0^{(1,\alpha)}\|v^0\|^2 + \|\delta_x u^0\|^2 + \tau\sum_{l=1}^{n}\frac{L^2}{6}\|\delta_x u^l\|^2 + \tau\sum_{l=1}^{n}\|p^l\|^2\right.\\
&\quad\left. + \tau\sum_{l=1}^{n}\|\delta_x^2 q^l\|^2\right),\quad 1\leqslant n\leqslant N.
\end{aligned}$$

The application of the Gronwall inequality will lead to (3.253). The proof ends.

□

3.7.4 Convergence of the difference scheme

Theorem 3.7.3. *Suppose* $\{U_i^n, V_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n, v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (3.220)–(3.223) *and the difference scheme* (3.235)–(3.239), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad z_i^n = V_i^n - v_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then it holds

$$\tau \sum_{k=1}^{n} \|z^k\|^2 + \|\delta_x e^n\|^2 \leqslant C(c_{10}^2 + c_{11}^2) LT(\tau^2 + h^2)^2, \quad 1 \leqslant n \leqslant N,$$

where the constant C is defined in Theorem 3.7.2.

Proof. Subtracting (3.235)–(3.239) from (3.227), (3.229)–(3.230), and (3.233)–(3.234) respectively, produces the system of error equations as follows:

$$\begin{cases} \sum\limits_{k=0}^{n-1} \hat{c}_k^{(n,\alpha)} (z_i^{n-k} - z_i^{n-k-1}) = \sigma \delta_x^2 e_i^n + (1-\sigma)\delta_x^2 e_i^{n-1} + (r_{10})_i^n, \\ \qquad\qquad\qquad\qquad\qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ \delta_t e_i^{\frac{1}{2}} = z_i^{\frac{1}{2}} + (r_{11})_i^1, \quad 0 \leqslant i \leqslant M, \\ D_{\bar{t}} e_i^n = \sigma z_i^n + (1-\sigma) z_i^{n-1} + (r_{11})_i^n, \quad 0 \leqslant i \leqslant M,\ 2 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad z_i^0 = 0, \quad 0 \leqslant i \leqslant M, \\ e_0^n = 0, \quad e_M^n = 0, \quad 1 \leqslant n \leqslant N. \end{cases}$$

Noticing (3.228) and (3.231)–(3.232), an immediate consequence of Theorem 3.7.2 into the system above is

$$\begin{aligned} &\tau \sum_{k=1}^{n} \|z^k\|^2 + \|\delta_x e^n\|^2 \\ &\leqslant C\left[\tau \hat{c}_0^{(1,\alpha)} \|z^0\|^2 + \|\delta_x e^0\|^2 + \tau \sum_{l=1}^{n} \|(r_{10})^l\|^2 + \tau \sum_{l=1}^{n} \|\delta_x^2 (r_{11})^l\|^2\right] \\ &\leqslant C\left[\tau \sum_{l=1}^{n} L[c_{10}(\tau^2 + h^2)]^2 + \tau \sum_{l=1}^{n} L(c_{11}\tau^2)^2\right] \\ &\leqslant C(c_{10}^2 + c_{11}^2) LT(\tau^2 + h^2)^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

The proof ends. □

3.8 The difference method for the time-fractional mixed diffusion and wave equations

Consider the difference method by using L1 approximation for the following initial-boundary value problem of the time-fractional mixed diffusion and wave equation

$$\begin{cases} {}_0^C D_t^{\gamma} u(x,t) + {}_0^C D_t^{\alpha} u(x,t) = u_{xx}(x,t) + f(x,t), \\ \qquad\qquad x \in (0,L),\ t \in (0,T], & (3.275) \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in (0,L), & (3.276) \\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), \quad t \in [0,T], & (3.277) \end{cases}$$

where the functions f, φ, ψ, μ and ν are given, $\varphi(0) = \mu(0)$, $\varphi(L) = \nu(0)$, $\psi(0) = \mu'(0)$, $\psi(L) = \nu'(0)$, ${}_0^C D_t^{\gamma} u(x,t)$ and ${}_0^C D_t^{\alpha} u(x,t)$ denote the time Caputo derivatives of u given by, respectively,

$$ {}_0^C D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, \quad 0 < \alpha < 1, $$

and

$$ {}_0^C D_t^{\gamma} u(x,t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{\partial^2 u(x,s)}{\partial s^2} \frac{ds}{(t-s)^{\gamma-1}}, \quad 1 < \gamma < 2. $$

Suppose the exact solution $u \in C^{(4,3)}([0,L] \times [0,T])$.

3.8.1 Derivation of the difference scheme

Denote

$$ U_i^n = u(x_i, t_n), \quad f_i^{n-\frac{1}{2}} = \frac{1}{2}[f(x_i,t_n) + f(x_i,t_{n-1})]. $$

Considering equation (3.275) at the points (x_i, t_n) and (x_i, t_{n-1}), respectively, and averaging the corresponding results yield

$$ \begin{aligned} &\frac{1}{2}\big[{}_0^C D_t^{\gamma} u(x_i,t_n) + {}_0^C D_t^{\gamma} u(x_i,t_{n-1})\big] + \frac{1}{2}\big[{}_0^C D_t^{\alpha} u(x_i,t_n) + {}_0^C D_t^{\alpha} u(x_i,t_{n-1})\big] \\ &= \frac{1}{2}\big[u_{xx}(x_i,t_n) + u_{xx}(x_i,t_{n-1})\big] + f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \qquad (3.278) \end{aligned} $$

With the help of Theorem 1.6.1 and Theorem 1.6.2, we have

$$ {}_0^C D_t^{\alpha} u(x_i,t_n) $$

$$= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)}U_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})U_i^k - a_{n-1}^{(\alpha)}U_i^0\right] + O(\tau^{2-\alpha})$$

$$= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n} a_{n-k}^{(\alpha)}\delta_t U_i^{k-\frac{1}{2}} + O(\tau^{2-\alpha}), \quad 1 \leqslant i \leqslant M-1,\ 0 \leqslant n \leqslant N \tag{3.279}$$

and

$$\frac{1}{2}\left[{}_0^C D_t^\gamma u(x_i,t_n) + {}_0^C D_t^\gamma u(x_i,t_{n-1})\right]$$
$$= \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\delta_t U_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t U_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}u_t(x_i,t_0)\right]$$
$$+ O(\tau^{3-\gamma}), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{3.280}$$

Applying (3.279) and (3.280) into (3.278), we obtain

$$\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\delta_t U_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t U_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_i\right]$$
$$+ \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\left[\frac{a_0^{(\alpha)}}{2}\delta_t U_i^{n-\frac{1}{2}} + \sum_{k=1}^{n-1}\frac{a_{n-k}^{(\alpha)} + a_{n-k-1}^{(\alpha)}}{2}\delta_t U_i^{k-\frac{1}{2}}\right]$$
$$= \delta_x^2 U_i^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}} + (r_{12})_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N \tag{3.281}$$

and there exists a positive constant c_{12} such that

$$\left|(r_{12})_i^{n-\frac{1}{2}}\right| \leqslant c_{12}(\tau^{\min\{2-\alpha,3-\gamma\}} + h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{3.282}$$

For simplicity, denote $s_\gamma = \tau^{\gamma-1}\Gamma(3-\gamma)$ and $s_\alpha = \tau^{\alpha-1}\Gamma(2-\alpha)$.

Noticing the initial-boundary value conditions (3.276)–(3.277), we have

$$\begin{cases} u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (3.283)\\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (3.284)\end{cases}$$

Dropping $(r_{12})_i^{n-\frac{1}{2}}$ in (3.281) and replacing U_i^n by u_i^n, we construct a finite difference scheme for solving the problem (3.275)–(3.277) as follows:

$$\begin{cases} \frac{1}{s_\gamma}\left[b_0^{(\gamma)}\delta_t u_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t u_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_i\right] \\ \quad + \frac{1}{s_\alpha}\left[\frac{a_0^{(\alpha)}}{2}\delta_t u_i^{n-\frac{1}{2}} + \sum_{k=1}^{n-1}\frac{a_{n-k}^{(\alpha)} + a_{n-1-k}^{(\alpha)}}{2}\delta_t u_i^{k-\frac{1}{2}}\right] \\ \quad = \delta_x^2 u_i^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.285)\\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (3.286)\\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. & (3.287)\end{cases}$$

3.8.2 Solvability of the difference scheme

Theorem 3.8.1. *The difference scheme* (3.285)–(3.287) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The value of u^0 is determined by (3.286)–(3.287). Now assume that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n can be obtained from (3.285) and (3.287). To show its unique solvability, it suffices to verify that the corresponding homogeneous one

$$\begin{cases} \left(\dfrac{b_0^{(\gamma)}}{s_\gamma} + \dfrac{a_0^{(\alpha)}}{2s_\alpha}\right)\dfrac{1}{\tau}u_i^n - \dfrac{1}{2}\delta_x^2 u_i^n = 0, \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = u_M^n = 0 \end{cases}$$

has only the trivial solution. By the maximum principle, the conclusion is obvious. The proof ends. □

3.8.3 Stability of the difference scheme

Before analyzing the stability of the difference scheme, the following three lemmas are provided at first.

Lemma 3.8.1. [52] Let $\{g_0, g_1, \ldots, g_n, \ldots\}$ be a sequence of real numbers with the properties

$$g_n \geqslant 0, \quad g_n - g_{n-1} \leqslant 0, \quad g_{n+1} - 2g_n + g_{n-1} \geqslant 0.$$

Then for any positive integer m and for any mesh functions $V_1, V_2, \ldots, V_m \in \mathring{\mathcal{U}}_h$, it holds that

$$\sum_{n=1}^{m}\left(\sum_{p=0}^{n-1} g_p V_{n-p}, V_n\right) \geqslant 0.$$

Lemma 3.8.2. *Let* $\{b_l^{(\gamma)}\}$ *be defined in* (1.64). *For any mesh functions* $\psi, V_1, V_2, \ldots, V_m \in \mathring{\mathcal{U}}_h$, *it holds that*

$$\sum_{n=1}^{m}\left(b_0^{(\gamma)}V^n - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})V^k - b_{n-1}^{(\gamma)}\psi, V^n\right)$$
$$\geqslant \frac{1}{2}\left(\sum_{k=1}^{m} b_{m-k}^{(\gamma)}\|V^k\|^2 - \sum_{n=1}^{m} b_{n-1}^{(\gamma)}\|\psi\|^2\right).$$

Proof.

$$\sum_{n=1}^{m}\left(b_0^{(\gamma)}V^n-\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})V^k-b_{n-1}^{(\gamma)}\psi,V^n\right)$$
$$=\sum_{n=1}^{m}\left[b_0^{(\gamma)}\|V^n\|^2-\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})(V^k,V^n)-b_{n-1}^{(\gamma)}(\psi,V^n)\right]$$
$$\geqslant\sum_{n=1}^{m}\left[b_0^{(\gamma)}\|V^n\|^2-\frac{1}{2}\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})(\|V^k\|^2+\|V^n\|^2)\right.$$
$$\left.-\frac{1}{2}b_{n-1}^{(\gamma)}(\|\psi\|^2+\|V^n\|^2)\right]$$
$$=\frac{1}{2}\sum_{n=1}^{m}\left(\sum_{k=1}^{n}b_{n-k}^{(\gamma)}\|V^k\|^2-\sum_{k=1}^{n-1}b_{n-k-1}^{(\gamma)}\|V^k\|^2-b_{n-1}^{(\gamma)}\|\psi\|^2\right)$$
$$=\frac{1}{2}\left(\sum_{k=1}^{m}b_{m-k}^{(\gamma)}\|V^k\|^2-\sum_{n=1}^{m}b_{n-1}^{(\gamma)}\|\psi\|^2\right).$$

□

Lemma 3.8.3. [77] *Let $\{a_l^{(\alpha)}\}$ be defined in* (1.57). *For any mesh functions $V_1,V_2,\ldots,V_m\in\mathring{\mathcal{U}}_h$, it holds that*

$$\sum_{n=1}^{m}\left(\frac{a_0^{(\alpha)}}{2}V^n+\sum_{k=1}^{n-1}\frac{a_{n-k}^{(\alpha)}+a_{n-1-k}^{(\alpha)}}{2}V^k,V^n\right)\geqslant-a_0^{(\alpha)}\sum_{n=1}^{m}\|V^n\|^2.$$

Proof. Notice the fact that

$$\sum_{n=1}^{m}\left(\frac{a_0^{(\alpha)}}{2}V^n+\sum_{k=1}^{n-1}\frac{a_{n-k}^{(\alpha)}+a_{n-1-k}^{(\alpha)}}{2}V^k,V^n\right)$$
$$=\sum_{n=1}^{m}\left(\frac{3}{2}a_0^{(\alpha)}V^n+\sum_{k=1}^{n-1}\frac{a_{n-k}^{(\alpha)}+a_{n-k-1}^{(\alpha)}}{2}V^k,V^n\right)-\sum_{n=1}^{m}a_0^{(\alpha)}\|V^n\|^2.$$

By some plain calculations, the coefficients of the first term on the right-hand side of the equality above satisfy the conditions of Lemma 3.8.1, hence it is nonnegative, which implies

$$\sum_{n=1}^{m}\left(\frac{a_0^{(\alpha)}}{2}V^n+\sum_{k=1}^{n-1}\frac{a_{n-k}^{(\alpha)}+a_{n-1-k}^{(\alpha)}}{2}V^k,V^n\right)\geqslant-a_0^{(\alpha)}\sum_{n=1}^{m}\|V^n\|^2.$$

The proof ends. □

Remark 3.8.1. In Lemma 3.8.3, a minor error in [77] has been corrected.

We have the following result on the stability of the difference scheme.

Theorem 3.8.2. *Let* $\{u_i^n \mid 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N\}$ *be the solution of the difference scheme* (3.285)–(3.287), *in which* $\mu(t) \equiv 0$, $\nu(t) \equiv 0$. *Denote*

$$\tau_0 = \left(\frac{T^{1-\gamma}\Gamma(2-\alpha)}{4\Gamma(2-\gamma)} \right)^{1/(1-\alpha)},$$

then, if $\tau \leqslant \tau_0$, *it holds that*

$$\begin{aligned} \|\delta_x u^n\|^2 \leqslant \|\delta_x u^0\|^2 &+ \frac{T^{2-\gamma}}{\Gamma(3-\gamma)}\|\psi\|^2 \\ &+ 2\Gamma(2-\gamma)T^{\gamma-1}\tau \sum_{k=1}^{n} \|f^{k-\frac{1}{2}}\|^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Proof. Taking the inner product on both hand sides of (3.285) with $\delta_t u^{n-\frac{1}{2}}$ and summing up for n from 1 to m, yield that

$$\begin{aligned} &\frac{1}{s_\gamma}\sum_{n=1}^{m}\Bigg[b_0^{(\gamma)}\|\delta_t u^{n-\frac{1}{2}}\|^2 - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\delta_t u^{k-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}) \\ &\quad - b_{n-1}^{(\gamma)}(\psi, \delta_t u^{n-\frac{1}{2}})\Bigg] \\ &\quad + \frac{1}{s_\alpha}\sum_{n=1}^{m}\left(\frac{a_0^{(\alpha)}}{2}\delta_t u^{n-\frac{1}{2}} + \sum_{k=1}^{n-1}\frac{a_{n-k}^{(\alpha)} + a_{n-k-1}^{(\alpha)}}{2}\delta_t u^{k-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}} \right) \\ &\quad + \frac{1}{2\tau}(\|\delta_x u^m\|^2 - \|\delta_x u^0\|^2) \\ &= \sum_{n=1}^{m}(f^{n-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}), \quad 1 \leqslant m \leqslant N. \end{aligned}$$

By Lemma 3.8.2 and Lemma 3.8.3, we can obtain

$$\begin{aligned} &\frac{1}{2s_\gamma}\left(\sum_{k=1}^{m} b_{m-k}^{(\gamma)}\|\delta_t u^{k-\frac{1}{2}}\|^2 - \sum_{n=1}^{m} b_{n-1}^{(\gamma)}\|\psi\|^2 \right) \\ &\quad - \frac{1}{s_\alpha}\sum_{n=1}^{m}\|\delta_t u^{n-\frac{1}{2}}\|^2 + \frac{1}{2\tau}(\|\delta_x u^m\|^2 - \|\delta_x u^0\|^2) \\ &\leqslant \sum_{n=1}^{m}(f^{n-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}), \quad 1 \leqslant m \leqslant N. \end{aligned}$$

Further we obtain

$$\begin{aligned} &\frac{1}{4s_\gamma}\sum_{k=1}^{m} b_{m-k}^{(\gamma)}\|\delta_t u^{k-\frac{1}{2}}\|^2 \\ &\quad + \left(\frac{1}{4s_\gamma}\sum_{k=1}^{m} b_{m-k}^{(\gamma)}\|\delta_t u^{k-\frac{1}{2}}\|^2 - \frac{1}{s_\alpha}\sum_{k=1}^{m}\|\delta_t u^{k-\frac{1}{2}}\|^2 \right) \end{aligned}$$

$$+\frac{1}{2\tau}(\|\delta_x u^m\|^2-\|\delta_x u^0\|^2)$$
$$\leqslant \frac{1}{2s_\gamma}\sum_{n=1}^{m} b_{n-1}^{(\gamma)}\|\psi\|^2+\sum_{n=1}^{m}(f^{n-\frac{1}{2}},\delta_t u^{n-\frac{1}{2}}),\quad 1\leqslant m\leqslant N. \tag{3.288}$$

For the second term on the left-hand side of (3.288), we have

$$\begin{aligned}
&\frac{1}{4s_\gamma}\sum_{k=1}^{m} b_{m-k}^{(\gamma)}\|\delta_t u^{k-\frac{1}{2}}\|^2-\frac{1}{s_\alpha}\sum_{k=1}^{m}\|\delta_t u^{k-\frac{1}{2}}\|^2\\
\geqslant{}& \frac{T^{1-\gamma}}{4\Gamma(2-\gamma)}\sum_{k=1}^{m}\|\delta_t u^{k-\frac{1}{2}}\|^2-\frac{1}{s_\alpha}\sum_{k=1}^{m}\|\delta_t u^{k-\frac{1}{2}}\|^2\\
={}&\left(\frac{T^{1-\gamma}}{4\Gamma(2-\gamma)}-\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\right)\sum_{k=1}^{m}\|\delta_t u^{k-\frac{1}{2}}\|^2,\quad 1\leqslant m\leqslant N,
\end{aligned}\tag{3.289}$$

in which the inequality

$$b_l^{(\gamma)}=(2-\gamma)(l+\theta_l)^{1-\gamma}\geqslant(2-\gamma)N^{1-\gamma},\quad \theta_l\in(0,1),\ 0\leqslant l\leqslant N-1$$

was used. A careful observation of (3.289) shows that when $\tau\leqslant\tau_0$, the right-hand side of (3.289) is nonnegative. Thus it follows from (3.288) that

$$\begin{aligned}
&\frac{T^{1-\gamma}}{4\Gamma(2-\gamma)}\sum_{k=1}^{m}\|\delta_t u^{k-\frac{1}{2}}\|^2+\frac{1}{2\tau}(\|\delta_x u^m\|^2-\|\delta_x u^0\|^2)\\
\leqslant{}& \frac{1}{2s_\gamma}\sum_{n=1}^{m} b_{n-1}^{(\gamma)}\|\psi\|^2+\sum_{n=1}^{m}(f^{n-\frac{1}{2}},\delta_t u^{n-\frac{1}{2}})\\
={}& \frac{1}{2s_\gamma}m^{2-\gamma}\|\psi\|^2+\sum_{n=1}^{m}(f^{n-\frac{1}{2}},\delta_t u^{n-\frac{1}{2}})\\
\leqslant{}& \frac{1}{2s_\gamma}m^{2-\gamma}\|\psi\|^2+\sum_{n=1}^{m}\left(\frac{T^{1-\gamma}}{4\Gamma(2-\gamma)}\|\delta_t u^{n-\frac{1}{2}}\|^2+\Gamma(2-\gamma)T^{\gamma-1}\|f^{n-\frac{1}{2}}\|^2\right),\quad 1\leqslant m\leqslant N.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\|\delta_x u^m\|^2&\leqslant\|\delta_x u^0\|^2+\frac{\tau^{2-\gamma}}{\Gamma(3-\gamma)}m^{2-\gamma}\|\psi\|^2+2\Gamma(2-\gamma)T^{\gamma-1}\tau\sum_{n=1}^{m}\|f^{n-\frac{1}{2}}\|^2\\
&\leqslant\|\delta_x u^0\|^2+\frac{T^{2-\gamma}}{\Gamma(3-\gamma)}\|\psi\|^2+2\Gamma(2-\gamma)T^{\gamma-1}\tau\sum_{n=1}^{m}\|f^{n-\frac{1}{2}}\|^2,\quad 1\leqslant m\leqslant N.
\end{aligned}$$

This completes the proof. □

3.8.4 Convergence of the difference scheme

With the help of Theorem 3.8.2, we can directly derive the following convergence theorem.

Theorem 3.8.3. *Suppose that* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of the problem* (3.275)–(3.277) *and* $\{u_i^n \mid 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N\}$ *is the solution the difference scheme* (3.285)–(3.287), *respectively. Denote*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N.$$

Then, if $\tau \leqslant \tau_0$, *we have*

$$\|\delta_x e^n\| \leqslant \sqrt{2\Gamma(2-\gamma)T^\gamma L} c_{12}(\tau^{\min\{2-\alpha,3-\gamma\}} + h^2), \quad 1 \leqslant n \leqslant N,$$

where c_{12} *is defined in* (3.282) *and* τ_0 *is defined in Theorem* 3.8.2.

Proof. Subtracting (3.285)–(3.287) from (3.281) and (3.283)–(3.284), respectively, we obtain the system of error equations as follows:

$$\begin{cases} \frac{1}{s_\gamma}\left(b_0^{(\gamma)}\delta_t e_i^{n-\frac{1}{2}} - \sum\limits_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t e_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\cdot 0\right) \\ \quad + \frac{1}{s_\alpha}\left(\frac{a_0^{(\alpha)}}{2}\delta_t e_i^{n-\frac{1}{2}} + \sum\limits_{k=1}^{n-1}\frac{a_{n-k}^{(\alpha)} + a_{n-1-k}^{(\alpha)}}{2}\delta_t e_i^{k-\frac{1}{2}}\right) \\ \quad = \delta_x^2 e_i^{n-\frac{1}{2}} + (r_{12})_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Combining (3.282) with Theorem 3.8.2, we obtain

$$\begin{aligned} \|\delta_x e^n\|^2 &\leqslant \|\delta_x e^0\|^2 + 2\Gamma(2-\gamma)T^{\gamma-1}\tau\sum_{k=1}^{n}\|(r_{12})^{k-\frac{1}{2}}\|^2 \\ &\leqslant 2\Gamma(2-\gamma)T^{\gamma-1}n\tau L[c_{12}(\tau^{\min\{2-\alpha,3-\gamma\}} + h^2)]^2 \\ &\leqslant 2\Gamma(2-\gamma)T^\gamma L c_{12}^2(\tau^{\min\{2-\alpha,3-\gamma\}} + h^2)^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

This completes the proof. □

3.9 The ADI method based on L1 approximation for 2D problem

Consider the following problem of the two-dimensional time-fractional wave equations:

$$\begin{cases} {}_0^C D_t^\gamma u(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) + f(x,y,t), \\ \qquad\qquad (x,y) \in \Omega, t \in (0,T], & (3.290) \\ u(x,y,0) = \varphi(x,y), \quad u_t(x,y,0) = \psi(x,y), \quad (x,y) \in \Omega, & (3.291) \\ u(x,y,t) = \mu(x,y,t), \quad (x,y) \in \partial\Omega,\ t \in [0,T], & (3.292) \end{cases}$$

where $\Omega = (0, L_1) \times (0, L_2)$, $\gamma \in (1, 2)$, the functions f, φ, ψ, μ are given and when $(x, y) \in \partial\Omega$, $\mu(x, y, 0) = \varphi(x, y)$, $\mu_t(x, y, 0) = \psi(x, y)$. Suppose the exact solution $u \in C^{(4,4,3)}(\bar{\Omega} \times [0, T])$.

Take the same mesh partition and notations as those in Section 2.10. For any mesh function $v = \{v_{ij}^k \mid (i,j) \in \bar{\omega}, 0 \leqslant k \leqslant N\}$ defined on $\Omega_h \times \Omega_\tau$, let

$$v_{ij}^{k-\frac{1}{2}} = \frac{1}{2}(v_{ij}^k + v_{ij}^{k-1}), \quad \delta_t v_{ij}^{k-\frac{1}{2}} = \frac{1}{\tau}(v_{ij}^k - v_{ij}^{k-1}).$$

Introduce the same mesh function spaces $\mathcal{V}_h$ and $\mathring{\mathcal{V}}_h$ as those in Section 2.10.

Denote

$$U_{ij}^n = u(x_i, y_j, t_n), \quad \psi_{ij} = \psi(x_i, y_j), \quad f_{ij}^n = f(x_i, y_j, t_n),$$
$$(i,j) \in \bar{\omega}, \ 0 \leqslant n \leqslant N.$$

3.9.1 Derivation of the difference scheme

Considering equation (3.290) at the point (x_i, y_j, t_n), we have

$${}_0^C D_t^\gamma u(x_i, y_j, t_n) = u_{xx}(x_i, y_j, t_n) + u_{yy}(x_i, y_j, t_n) + f_{ij}^n, \quad (i,j) \in \omega, \ 0 \leqslant n \leqslant N.$$

Taking an average on two adjacent time levels and applying the L1 formula (1.69) and the second-order central difference quotient to approximate the time-fractional derivative and the spatial second-order derivative, respectively, we have from Theorem 1.6.2 and Lemma 2.1.3 that

$$\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\delta_t U_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t U_{ij}^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_{ij}\right]$$
$$= \delta_x^2 U_{ij}^{n-\frac{1}{2}} + \delta_y^2 U_{ij}^{n-\frac{1}{2}} + f_{ij}^{n-\frac{1}{2}} + (r_{13})_{ij}^{n-\frac{1}{2}}, \quad (i,j) \in \omega, \ 1 \leqslant n \leqslant N, \tag{3.293}$$

and there is positive constant c_{13} such that

$$\left|(r_{13})_{ij}^{n-\frac{1}{2}}\right| \leqslant c_{13}(\tau^{3-\gamma} + h_1^2 + h_2^2), \quad (i,j) \in \omega, \ 1 \leqslant n \leqslant N,$$

where $\{b_l^{(\gamma)}\}$ is defined in (1.64).

Adding a small term $\frac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t U_{ij}^{n-\frac{1}{2}}$ into both hand sides of (3.293) produces

$$\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\delta_t U_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t U_{ij}^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_{ij}\right]$$
$$+ \frac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t U_{ij}^{n-\frac{1}{2}} = \delta_x^2 U_{ij}^{n-\frac{1}{2}} + \delta_y^2 U_{ij}^{n-\frac{1}{2}}$$

$$+f_{ij}^{n-\frac{1}{2}} + (r_{14})_{ij}^{n-\frac{1}{2}}, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \tag{3.294}$$

where

$$(r_{14})_{ij}^{n-\frac{1}{2}} = (r_{13})_{ij}^{n-\frac{1}{2}} + \frac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t U_{ij}^{n-\frac{1}{2}},$$

and there is a positive constant c_{14} such that

$$\left|(r_{14})_{ij}^{n-\frac{1}{2}}\right| \leqslant c_{14}(\tau^{3-\gamma} + h_1^2 + h_2^2), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \tag{3.295}$$

Noticing the initial-boundary value conditions (3.291)–(3.292), we have

$$\begin{cases} U_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (3.296)\\ U_{ij}^n = \mu(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (3.297) \end{cases}$$

Omitting the small term $(r_{14})_{ij}^{n-\frac{1}{2}}$ in (3.294) and replacing the exact solution U_{ij}^n with its numerical one u_{ij}^n give the difference scheme for (3.290)–(3.292) as follows:

$$\begin{cases} \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\delta_t u_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t u_{ij}^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_{ij}\right] \\ \quad + \frac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t u_{ij}^{n-\frac{1}{2}} = \delta_x^2 u_{ij}^{n-\frac{1}{2}} + \delta_y^2 u_{ij}^{n-\frac{1}{2}} + f_{ij}^{n-\frac{1}{2}}, \\ \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (3.298)\\ u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (3.299)\\ u_{ij}^n = \mu(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (3.300) \end{cases}$$

Denote

$$\eta = \tau^{\gamma-1}\Gamma(3-\gamma).$$

Equation (3.298) can be rearranged as

$$\begin{aligned} &\delta_t u_{ij}^{n-\frac{1}{2}} - \eta\delta_x^2 u_{ij}^{n-\frac{1}{2}} - \eta\delta_y^2 u_{ij}^{n-\frac{1}{2}} + \frac{1}{4}\eta^2\tau^2\delta_x^2\delta_y^2\delta_t u_{ij}^{n-\frac{1}{2}} \\ &= \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t u_{ij}^{k-\frac{1}{2}} + b_{n-1}^{(\gamma)}\psi_{ij} + \eta f_{ij}^{n-\frac{1}{2}}, \end{aligned}$$

or

$$\begin{aligned} &u_{ij}^n - \frac{\eta}{2}\tau\delta_x^2 u_{ij}^n - \frac{\eta}{2}\tau\delta_y^2 u_{ij}^n + \frac{1}{4}\eta^2\tau^2\delta_x^2\delta_y^2 u_{ij}^n \\ &= u_{ij}^{n-1} + \frac{\eta}{2}\tau\delta_x^2 u_{ij}^{n-1} + \frac{\eta}{2}\tau\delta_y^2 u_{ij}^{n-1} + \frac{1}{4}\eta^2\tau^2\delta_x^2\delta_y^2 u_{ij}^{n-1} \\ &\quad + \tau\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t u_{ij}^{k-\frac{1}{2}} + \tau b_{n-1}^{(\gamma)}\psi_{ij} + \tau\eta f_{ij}^{n-\frac{1}{2}}, \end{aligned}$$

which is

$$\left(\mathcal{I}-\frac{\eta}{2}\tau\delta_x^2\right)\left(\mathcal{I}-\frac{\eta}{2}\tau\delta_y^2\right)u_{ij}^n=\left(\mathcal{I}+\frac{\eta}{2}\tau\delta_x^2\right)\left(\mathcal{I}+\frac{\eta}{2}\tau\delta_y^2\right)u_{ij}^{n-1}$$
$$+\tau\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\delta_t u_{ij}^{k-\frac{1}{2}}+\tau b_{n-1}^{(\gamma)}\psi_{ij}+\tau\eta f_{ij}^{n-\frac{1}{2}}.$$

Let

$$u_{ij}^*=\left(\mathcal{I}-\frac{\eta}{2}\tau\delta_y^2\right)u_{ij}^n,$$

then the difference scheme (3.298)–(3.300) can be written as the following ADI form:

On each time level $t=t_n$ $(1\leqslant n\leqslant N)$, firstly, for any fixed j from 1 to M_2-1, solve a series of linear systems in $\{u_{ij}^*\mid 0\leqslant i\leqslant M_1\}$ in x direction

$$\begin{cases}\left(\mathcal{I}-\frac{\eta}{2}\tau\delta_x^2\right)u_{ij}^*=\left(\mathcal{I}+\frac{\eta}{2}\tau\delta_x^2\right)\left(\mathcal{I}+\frac{\eta}{2}\tau\delta_y^2\right)u_{ij}^{n-1}\\ \quad+\tau\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\delta_t u_{ij}^{k-\frac{1}{2}}+\tau b_{n-1}^{(\gamma)}\psi_{ij}+\tau\eta f_{ij}^{n-\frac{1}{2}},\quad 1\leqslant i\leqslant M_1-1,\\ u_{0j}^*=\left(\mathcal{I}-\frac{\eta}{2}\tau\delta_y^2\right)u_{0j}^n,\quad u_{M_1,j}^*=\left(\mathcal{I}-\frac{\eta}{2}\tau\delta_y^2\right)u_{M_1,j}^n\end{cases}$$

to obtain the value of

$$\{u_{ij}^*\mid 1\leqslant i\leqslant M_1-1\}$$

on the intermediate time level.

Then, for any fixed i from 1 to M_1-1, carry out some calculations for the unknown $\{u_{ij}^n\mid 0\leqslant j\leqslant M_2\}$ in y direction

$$\begin{cases}\left(\mathcal{I}-\frac{\eta}{2}\tau\delta_y^2\right)u_{ij}^n=u_{ij}^*,\quad 1\leqslant j\leqslant M_2-1,\\ u_{i0}^n=\mu(x_i,y_0,t_n),\quad u_{i,M_2}^n=\mu(x_i,y_{M_2},t_n)\end{cases}$$

to produce the desired value of

$$\{u_{ij}^n\mid 1\leqslant j\leqslant M_2-1\}.$$

Next, the corresponding theoretical analysis on the difference scheme (3.298)–(3.300) will be studied.

3.9.2 Solvability of the difference scheme

Theorem 3.9.1. *The difference scheme* (3.298)–(3.300) *is uniquely solvable.*

Proof. Let

$$u^n = \{u_{ij}^n \mid (i,j) \in \bar{\omega}\}.$$

The value of u^0 is determined by (3.299)–(3.300). Now assume that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n is obtained from (3.298) and (3.300). To show its unique solvability, it is sufficient to verify that the corresponding homogeneous one

$$\begin{cases} \frac{1}{\eta\tau}u_{ij}^n + \frac{\Gamma(3-\gamma)}{4}\tau^\gamma \delta_x^2\delta_y^2 u_{ij}^n = \frac{1}{2}(\delta_x^2 u_{ij}^n + \delta_y^2 u_{ij}^n), \quad (i,j) \in \omega, & (3.301) \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega & (3.302) \end{cases}$$

has only the trivial solution.

Making the inner product on both hand sides of (3.301) with u^n, it follows from (2.200)–(2.202) that

$$\frac{1}{\eta\tau}(u^n, u^n) + \frac{\Gamma(3-\gamma)}{4}\tau^\gamma(\delta_x\delta_y u^n, \delta_x\delta_y u^n) = -\frac{1}{2}[(\delta_x u^n, \delta_x u^n) + (\delta_y u^n, \delta_y u^n)],$$

that is,

$$\frac{1}{\eta\tau}\|u^n\|^2 + \frac{\Gamma(3-\gamma)}{4}\tau^\gamma\|\delta_x\delta_y u^n\|^2 = -\frac{1}{2}\|\nabla_h u^n\|^2 \leqslant 0,$$

thus $\|u^n\| = 0$. Noticing (3.302), we have $u^n = 0$.

By the principle of induction, the difference scheme (3.298)–(3.300) is uniquely solvable. The proof ends. □

3.9.3 Stability of the difference scheme

Theorem 3.9.2. *Suppose $\{v_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ is the solution of the difference scheme*

$$\begin{cases} \frac{1}{\eta}\left[b_0^{(\gamma)}\delta_t v_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t v_{ij}^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_{ij}\right] \\ \quad + \frac{\tau^2}{4}\eta\delta_x^2\delta_y^2\delta_t v_{ij}^{n-\frac{1}{2}} = \delta_x^2 v_{ij}^{n-\frac{1}{2}} + \delta_y^2 v_{ij}^{n-\frac{1}{2}} + f_{ij}^{n-\frac{1}{2}}, \\ \qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (3.303) \\ v_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, \\ v_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (3.304) \end{cases}$$

Then it holds

$$\|\nabla_h v^n\|^2 \leqslant \|\nabla_h v^0\|^2 + \frac{t_n^{2-\gamma}}{\Gamma(3-\gamma)}\|\psi\|^2 + t_n^{\gamma-1}\Gamma(2-\gamma)\tau\sum_{k=1}^{n}\|f^{k-\frac{1}{2}}\|^2, \quad 1 \leqslant n \leqslant N, \qquad (3.305)$$

where

$$\|\psi\|^2 = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \psi_{ij}^2, \quad \|f^{k-\frac{1}{2}}\|^2 = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (f_{ij}^{k-\frac{1}{2}})^2.$$

Proof. Taking the inner product on both hand sides of (3.303) with $\eta\delta_t v^{n-\frac{1}{2}}$ yields

$$\begin{aligned}
& b_0^{(\gamma)}\|\delta_t v^{n-\frac{1}{2}}\|^2 - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\delta_t v^{k-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) - b_{n-1}^{(\gamma)}(\psi, \delta_t v^{n-\frac{1}{2}}) \\
& + \frac{1}{4}\eta^2\tau^2(\delta_x^2\delta_y^2\delta_t v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) = \eta(\delta_x^2 v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) \\
& + \eta(\delta_y^2 v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) + \eta(f^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N.
\end{aligned}$$

Noticing (3.304), it follows from (2.200)–(2.202) that

$$\begin{aligned}
& (\delta_x^2\delta_y^2\delta_t v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) = (\delta_x\delta_y\delta_t v^{n-\frac{1}{2}}, \delta_x\delta_y\delta_t v^{n-\frac{1}{2}}) \geqslant 0, \\
& (\delta_x^2 v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) = -(\delta_x v^{n-\frac{1}{2}}, \delta_x\delta_t v^{n-\frac{1}{2}}) = -\frac{1}{2\tau}(\|\delta_x v^n\|^2 - \|\delta_x v^{n-1}\|^2), \\
& (\delta_y^2 v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) = -(\delta_y v^{n-\frac{1}{2}}, \delta_y\delta_t v^{n-\frac{1}{2}}) = -\frac{1}{2\tau}(\|\delta_y v^n\|^2 - \|\delta_y v^{n-1}\|^2),
\end{aligned}$$

hence,

$$\begin{aligned}
& b_0^{(\gamma)}\|\delta_t v^{n-\frac{1}{2}}\|^2 + \frac{\eta}{2\tau}[(\|\delta_x v^n\|^2 + \|\delta_y v^n\|^2) - (\|\delta_x v^{n-1}\|^2 + \|\delta_y v^{n-1}\|^2)] \\
\leqslant & \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\delta_t v^{k-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) + b_{n-1}^{(\gamma)}(\psi, \delta_t v^{n-\frac{1}{2}}) \\
& + \eta(f^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N.
\end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
& b_0^{(\gamma)}\|\delta_t v^{n-\frac{1}{2}}\|^2 + \frac{\eta}{2\tau}(\|\nabla_h v^n\|^2 - \|\nabla_h v^{n-1}\|^2) \\
\leqslant & \frac{1}{2}\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\|\delta_t v^{k-\frac{1}{2}}\|^2 + \|\delta_t v^{n-\frac{1}{2}}\|^2) \\
& + \frac{1}{2}b_{n-1}^{(\gamma)}(\|\psi\|^2 + \|\delta_t v^{n-\frac{1}{2}}\|^2) + \eta(f^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N,
\end{aligned}$$

which can be reduced to

$$\begin{aligned}
& \|\nabla_h v^n\|^2 + \frac{\tau}{\eta}\sum_{k=1}^{n} b_{n-k}^{(\gamma)}\|\delta_t v^{k-\frac{1}{2}}\|^2 \\
\leqslant & \|\nabla_h v^{n-1}\|^2 + \frac{\tau}{\eta}\sum_{k=1}^{n-1} b_{n-k-1}^{(\gamma)}\|\delta_t v^{k-\frac{1}{2}}\|^2
\end{aligned}$$

$$+\frac{\tau}{\eta}b_{n-1}^{(\gamma)}\|\psi\|^2+2\tau\big(f^{n-\frac{1}{2}},\delta_t v^{n-\frac{1}{2}}\big),\quad 1\leqslant n\leqslant N.$$

The recursive process will produce

$$\begin{aligned}&\|\nabla_h v^n\|^2+\frac{\tau}{\eta}\sum_{k=1}^{n}b_{n-k}^{(\gamma)}\|\delta_t v^{k-\frac{1}{2}}\|^2\\ &\leqslant\|\nabla_h v^0\|^2+\frac{\tau}{\eta}\sum_{k=0}^{n-1}b_k^{(\gamma)}\|\psi\|^2+2\tau\sum_{k=1}^{n}\big(f^{k-\frac{1}{2}},\delta_t v^{k-\frac{1}{2}}\big)\\ &\leqslant\|\nabla_h v^0\|^2+\frac{\tau}{\eta}\sum_{k=0}^{n-1}b_k^{(\gamma)}\|\psi\|^2\\ &\quad+\tau\sum_{k=1}^{n}\bigg(\frac{b_{n-k}^{(\gamma)}}{\eta}\|\delta_t v^{k-\frac{1}{2}}\|^2+\frac{\eta}{b_{n-k}^{(\gamma)}}\|f^{k-\frac{1}{2}}\|^2\bigg),\quad 1\leqslant n\leqslant N,\end{aligned}$$

that is,

$$\|\nabla_h v^n\|^2\leqslant\|\nabla_h v^0\|^2+\frac{\tau}{\eta}\sum_{k=0}^{n-1}b_k^{(\gamma)}\|\psi\|^2+\tau\sum_{k=1}^{n}\frac{\eta}{b_{n-k}^{(\gamma)}}\|f^{k-\frac{1}{2}}\|^2,\quad 1\leqslant n\leqslant N. \tag{3.306}$$

By (3.20) and (3.22), it follows (3.305) from (3.306). The proof ends. □

3.9.4 Convergence of the difference scheme

Theorem 3.9.3. *Suppose* $\{U_{ij}^n \mid (i,j)\in\bar{\omega}, 0\leqslant n\leqslant N\}$ *and* $\{u_{ij}^n \mid (i,j)\in\bar{\omega}, 0\leqslant n\leqslant N\}$ *are solutions of the problem* (3.290)–(3.292) *and the difference scheme* (3.298)–(3.300), *respectively. Let*

$$e_{ij}^n=U_{ij}^n-u_{ij}^n,\quad (i,j)\in\bar{\omega},\ 0\leqslant n\leqslant N,$$

then it holds

$$\|\nabla_h e^n\|\leqslant\sqrt{T^\gamma\Gamma(2-\gamma)L_1L_2}\,c_{14}(\tau^{3-\gamma}+h_1^2+h_2^2),\quad 1\leqslant n\leqslant N.$$

Proof. The subtraction of (3.298)–(3.300) from (3.294), (3.296)–(3.297), respectively, produces the system of error equations as follows:

$$\begin{cases}\dfrac{1}{\eta}\bigg[b_0^{(\gamma)}\delta_t e_{ij}^{n-\frac{1}{2}}-\displaystyle\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\delta_t e_{ij}^{k-\frac{1}{2}}-b_{n-1}^{(\gamma)}\cdot 0\bigg]\\ \quad+\dfrac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t e_{ij}^{n-\frac{1}{2}}\\ \quad=\delta_x^2 e_{ij}^{n-\frac{1}{2}}+\delta_y^2 e_{ij}^{n-\frac{1}{2}}+(r_{14})_{ij}^{n-\frac{1}{2}},\quad (i,j)\in\omega,\ 1\leqslant n\leqslant N,\\ e_{ij}^0=0,\quad (i,j)\in\omega,\\ e_{ij}^n=0,\quad (i,j)\in\partial\omega,\ 0\leqslant n\leqslant N.\end{cases}$$

Noticing (3.295), the application of Theorem 3.9.2 will lead to

$$\begin{aligned}\|\nabla_h e^n\|^2 &\leqslant t_n^{\gamma-1}\Gamma(2-\gamma)\tau\sum_{k=1}^{n}\|(r_{14})^{k-\frac{1}{2}}\|^2\\ &\leqslant T^{\gamma}\Gamma(2-\gamma)L_1L_2c_{14}^2\big(\tau^{3-\gamma}+h_1^2+h_2^2\big)^2,\quad 1\leqslant n\leqslant N.\end{aligned}$$

Taking the square root on both hand sides of the inequality above will reach the desired result. The proof ends. □

3.10 The compact ADI method based on L1 approximation for 2D problem

In this section, the spatial compact ADI difference method will be developed for (3.290)–(3.292). Suppose $u \in C^{(6,6,3)}(\bar{\Omega}\times[0,T])$. Take the same mesh partition, mesh function spaces and notations as those in last section. In addition, for any mesh function $u \in \mathcal{V}_h$, define the average operators as follows:

$$\mathcal{A}_x u_{ij}=\begin{cases}(\mathcal{I}+\frac{h_1^2}{12}\delta_x^2)u_{ij}, & 1\leqslant i\leqslant M_1-1,\\ u_{ij}, & i=0 \text{ or } i=M_1,\end{cases}\qquad 0\leqslant j\leqslant M_2,$$

$$\mathcal{A}_y u_{ij}=\begin{cases}(\mathcal{I}+\frac{h_2^2}{12}\delta_y^2)u_{ij}, & 1\leqslant j\leqslant M_2-1,\\ u_{ij}, & j=0 \text{ or } j=M_2,\end{cases}\qquad 0\leqslant i\leqslant M_1.$$

3.10.1 Derivation of the difference scheme

Considering equation (3.290) at the point (x_i, y_j, t_n), we have

$${}_0^CD_t^{\gamma}u(x_i,y_j,t_n)=u_{xx}(x_i,y_j,t_n)+u_{yy}(x_i,y_j,t_n)+f_{ij}^n,\quad (i,j)\in\bar{\omega},\ 0\leqslant n\leqslant N.$$

Performing the operator $\mathcal{A}_x\mathcal{A}_y$ to both hand sides of the equality above, it follows from Theorem 1.6.2 and Lemma 2.1.3 that

$$\begin{aligned}\mathcal{A}_x\mathcal{A}_y{}_0^CD_t^{\gamma}u(x_i,y_j,t_n) &= \mathcal{A}_y\big(\mathcal{A}_x u_{xx}(x_i,y_j,t_n)\big)\\ &\quad+\mathcal{A}_x\big(\mathcal{A}_y u_{yy}(x_i,y_j,t_n)\big)+\mathcal{A}_x\mathcal{A}_y f_{ij}^n\\ &= \mathcal{A}_y\delta_x^2U_{ij}^n+\mathcal{A}_x\delta_y^2U_{ij}^n+\mathcal{A}_x\mathcal{A}_y f_{ij}^n\\ &\quad+O(h_1^4+h_2^4),\quad (i,j)\in\omega,\ 0\leqslant n\leqslant N.\end{aligned}$$

Taking an average on two adjacent time levels and applying the L1 formula (1.69) to approximate the temporal Caputo fractional derivative, it follows from Theorem 1.6.2 that

$$\mathcal{A}_x\mathcal{A}_y\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\delta_tU_{ij}^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}\big(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)}\big)\delta_tU_{ij}^{k-\frac{1}{2}}-b_{n-1}^{(\gamma)}\psi_{ij}\right]$$

$$= \mathcal{A}_y\delta_x^2 U_{ij}^{n-\frac{1}{2}} + \mathcal{A}_x\delta_y^2 U_{ij}^{n-\frac{1}{2}} + \mathcal{A}_x\mathcal{A}_y f_{ij}^{n-\frac{1}{2}} + (r_{15})_{ij}^{n-\frac{1}{2}}, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \tag{3.307}$$

and there is a positive constant c_{15} such that

$$\left|(r_{15})_{ij}^{n-\frac{1}{2}}\right| \leqslant c_{15}(\tau^{3-\gamma} + h_1^4 + h_2^4), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N,$$

where $\{b_l^{(\gamma)}\}$ is defined in (1.64). Adding a small term $\frac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t U_{ij}^{n-\frac{1}{2}}$ into both hand sides of (3.307) arrives at

$$\begin{aligned}
&\mathcal{A}_x\mathcal{A}_y\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\delta_t U_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t U_{ij}^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_{ij}\right] \\
&+ \frac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t U_{ij}^{n-\frac{1}{2}} = \mathcal{A}_y\delta_x^2 U_{ij}^{n-\frac{1}{2}} + \mathcal{A}_x\delta_y^2 U_{ij}^{n-\frac{1}{2}} \\
&+ \mathcal{A}_x\mathcal{A}_y f_{ij}^{n-\frac{1}{2}} + (r_{16})_{ij}^{n-\frac{1}{2}}, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N,
\end{aligned} \tag{3.308}$$

where

$$(r_{16})_{ij}^{n-\frac{1}{2}} = (r_{15})_{ij}^{n-\frac{1}{2}} + \frac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t U_{ij}^{n-\frac{1}{2}},$$

and there is a positive constant c_{16} such that

$$\left|(r_{16})_{ij}^{n-\frac{1}{2}}\right| \leqslant c_{16}(\tau^{3-\gamma} + h_1^4 + h_2^4), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \tag{3.309}$$

Noticing the initial-boundary value conditions (3.291)–(3.292), we have

$$\begin{cases} U_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (3.310) \\ U_{ij}^n = \mu(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (3.311) \end{cases}$$

Omitting the small term $(r_{16})_{ij}^{n-\frac{1}{2}}$ in (3.308) and replacing the exact solution U_{ij}^n with its numerical one u_{ij}^n produce another difference scheme for (3.290)–(3.292) as follows:

$$\begin{cases}
\mathcal{A}_x\mathcal{A}_y\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[b_0^{(\gamma)}\delta_t u_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t u_{ij}^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_{ij}\right] \\
\quad + \frac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t u_{ij}^{n-\frac{1}{2}} = \mathcal{A}_y\delta_x^2 u_{ij}^{n-\frac{1}{2}} + \mathcal{A}_x\delta_y^2 u_{ij}^{n-\frac{1}{2}} + \mathcal{A}_x\mathcal{A}_y f_{ij}^{n-\frac{1}{2}}, \\
\qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (3.312) \\
u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (3.313) \\
u_{ij}^n = \mu(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (3.314)
\end{cases}$$

Denote

$$\eta = \tau^{\gamma-1}\Gamma(3-\gamma).$$

Rewrite (3.312) as

$$\begin{aligned}&\Big(\mathcal{A}_x - \frac{\eta\tau}{2}\delta_x^2\Big)\Big(\mathcal{A}_y - \frac{\eta\tau}{2}\delta_y^2\Big)u_{ij}^n\\ =&\Big(\mathcal{A}_x + \frac{\eta\tau}{2}\delta_x^2\Big)\Big(\mathcal{A}_y + \frac{\eta\tau}{2}\delta_y^2\Big)u_{ij}^{n-1}\\ &+\tau\mathcal{A}_x\mathcal{A}_y\Big[\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\delta_t u_{ij}^{k-\frac{1}{2}}+b_{n-1}^{(\gamma)}\psi_{ij}\Big]+\eta\tau\mathcal{A}_x\mathcal{A}_y f_{ij}^{n-\frac{1}{2}}.\end{aligned}$$

Let

$$u_{ij}^* = \Big(\mathcal{A}_y - \frac{\eta\tau}{2}\delta_y^2\Big)u_{ij}^n,$$

then the difference scheme (3.312)–(3.314) can be written into the following ADI form:

On each time level $t = t_n$ $(1 \leqslant n \leqslant N)$, firstly, for any fixed j from 1 to $M_2 - 1$, solve a series of linear systems in the unknown $\{u_{ij}^* \mid 0 \leqslant i \leqslant M_1\}$ in x direction

$$\begin{cases}\Big(\mathcal{A}_x - \frac{\eta\tau}{2}\delta_x^2\Big)u_{ij}^* = \Big(\mathcal{A}_x + \frac{\eta\tau}{2}\delta_x^2\Big)\Big(\mathcal{A}_y + \frac{\eta\tau}{2}\delta_y^2\Big)u_{ij}^{n-1}\\ \qquad\qquad +\tau\mathcal{A}_x\mathcal{A}_y\Big[\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)}-b_{n-k}^{(\gamma)})\delta_t u_{ij}^{k-\frac{1}{2}}+b_{n-1}^{(\gamma)}\psi_{ij}\Big]\\ \qquad\qquad +\eta\tau\mathcal{A}_x\mathcal{A}_y f_{ij}^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M_1 - 1,\\ u_{0j}^* = \Big(\mathcal{A}_y - \frac{\eta\tau}{2}\delta_y^2\Big)u_{0j}^n, \quad u_{M_1,j}^* = \Big(\mathcal{A}_y - \frac{\eta\tau}{2}\delta_y^2\Big)u_{M_1,j}^n\end{cases}$$

to obtain the value of

$$\{u_{ij}^* \mid 1 \leqslant i \leqslant M_1 - 1\}$$

on the intermediate time level.

Then, for any fixed i from 1 to $M_1 - 1$, carry out some calculations about the unknown $\{u_{ij}^n \mid 0 \leqslant j \leqslant M_2\}$ in y direction

$$\begin{cases}\Big(\mathcal{A}_y - \frac{\eta\tau}{2}\delta_y^2\Big)u_{ij}^n = u_{ij}^*, \quad 1 \leqslant j \leqslant M_2 - 1,\\ u_{i0}^n = \mu(x_i, y_0, t_n), \quad u_{i,M_2}^n = \mu(x_i, y_{M_2}, t_n)\end{cases}$$

to get the desired value of

$$\{u_{ij}^n \mid 1 \leqslant j \leqslant M_2 - 1\}.$$

3.10.2 Solvability of the difference scheme

Some preparation work is firstly done in order to show the unique solvability of the difference scheme (3.312)–(3.314).

For any mesh functions $u, v \in \mathring{\mathcal{V}}_h$, define

$$J(u, v) \equiv (\mathcal{A}_x \mathcal{A}_y u, v).$$

Noticing (2.200)-(2.202), it follows

$$\begin{aligned} J(u,v) &= \left(\left(\mathcal{I} + \frac{h_1^2}{12}\delta_x^2\right)\left(\mathcal{I} + \frac{h_2^2}{12}\delta_y^2\right)u, v\right) \\ &= (u,v) - \frac{h_1^2}{12}(\delta_x u, \delta_x v) - \frac{h_2^2}{12}(\delta_y u, \delta_y v) + \frac{h_1^2 h_2^2}{144}(\delta_x\delta_y u, \delta_x\delta_y v). \end{aligned}$$

It is easy to verify that

$$\frac{1}{3}\|u\|^2 \leqslant J(u,u) \leqslant \|u\|^2, \tag{3.315}$$

which reveals that $J(u, v)$ is an inner product defined on $\mathring{\mathcal{V}}_h$. Denote

$$(u, v)_A = J(u, v), \quad \|u\|_A = \sqrt{(u,u)_A}\,.$$

In view of (3.315), the following lemma is true.

Lemma 3.10.1. *For any mesh function $u \in \mathring{\mathcal{V}}_h$, it holds*

$$\frac{1}{3}\|u\|^2 \leqslant \|u\|_A^2 \leqslant \|u\|^2.$$

Now the existence and uniqueness of the solution to the difference scheme (3.312)–(3.314) will be proved.

Theorem 3.10.1. *The difference scheme* (3.312)–(3.314) *is uniquely solvable.*

Proof. Let

$$u^n = \{u_{ij}^n \mid (i,j) \in \bar{\omega}\}.$$

The value of u^0 is determined by (3.313)–(3.314).

Suppose the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the system in u^n can be obtained from (3.312) and (3.314). To show its unique solvability, it is sufficient to prove that the corresponding homogeneous one

$$\begin{cases} \dfrac{1}{\eta\tau}\mathcal{A}_x\mathcal{A}_y u_{ij}^n + \dfrac{\Gamma(3-\gamma)}{4}\tau^\gamma \delta_x^2\delta_y^2 u_{ij}^n = \dfrac{1}{2}(\mathcal{A}_y\delta_x^2 u_{ij}^n + \mathcal{A}_x\delta_y^2 u_{ij}^n), \quad (i,j) \in \omega, & (3.316) \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega & (3.317) \end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (3.316) with u^n yields

$$\frac{1}{\eta\tau}(\mathcal{A}_x\mathcal{A}_y u^n, u^n) + \frac{\Gamma(3-\gamma)}{4}\tau^\gamma(\delta_x^2\delta_y^2 u^n, u^n) = \frac{1}{2}[(\mathcal{A}_y\delta_x^2 u^n, u^n) + (\mathcal{A}_x\delta_y^2 u^n, u^n)]. \quad (3.318)$$

Noticing (3.317), it follows from Lemma 3.10.1 that

$$(\mathcal{A}_x\mathcal{A}_y u^n, u^n) = (u^n, u^n)_A = \|u^n\|_A^2 \geqslant \frac{1}{3}\|u^n\|^2. \quad (3.319)$$

Noticing (2.200)–(2.202), we have

$$(\delta_x^2\delta_y^2 u^n, u^n) = (\delta_x\delta_y u^n, \delta_x\delta_y u^n) = \|\delta_x\delta_y u^n\|^2, \quad (3.320)$$

$$\begin{aligned}(\mathcal{A}_y\delta_x^2 u^n, u^n) &= \left(\left(\mathcal{I} + \frac{h_2^2}{12}\delta_y^2\right)\delta_x^2 u^n, u^n\right)\\ &= -\|\delta_x u^n\|^2 + \frac{h_2^2}{12}\|\delta_x\delta_y u^n\|^2 \leqslant -\frac{2}{3}\|\delta_x u^n\|^2, \end{aligned} \quad (3.321)$$

$$\begin{aligned}(\mathcal{A}_x\delta_y^2 u^n, u^n) &= \left(\left(\mathcal{I} + \frac{h_1^2}{12}\delta_x^2\right)\delta_y^2 u^n, u^n\right)\\ &= -\|\delta_y u^n\|^2 + \frac{h_1^2}{12}\|\delta_x\delta_y u^n\|^2 \leqslant -\frac{2}{3}\|\delta_y u^n\|^2. \end{aligned} \quad (3.322)$$

Substituting (3.319)–(3.322) into (3.318) arrives at

$$\frac{1}{3\eta\tau}\|u^n\|^2 + \frac{\Gamma(3-\gamma)}{4}\tau^\gamma\|\delta_x\delta_y u^n\|^2 \leqslant -\frac{1}{3}\|\nabla_h u^n\|^2 \leqslant 0,$$

thus $\|u^n\| = 0$. Noticing (3.317), we have $u^n = 0$.

By the principle of induction, the difference scheme (3.312)–(3.314) is uniquely solvable. The proof ends. □

3.10.3 Stability of the difference scheme

For any mesh function $u \in \mathring{\mathcal{V}}_h$, define

$$\|\nabla_h u\|_A^2 = \left(\|\delta_x u\|^2 - \frac{h_2^2}{12}\|\delta_y\delta_x u\|^2\right) + \left(\|\delta_y u\|^2 - \frac{h_1^2}{12}\|\delta_x\delta_y u\|^2\right).$$

The following lemma will state the equivalence between $\|\nabla_h \cdot\|_A$ and $\|\nabla_h \cdot\|$.

Lemma 3.10.2. *For any mesh function $u \in \mathring{\mathcal{V}}_h$, it holds*

$$\frac{2}{3}\|\nabla_h u\|^2 \leqslant \|\nabla_h u\|_A^2 \leqslant \|\nabla_h u\|^2.$$

Proof. First, it follows from the inequalities of inverse estimate

$$\|\delta_y\delta_x u\|^2 \leqslant \frac{4}{h_2^2}\|\delta_x u\|^2, \quad \|\delta_x\delta_y u\|^2 \leqslant \frac{4}{h_1^2}\|\delta_y u\|^2$$

that

$$\|\nabla_h u\|_A^2 \geqslant \frac{2}{3}(\|\delta_x u\|^2 + \|\delta_y u\|^2) = \frac{2}{3}\|\nabla_h u\|^2.$$

Second, it is easy to know that

$$\|\nabla_h u\|_A^2 \leqslant \|\delta_x u\|^2 + \|\delta_y u\|^2 = \|\nabla_h u\|^2.$$

The proof ends. □

Another lemma will be needed.

Lemma 3.10.3. *For any mesh function $u \in \mathring{\mathcal{V}}_h$, it holds*

$$(\mathcal{A}_y\delta_x^2 u^{n-\frac{1}{2}} + \mathcal{A}_x\delta_y^2 u^{n-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}) = -\frac{1}{2\tau}(\|\nabla_h u^n\|_A^2 - \|\nabla_h u^{n-1}\|_A^2).$$

Proof. Noticing (2.200)–(2.202) and applying the summation by parts arrive at

$$\begin{aligned}
&(\mathcal{A}_y\delta_x^2 u^{n-\frac{1}{2}} + \mathcal{A}_x\delta_y^2 u^{n-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}) \\
&= \left(\left(\mathcal{I} + \frac{h_2^2}{12}\delta_y^2\right)\delta_x^2 u^{n-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}\right) + \left(\left(\mathcal{I} + \frac{h_1^2}{12}\delta_x^2\right)\delta_y^2 u^{n-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}\right) \\
&= (\delta_x^2 u^{n-\frac{1}{2}} + \delta_y^2 u^{n-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}) \\
&\quad + \frac{h_2^2}{12}(\delta_x\delta_y u^{n-\frac{1}{2}}, \delta_x\delta_y\delta_t u^{n-\frac{1}{2}}) + \frac{h_1^2}{12}(\delta_x\delta_y u^{n-\frac{1}{2}}, \delta_x\delta_y\delta_t u^{n-\frac{1}{2}}) \\
&= -\frac{1}{2\tau}[(\|\delta_x u^n\|^2 + \|\delta_y u^n\|^2) - (\|\delta_x u^{n-1}\|^2 + \|\delta_y u^{n-1}\|^2)] \\
&\quad + \frac{h_2^2}{12}\cdot\frac{1}{2\tau}(\|\delta_y\delta_x u^n\|^2 - \|\delta_y\delta_x u^{n-1}\|^2) \\
&\quad + \frac{h_1^2}{12}\cdot\frac{1}{2\tau}(\|\delta_x\delta_y u^n\|^2 - \|\delta_x\delta_y u^{n-1}\|^2) \\
&= -\frac{1}{2\tau}\left\{\left[\left(\|\delta_x u^n\|^2 - \frac{h_2^2}{12}\|\delta_y\delta_x u^n\|^2\right) + \left(\|\delta_y u^n\|^2 - \frac{h_1^2}{12}\|\delta_x\delta_y u^n\|^2\right)\right]\right. \\
&\quad \left. - \left[\left(\|\delta_x u^{n-1}\|^2 - \frac{h_2^2}{12}\|\delta_y\delta_x u^{n-1}\|^2\right) + \left(\|\delta_y u^{n-1}\|^2 - \frac{h_1^2}{12}\|\delta_x\delta_y u^{n-1}\|^2\right)\right]\right\} \\
&= -\frac{1}{2\tau}(\|\nabla_h u^n\|_A^2 - \|\nabla_h u^{n-1}\|_A^2).
\end{aligned}$$

The proof ends. □

Next, the stability result of the difference scheme (3.312)–(3.314) will be shown.

Theorem 3.10.2. *Suppose $\{v_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ is the solution of the difference scheme*

$$\begin{cases} \dfrac{1}{\eta}\mathcal{A}_x\mathcal{A}_y\left[b_0^{(\gamma)}\delta_t v_{ij}^{n-\frac{1}{2}} - \sum\limits_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t v_{ij}^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\psi_{ij}\right] + \dfrac{\tau^2}{4}\eta\delta_x^2\delta_y^2\delta_t v_{ij}^{n-\frac{1}{2}} \\ = \mathcal{A}_y\delta_x^2 v_{ij}^{n-\frac{1}{2}} + \mathcal{A}_x\delta_y^2 v_{ij}^{n-\frac{1}{2}} + g_{ij}^{n-\frac{1}{2}}, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (3.323) \\ v_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (3.324) \\ v_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N, & (3.325) \end{cases}$$

where $\psi_{ij}|_{(i,j)\in\partial\omega} = 0$. Then it holds

$$\|\nabla_h v^n\|_A^2 \leqslant \|\nabla_h v^0\|_A^2 + \frac{t_n^{2-\gamma}}{\Gamma(3-\gamma)}\|\psi\|^2 + 3t_n^{\gamma-1}\Gamma(2-\gamma)\tau\sum_{k=1}^{n}\|g^{k-\frac{1}{2}}\|^2, \quad 1 \leqslant n \leqslant N, \tag{3.326}$$

with

$$\|\psi\|^2 = h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(\psi_{ij})^2, \quad \|g^{k-\frac{1}{2}}\|^2 = h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(g_{ij}^{k-\frac{1}{2}})^2.$$

Proof. Taking the inner product on both hand sides of (3.323) with $\eta\delta_t v^{n-\frac{1}{2}}$ yields

$$\begin{aligned} & b_0^{(\gamma)}\|\delta_t v^{n-\frac{1}{2}}\|_A^2 - \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\delta_t v^{k-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}})_A \\ & - b_{n-1}^{(\gamma)}(\psi, \delta_t v^{n-\frac{1}{2}})_A + \frac{\tau^2}{4}\eta^2(\delta_x^2\delta_y^2\delta_t v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) \\ = {} & \eta(\mathcal{A}_y\delta_x^2 v^{n-\frac{1}{2}} + \mathcal{A}_x\delta_y^2 v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) + \eta(g^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}). \end{aligned} \tag{3.327}$$

It is easy to see that

$$(\delta_x^2\delta_y^2\delta_t v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) = \|\delta_x\delta_y\delta_t v^{n-\frac{1}{2}}\|^2 \geqslant 0, \tag{3.328}$$

and it follows from Lemma 3.10.3 that

$$(\mathcal{A}_y\delta_x^2 v^{n-\frac{1}{2}} + \mathcal{A}_x\delta_y^2 v^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) = -\frac{1}{2\tau}(\|\nabla_h v^n\|_A^2 - \|\nabla_h v^{n-1}\|_A^2). \tag{3.329}$$

Substituting (3.328) and (3.329) into (3.327), by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} & b_0^{(\gamma)}\|\delta_t v^{n-\frac{1}{2}}\|_A^2 + \frac{\eta}{2\tau}(\|\nabla_h v^n\|_A^2 - \|\nabla_h v^{n-1}\|_A^2) \\ \leqslant {} & \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\delta_t v^{k-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}})_A + b_{n-1}^{(\gamma)}(\psi, \delta_t v^{n-\frac{1}{2}})_A + \eta(g^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}) \end{aligned}$$

$$\leqslant \frac{1}{2}\sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\|\delta_t v^{k-\frac{1}{2}}\|_A^2 + \|\delta_t v^{n-\frac{1}{2}}\|_A^2)$$
$$+ \frac{1}{2}b_{n-1}^{(\gamma)}(\|\psi\|_A^2 + \|\delta_t v^{n-\frac{1}{2}}\|_A^2) + \eta(g^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N,$$

which gives

$$b_0^{(\gamma)}\|\delta_t v^{n-\frac{1}{2}}\|_A^2 + \frac{\eta}{\tau}(\|\nabla_h v^n\|_A^2 - \|\nabla_h v^{n-1}\|_A^2)$$
$$\leqslant \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\|\delta_t v^{k-\frac{1}{2}}\|_A^2 + b_{n-1}^{(\gamma)}\|\psi\|_A^2 + 2\eta(g^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N. \tag{3.330}$$

Let

$$H^0 = \|\nabla_h v^0\|_A^2, \quad H^n = \|\nabla_h v^n\|_A^2 + \frac{\tau}{\eta}\sum_{k=1}^{n} b_{n-k}^{(\gamma)}\|\delta_t v^{k-\frac{1}{2}}\|_A^2, \quad 1 \leqslant n \leqslant N,$$

then it follows from (3.330) that

$$H^n \leqslant H^{n-1} + \frac{\tau}{\eta}b_{n-1}^{(\gamma)}\|\psi\|^2 + 2\tau(g^{n-\frac{1}{2}}, \delta_t v^{n-\frac{1}{2}}), \quad 1 \leqslant n \leqslant N.$$

The recursive process leads to

$$H^n \leqslant H^0 + \frac{\tau}{\eta}\sum_{k=0}^{n-1} b_k^{(\gamma)}\|\psi\|^2 + 2\tau\sum_{k=1}^{n}(g^{k-\frac{1}{2}}, \delta_t v^{k-\frac{1}{2}})$$
$$\leqslant H^0 + \frac{\tau}{\eta}\sum_{k=0}^{n-1} b_k^{(\gamma)}\|\psi\|^2 + \tau\sum_{k=1}^{n}\left(\frac{3\eta}{b_{n-k}^{(\gamma)}}\|g^{k-\frac{1}{2}}\|^2 + \frac{b_{n-k}^{(\gamma)}}{3\eta}\|\delta_t v^{k-\frac{1}{2}}\|^2\right)$$
$$\leqslant H^0 + \frac{\tau}{\eta}\sum_{k=0}^{n-1} b_k^{(\gamma)}\|\psi\|^2 + \tau\sum_{k=1}^{n}\left(\frac{3\eta}{b_{n-k}^{(\gamma)}}\|g^{k-\frac{1}{2}}\|^2 + \frac{b_{n-k}^{(\gamma)}}{\eta}\|\delta_t v^{k-\frac{1}{2}}\|_A^2\right), \quad 1 \leqslant n \leqslant N,$$

that is,

$$\|\nabla_h v^n\|_A^2 \leqslant \|\nabla_h v^0\|_A^2 + \frac{\tau}{\eta}\sum_{k=0}^{n-1} b_k^{(\gamma)}\|\psi\|^2 + 3\tau\sum_{k=1}^{n}\frac{\eta}{b_{n-k}^{(\gamma)}}\|g^{k-\frac{1}{2}}\|^2, \quad 1 \leqslant n \leqslant N. \tag{3.331}$$

By (3.20) and (3.22), the desired result (3.326) can be obtained from (3.331). The proof ends. □

3.10.4 Convergence of the difference scheme

Theorem 3.10.3. *Suppose* $\{U_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *and* $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (3.290)–(3.292) *and the difference scheme* (3.312)–(3.314), *respectively. Let*

$$e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j) \in \bar{\omega}, \ 0 \leqslant n \leqslant N,$$

then it holds

$$\|\nabla_h e^n\| \leqslant \frac{3}{2}\sqrt{2T^\gamma\Gamma(2-\gamma)L_1L_2}\, c_{16}(\tau^{3-\gamma}+h_1^4+h_2^4), \quad 1\leqslant n\leqslant N.$$

Proof. Subtracting (3.312)–(3.314) from (3.308), (3.310)–(3.311), respectively, the system of error equations can be obtained as follows:

$$\begin{cases} \dfrac{1}{\eta}\mathcal{A}_x\mathcal{A}_y\left[b_0^{(\gamma)}\delta_t e_{ij}^{n-\frac{1}{2}} - \sum\limits_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\delta_t e_{ij}^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)}\cdot 0\right] \\ \quad + \dfrac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t e_{ij}^{n-\frac{1}{2}} = \mathcal{A}_y\delta_x^2 e_{ij}^{n-\frac{1}{2}} + \mathcal{A}_x\delta_y^2 e_{ij}^{n-\frac{1}{2}} + (r_{16})_{ij}^{n-\frac{1}{2}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i,j)\in\omega,\ 1\leqslant n\leqslant N, \\ e_{ij}^0 = 0, \quad (i,j)\in\omega, \\ e_{ij}^n = 0, \quad (i,j)\in\partial\omega,\ 0\leqslant n\leqslant N. \end{cases}$$

Noticing (3.309), the application of Theorem 3.10.2 will yield

$$\begin{aligned} \|\nabla_h e^n\|_A^2 &\leqslant 3t_n^{\gamma-1}\Gamma(2-\gamma)\tau\sum_{k=1}^{n}\|(r_{16})^{k-\frac{1}{2}}\|^2 \\ &\leqslant 3T^\gamma\Gamma(2-\gamma)L_1L_2c_{16}^2(\tau^{3-\gamma}+h_1^4+h_2^4)^2, \quad 1\leqslant n\leqslant N. \end{aligned}$$

Taking the square root on both hand sides of the inequality above, it follows from Lemma 3.10.2 that

$$\|\nabla_h e^n\| \leqslant \sqrt{\frac{3}{2}}\|\nabla_h e^n\|_A \leqslant \frac{3}{2}\sqrt{2T^\gamma\Gamma(2-\gamma)L_1L_2}\, c_{16}(\tau^{3-\gamma}+h_1^4+h_2^4), \quad 1\leqslant n\leqslant N.$$

The proof ends. □

3.11 Supplementary remarks and discussions

1. In this chapter, the finite difference methods for solving the initial-boundary value problems of time-fractional wave equations were discussed. The time Caputo fractional derivative was approximated by the L1 approximation, the fast L1 approximation, the L2-1_σ approximation or the fast L2-1_σ approximation. The spatial second-order derivative was discretized by the second-order central difference quotient or the compact approximation. Several difference schemes were derived. The unique solvability, unconditional stability and convergence of each scheme were analyzed. For 2D problem, the ADI and compact ADI methods were mainly addressed.

2. The concerned problems in this chapter were the time-fractional wave equations in the differential form and the difference schemes were derived using the fractional numerical differentiation formulae to directly approximate the fractional

derivatives. It is worth to mention that in 1993, Tang[87] investigated the difference method for a class of pseudo-differential equations with the weak singularity kernel, which are precisely the integral form of the fractional wave equations of order 3/2. There are also some results on the difference methods for the integral forms of fractional differential equations. Huang et al.[38] performed the fractional integral operator ${}_0D_t^{1-\alpha}$ on both hand sides of (3.1) and obtained an equivalent form of (3.1) as follows:

$$u_t(x,t) = \psi(x) + {}_0D_t^{1-\alpha}u_{xx}(x,t) + {}_0D_t^{1-\alpha}f(x,t). \tag{3.332}$$

Then the first-order G-L formula was used to derive the difference scheme. Wang and Vong[96] also studied the difference method for solving (3.332) and the second-order convergent difference scheme was established, where the second-order G-L formula was derived to approximate the R-L integral. Huang et al.[39] continued to study the alternating direction implicit scheme for the two-dimensional time fractional nonlinear super-diffusion equations by the above transformation.

3. All problems in this chapter are ones with the Dirichlet boundary conditions. Ren and Sun[66] studied the high-order difference method for this kind of equations with the Neumann boundary conditions. Later on, Vong and Wang[90] also discussed the difference method for this kind of problem. Besides, the difference method for the time-fractional wave equations in unbounded domains was focused on by Brunner et al.[3].

4. In Section 3.6 and Section 3.7, a class of multiterm time-fractional wave equations was briefly studied. For the numerical method solving this kind of problems, Gao and Liu [20] discussed the fourth-order multi-term time fractional wave equations.; Liu[51] also made some research results on the 1D problem. For the 2D problem, the compact ADI method was developed in [68] and the energy method was used to make some analyses.

5. Sun et al.[81] obtained a temporal second-order difference scheme for the fractional wave equation by using the method of order reduction and L2-1_σ approximation. Then Sun et al.[83] combined the method of order reduction with the method in [19] to construct the temporal second-order difference scheme for the multiterm time fractional wave equation. Sun and Sun [79] developed a fast temporal second-order difference scheme for the multiterm time fractional wave equation.

6. Feng et al.[18], Sun et al.[77] have studied the difference methods based on L1 approximation for the time fractional mixed diffusion and wave equations. Du and Sun[17] have studied the difference methods based on L2-1_σ approximation with the method of order reduction for the time fractional mixed diffusion and wave equations.

Exercises 3

3.1 Consider the problem

$$\begin{cases} {}_0\mathbf{D}_t^{\gamma} u(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], & (3.333)\\ u(x,0) = 0,\ u_t(x,0) = 0, \quad x \in (0,L), & (3.334)\\ u(0,t) = \mu(t),\ u(L,t) = \nu(t), \quad t \in [0,T], & (3.335) \end{cases}$$

where $\gamma \in (1,2)$, the functions f, μ, ν are given and $\mu(0) = \nu(0) = \mu'(0) = \nu'(0) = 0$. For any fixed $x \in [0,L]$, define the function

$$\hat{u}(x,t) = \begin{cases} 0, & t < 0,\\ u(x,t), & 0 \leqslant t \leqslant T,\\ v(x,t), & T < t < 2T,\\ 0, & t \geqslant 2T, \end{cases}$$

where $v(x,t)$ is a smooth function satisfying $\frac{\partial^k v(x,t)}{\partial t^k}|_{t=T} = \frac{\partial^k u(x,t)}{\partial t^k}|_{t=T}, \frac{\partial^k v(x,t)}{\partial t^k}|_{t=2T} = 0$, $k = 0,1,2,3$. Suppose $\hat{u}(x,\cdot) \in \mathscr{C}^{1+\gamma}(\mathcal{R})$.
For the problem, construct the difference scheme as follows:

$$\begin{cases} \tau^{-(\gamma-1)} \sum_{k=0}^{n-1} g_k^{(\gamma-1)} \delta_t u_i^{n-k-\frac{1}{2}} = \delta_x^2 u_i^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}},\\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N,\\ u_i^0 = 0, \quad 1 \leqslant i \leqslant M-1,\\ u_0^n = \mu(t_n),\ u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. \end{cases}$$

For this difference scheme, try to
(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the function f;
(4) show the convergence.

3.2 For the problem (3.333)–(3.335), construct the following difference scheme:

$$\begin{cases} \mathcal{A}\left(\tau^{-(\gamma-1)} \sum_{k=0}^{n-1} g_k^{(\gamma-1)} \delta_t u_i^{n-k-\frac{1}{2}}\right) = \delta_x^2 u_i^{n-\frac{1}{2}} + \mathcal{A} f_i^{n-\frac{1}{2}},\\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N,\\ u_i^0 = 0, \quad 1 \leqslant i \leqslant M-1,\\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. \end{cases}$$

For this difference scheme, try to
(1) analyze the truncation error;
(2) show the unique solvability;

(3) show the stability with respect to the function f;
(4) show the convergence.

3.3 For the problem (3.1)–(3.3), apply the H2N2 approximation (1.106) in Subsection 1.6.5 to construct the following difference scheme:

$$\begin{cases} \dfrac{1}{\Gamma(2-\gamma)}\left[\hat{b}_0^{(n,\gamma)}\delta_t u_i^{n-\frac{1}{2}} - \sum\limits_{k=1}^{n-1}(\hat{b}_{n-k-1}^{(n,\gamma)} - \hat{b}_{n-k}^{(n,\gamma)})\delta_t u_i^{k-\frac{1}{2}} - \hat{b}_{n-1}^{(n,\gamma)}\psi_i\right] \\ = \delta_x^2 u_i^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. \end{cases}$$

For this difference scheme, try to
(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the initial value φ and the function f;
(4) show the convergence.

3.4 For the problem (3.1)–(3.3), apply the fast H2N2 approximation (1.154)–(1.156) in Subsection 1.7.3 to construct the following difference scheme:

$$\begin{cases} \dfrac{1}{\Gamma(2-\gamma)}\hat{b}_0^{(1,\gamma)}(\delta_t u_i^{\frac{1}{2}} - \psi_i) = \delta_x^2 u_i^{\frac{1}{2}} + f_i^{\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1, \\ \dfrac{1}{\Gamma(2-\gamma)}\left[\sum\limits_{l=1}^{N_{\exp}}\omega_l F_{l,i}^n + \dfrac{\tau^{2-\gamma}}{2-\gamma}\delta_t u_i^{n-1}\right] = \delta_x^2 u_i^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}}, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 2 \leqslant n \leqslant N, \\ F_{l,i}^2 = \dfrac{2}{\tau}\displaystyle\int_{t_0}^{t_{\frac{1}{2}}} e^{-s_l(t_{\frac{3}{2}}-t)}\mathrm{d}t(\delta_t u_i^{\frac{1}{2}} - \psi_i), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant l \leqslant N_{\exp}, \\ F_{l,i}^n = e^{-s_l\tau}F_{l,i}^{n-1} + B_l(\delta_t u_i^{n-\frac{3}{2}} - \delta_t u_i^{n-\frac{5}{2}}), \\ 1 \leqslant l \leqslant N_{\exp}, \quad 1 \leqslant i \leqslant M-1,\ 3 \leqslant n \leqslant N, \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. \end{cases}$$

For this difference scheme, try to
(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the initial value φ and the function f;
(4) show the convergence.

3.5 Consider the problem

$$\begin{cases} {}_0\mathbf{D}_t^{\gamma_1}u(x,t) + {}_0\mathbf{D}_t^{\gamma}u(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], \\ u(x,0) = 0,\ u_t(x,0) = 0, \quad x \in (0,L), \\ u(0,t) = \mu(t),\ u(L,t) = \nu(t), \quad t \in [0,T], \end{cases}$$

where $1 < \gamma_1 < \gamma < 2$, the functions f, μ, ν are given and $\mu(0) = \nu(0) = \mu'(0) = \nu'(0) = 0$.

Define the function $\hat{u}(x,t)$ like that in Exercise 3.1, and suppose $\hat{u}(x,\cdot) \in \mathscr{C}^{1+\gamma}(\mathcal{R})$. For the problem, construct the difference scheme as follows:

$$\begin{cases} \tau^{-(\gamma_1-1)} \sum_{k=0}^{n-1} g_k^{(\gamma_1-1)} \delta_t u_i^{n-k-\frac{1}{2}} + \tau^{-(\gamma-1)} \sum_{k=0}^{n-1} g_k^{(\gamma-1)} \delta_t u_i^{n-k-\frac{1}{2}} \\ = \delta_x^2 u_i^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = \mu(t_n),\ u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. \end{cases}$$

For this difference scheme, try to

(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the function f;
(4) show the convergence.

3.6 For the following problem of the time-fractional mixed diffusion and wave equations

$$\begin{cases} {}_0^C D_t^{\gamma} u(x,t) + {}_0^C D_t^{\alpha} u(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x), \quad x \in (0,L), \\ u(0,t) = \mu(t),\ u(L,t) = \nu(t), \quad t \in [0,T], \end{cases}$$

where $1 < \gamma < 2$, $0 < \alpha < 1$, the functions f, φ, ψ, μ, ν are given and $\mu(0) = \varphi(0)$, $\nu(0) = \varphi(L)$, $\mu'(0) = \psi(0)$, $\nu'(0) = \psi(L)$, construct the following compact difference scheme:

$$\begin{cases} \mathcal{A}\left\{\frac{1}{s_\gamma}\left[b_0^{(\gamma)} \delta_t u_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)}) \delta_t u_i^{k-\frac{1}{2}} - b_{n-1}^{(\gamma)} \psi_i\right]\right\} \\ + \mathcal{A}\left\{\frac{1}{s_\alpha}\left[\frac{a_0^{(\alpha)}}{2} \delta_t u_i^{n-\frac{1}{2}} + \sum_{k=1}^{n-1} \frac{a_{n-k}^{(\alpha)} + a_{n-1-k}^{(\alpha)}}{2} \delta_t u_i^{k-\frac{1}{2}}\right]\right\} \\ = \delta_x^2 u_i^{n-\frac{1}{2}} + \mathcal{A} f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = \mu(t_n), \quad u_M^n = \nu(t_n), \quad 0 \leqslant n \leqslant N. \end{cases}$$

For this difference scheme, try to

(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the initial value φ and the function f;
(4) show the convergence.

3.7 Consider the problem

$$\begin{cases} {}_0\mathbf{D}_t^{\gamma} u(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) + f(x,y,t), \\ \qquad\qquad (x,y) \in \Omega,\ t \in (0,T], & (3.336) \\ u(x,y,0) = 0, \quad u_t(x,y,0) = 0, \quad (x,y) \in \Omega, & (3.337) \\ u(x,y,t) = \mu(x,y,t), \quad (x,y) \in \partial\Omega,\ t \in [0,T], & (3.338) \end{cases}$$

where $\Omega = (0,L_1) \times (0,L_2)$, $\gamma \in (1,2)$, the functions f, μ are given and $\mu(x,y,0)|_{(x,y)\in\partial\Omega} = 0$, $\mu_t(x,y,0)|_{(x,y)\in\partial\Omega} = 0$.
For any fixed $(x,y) \in \bar{\Omega}$, define the function

$$\hat{u}(x,y,t) = \begin{cases} 0, & t < 0, \\ u(x,y,t), & 0 \leqslant t \leqslant T, \\ v(x,y,t), & T < t < 2T, \\ 0, & t \geqslant 2T, \end{cases}$$

where $v(x,y,t)$ is a smooth function satisfying $\frac{\partial^k v(x,y,t)}{\partial t^k}|_{t=T} = \frac{\partial^k u(x,y,t)}{\partial t^k}|_{t=T}$, $\frac{\partial^k v(x,y,t)}{\partial t^k}|_{t=2T} = 0$, $k = 0,1,2,3$. Suppose $\hat{u}(x,y,\cdot) \in \mathscr{C}^{1+\gamma}(\mathcal{R})$.
For the problem, construct the difference scheme as follows:

$$\begin{cases} \tau^{-(\gamma-1)} \sum_{k=0}^{n-1} g_k^{(\gamma-1)} \delta_t u_{ij}^{n-k-\frac{1}{2}} = \delta_x^2 u_{ij}^{n-\frac{1}{2}} + \delta_y^2 u_{ij}^{n-\frac{1}{2}} + f_{ij}^{n-\frac{1}{2}}, \\ \qquad\qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ u_{ij}^0 = 0, \quad (i,j) \in \omega, \\ u_{ij}^n = \mu(x_i,y_j,t_n), \quad (i,j) \in \partial\omega, \quad 0 \leqslant n \leqslant N. \end{cases}$$

For this difference scheme, try to
(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the function f;
(4) show the convergence.

3.8 Similar to Exercise 3.7, define $\hat{u}(x,y,t)$ and suppose $\hat{u}(x,y,\cdot) \in \mathscr{C}^{1+\gamma}(\mathcal{R})$. For the problem (3.336)–(3.338), construct the following difference scheme:

$$\begin{cases} \mathcal{A}_x\mathcal{A}_y\left(\tau^{-(\gamma-1)} \sum_{k=0}^{n-1} g_k^{(\gamma-1)} \delta_t u_{ij}^{n-k-\frac{1}{2}} \right) \\ = \mathcal{A}_y \delta_x^2 u_{ij}^{n-\frac{1}{2}} + \mathcal{A}_x \delta_y^2 u_{ij}^{n-\frac{1}{2}} + \mathcal{A}_x\mathcal{A}_y f_{ij}^{n-\frac{1}{2}}, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ u_{ij}^0 = 0, \quad (i,j) \in \omega, \\ u_{ij}^n = \mu(x_i,y_j,t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

For this difference scheme, try to

(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the function f;
(4) show the convergence.

4 Difference methods for the space-fractional partial differential equations

The space-fractional differential equation can be used to model the diffusion process related to the whole space. In mathematics, it can be obtained by replacing the spatial integer-order derivative in the classical diffusion equation with the R-L fractional derivative or the Riesz fractional derivative. In this chapter, we mainly focus on the numerical method for the space-fractional partial differential equations in R-L type. It is a natural idea to approximate the R-L fractional derivative using the standard G-L formula, whereas, the resultant difference scheme is unstable. Meerschaert and Tadjeran[57, 58, 59] applied the shifted G-L formula to study the numerical solution of the space-fractional differential equation and some stable difference schemes were derived. In this chapter, for the one-dimensional space-fractional partial differential equation, the first-order method in both time and space based on the shifted G-L formula, the second-order method in both time and space based on the WSGL formula, the method of order two in time and order four in space will be introduced successively. For the 2D case, the method of order two in time and order four in space will be developed. The whole chapter consists of 5 sections.

4.1 The first-order method based on the shifted G-L formula for 1D problem

Consider the initial-boundary value problem of the space-fractional partial differential equations

$$\begin{cases} u_t(x,t) = K_1\, {}_0\mathbf{D}_x^{\beta} u(x,t) + K_2\, {}_x\mathbf{D}_L^{\beta} u(x,t) + f(x,t), \\ \qquad\qquad x \in (0,L),\ t \in (0,T], & (4.1) \\ u(x,0) = \varphi(x), \quad x \in (0,L), & (4.2) \\ u(0,t) = 0, \quad u(L,t) = 0, \quad t \in [0,T], & (4.3) \end{cases}$$

where K_1, K_2 are two nonnegative constants, $K_1 + K_2 > 0$, $\beta \in (1,2)$, the functions f, φ are given and $\varphi(0) = \varphi(L) = 0$.

Introduce the same mesh partition, notations and the mesh function spaces $\mathcal{U}_h$, $\mathring{\mathcal{U}}_h$ as those in Section 2.1 and Section 3.1. In addition, for any fixed $t \in [0,T]$, define the function

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & 0 \leqslant x \leqslant L, \\ 0, & x \notin [0,L]. \end{cases}$$

Suppose $u(x,\cdot) \in C^2[0,T]$ and the function $\tilde{u}(x,t)$ satisfies the conditions of Theorem 1.4.2, that is, $\tilde{u}(\cdot,t) \in \mathscr{C}^{1+\beta}(\mathcal{R})$.

https://doi.org/10.1515/9783110616064-004

Define the mesh functions

$$U_i^n = u(x_i, t_n), \quad f_i^n = f(x_i, t_n), \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N.$$

4.1.1 Derivation of the difference scheme

Considering equation (4.1) at the point (x_i, t_n), we have

$$u_t(x_i, t_n) = K_1\, {}_0\mathbf{D}_x^\beta u(x_i, t_n) + K_2\, {}_x\mathbf{D}_L^\beta u(x_i, t_n) + f_i^n, \\ 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{4.4}$$

For the space-fractional derivatives on the right-hand side of (4.4), by the shifted G-L formula (1.19), it follows

$${}_0\mathbf{D}_x^\beta u(x_i, t_n) = h^{-\beta} \sum_{k=0}^{i+1} g_k^{(\beta)} U_{i-k+1}^n + O(h), \tag{4.5}$$

$${}_x\mathbf{D}_L^\beta u(x_i, t_n) = h^{-\beta} \sum_{k=0}^{M-i+1} g_k^{(\beta)} U_{i+k-1}^n + O(h). \tag{4.6}$$

For the time first-order derivative on the left-hand side of (4.4), we have

$$u_t(x_i, t_n) = \frac{1}{\tau}(U_i^n - U_i^{n-1}) + O(\tau). \tag{4.7}$$

The substitution of (4.5)–(4.7) into (4.4) gives

$$\frac{1}{\tau}(U_i^n - U_i^{n-1}) = K_1\, h^{-\beta} \sum_{k=0}^{i+1} g_k^{(\beta)} U_{i-k+1}^n + K_2\, h^{-\beta} \sum_{k=0}^{M-i+1} g_k^{(\beta)} U_{i+k-1}^n + f_i^n + (r_1)_i^n, \\ 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{4.8}$$

and there is a positive constant c_1 such that

$$\left|(r_1)_i^n\right| \leqslant c_1(\tau + h), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{4.9}$$

Noticing the initial-boundary value conditions (4.2)–(4.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (4.10)\\ U_0^n = 0, \quad U_M^n = 0, \quad 0 \leqslant n \leqslant N. & (4.11)\end{cases}$$

Neglecting the small term $(r_1)_i^n$ in (4.8) and replacing the exact solution U_i^n with its numerical one u_i^n produce a difference scheme for solving (4.1)–(4.3) as follows:

$$\begin{cases} \dfrac{1}{\tau}(u_i^n - u_i^{n-1}) = K_1\, h^{-\beta} \displaystyle\sum_{k=0}^{i+1} g_k^{(\beta)} u_{i-k+1}^n + K_2\, h^{-\beta} \sum_{k=0}^{M-i+1} g_k^{(\beta)} u_{i+k-1}^n + f_i^n, & \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (4.12)\\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (4.13)\\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N. & (4.14)\end{cases}$$

Denote

$$\lambda_1 = K_1 \frac{\tau}{h^\beta}, \quad \lambda_2 = K_2 \frac{\tau}{h^\beta}.$$

4.1.2 Solvability of the difference scheme

Theorem 4.1.1. *The difference scheme* (4.12)–(4.14) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The value of u^0 is obviously determined by (4.13)–(4.14). Suppose that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined. From (4.12) and (4.14), we can obtain the linear system in the unknown u^n. To prove its unique solvability, it suffices to show the corresponding homogeneous one

$$\begin{cases} \frac{1}{\tau} u_i^n = K_1\, h^{-\beta} \sum\limits_{k=0}^{i+1} g_k^{(\beta)} u_{i-k+1}^n + K_2\, h^{-\beta} \sum\limits_{k=0}^{M-i+1} g_k^{(\beta)} u_{i+k-1}^n, \\ \qquad\qquad\qquad\qquad 1 \leqslant i \leqslant M-1, & (4.15) \\ u_0^n = u_M^n = 0 & (4.16) \end{cases}$$

has only the trivial solution.

When $1 < \beta < 2$, it follows from Lemma 1.4.1 that

$$g_0^{(\beta)} = 1, \quad g_1^{(\beta)} = -\beta, \quad g_2^{(\beta)} > g_3^{(\beta)} > \cdots > 0,$$
$$\sum_{k=0}^{\infty} g_k^{(\beta)} = 0, \quad \sum_{k=0}^{m} g_k^{(\beta)} < 0, \quad m \geqslant 1.$$

Rewrite equation (4.15) as

$$\begin{aligned} &[1 + \lambda_1(-g_1^{(\beta)}) + \lambda_2(-g_1^{(\beta)})] u_i^n \\ &= \lambda_1 \sum_{\substack{k=0 \\ k\neq 1}}^{i+1} g_k^{(\beta)} u_{i-k+1}^n + \lambda_2 \sum_{\substack{k=0 \\ k\neq 1}}^{M-i+1} g_k^{(\beta)} u_{i+k-1}^n, \quad 1 \leqslant i \leqslant M-1. \end{aligned}$$

Suppose $\|u^n\|_\infty = |u_{i_n}^n|$, where $i_n \in \{1, 2, \ldots, M-1\}$. Letting $i = i_n$ in the equality above and taking the absolute value of both hand sides, it follows from the triangle inequality that

$$\begin{aligned} &[1 + \lambda_1(-g_1^{(\beta)}) + \lambda_2(-g_1^{(\beta)})] \|u^n\|_\infty \\ &\leqslant \lambda_1 \sum_{\substack{k=0 \\ k\neq 1}}^{i_n+1} g_k^{(\beta)} |u_{i_n-k+1}^n| + \lambda_2 \sum_{\substack{k=0 \\ k\neq 1}}^{M-i_n+1} g_k^{(\beta)} |u_{i_n+k-1}^n| \end{aligned}$$

$$\leqslant \lambda_1 \sum_{\substack{k=0\\k\neq 1}}^{i_n+1} g_k^{(\beta)} \|u^n\|_\infty + \lambda_2 \sum_{\substack{k=0\\k\neq 1}}^{M-i_n+1} g_k^{(\beta)} \|u^n\|_\infty.$$

Noticing $\sum_{k=0}^{m} g_k^{(\beta)} < 0$ $(m \geqslant 1)$, we have

$$\|u^n\|_\infty \leqslant \lambda_1 \sum_{k=0}^{i_n+1} g_k^{(\beta)} \|u^n\|_\infty + \lambda_2 \sum_{k=0}^{M-i_n+1} g_k^{(\beta)} \|u^n\|_\infty \leqslant 0,$$

thus $\|u^n\|_\infty = 0$, which implies that (4.15)–(4.16) has only the trivial solution.

By the principle of induction, the difference scheme (4.12)–(4.14) is uniquely solvable. The proof ends. □

4.1.3 Stability of the difference scheme

Theorem 4.1.2. *The solution of the difference scheme* (4.12)–(4.14) *is stable with respect to both the initial value φ and the right hand function f. More precisely, it holds*

$$\|u^n\|_\infty \leqslant \|u^0\|_\infty + \tau \sum_{m=1}^{n} \|f^m\|_\infty, \quad 1 \leqslant n \leqslant N,$$

where

$$\|f^m\|_\infty = \max_{1\leqslant i\leqslant M-1} |f_i^m|.$$

Proof. Rewrite equation (4.12) as follows:

$$\begin{aligned}&[1 + \lambda_1(-g_1^{(\beta)}) + \lambda_2(-g_1^{(\beta)})]u_i^n\\&= u_i^{n-1} + \lambda_1 \sum_{\substack{k=0\\k\neq 1}}^{i+1} g_k^{(\beta)} u_{i-k+1}^n + \lambda_2 \sum_{\substack{k=0\\k\neq 1}}^{M-i+1} g_k^{(\beta)} u_{i+k-1}^n + \tau f_i^n,\\&\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.\end{aligned}$$

Suppose $\|u^n\|_\infty = |u_{i_n}^n|$, where $i_n \in \{1, 2, \ldots, M-1\}$. Letting $i = i_n$ in the equality above and taking the absolute value of both hand sides, the application of the triangle inequality leads to

$$\begin{aligned}&[1 + \lambda_1(-g_1^{(\beta)}) + \lambda_2(-g_1^{(\beta)})]\|u^n\|_\infty\\&\leqslant \|u^{n-1}\|_\infty + \lambda_1 \sum_{\substack{k=0\\k\neq 1}}^{i_n+1} g_k^{(\beta)} \|u^n\|_\infty + \lambda_2 \sum_{\substack{k=0\\k\neq 1}}^{M-i_n+1} g_k^{(\beta)} \|u^n\|_\infty + \tau\|f^n\|_\infty, \quad 1 \leqslant n \leqslant N,\end{aligned}$$

that is,

$$\|u^n\|_\infty \leqslant \|u^{n-1}\|_\infty + \lambda_1 \sum_{k=0}^{i_n+1} g_k^{(\beta)} \|u^n\|_\infty + \lambda_2 \sum_{k=0}^{M-i_n+1} g_k^{(\beta)} \|u^n\|_\infty + \tau\|f^n\|_\infty$$

$$\leqslant \|u^{n-1}\|_\infty + \tau\|f^n\|_\infty, \quad 1 \leqslant n \leqslant N.$$

The recursive process will produce

$$\|u^n\|_\infty \leqslant \|u^0\|_\infty + \tau \sum_{m=1}^{n} \|f^m\|_\infty, \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

4.1.4 Convergence of the difference scheme

Theorem 4.1.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (4.1)–(4.3) *and the difference scheme* (4.12)–(4.14), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then it holds

$$\|e^n\|_\infty \leqslant c_1 T(\tau + h), \quad 1 \leqslant n \leqslant N.$$

Proof. The subtraction of (4.12)–(4.14) from (4.8), (4.10)–(4.11), respectively, yields the system of error equations as follows:

$$\begin{cases} \dfrac{1}{\tau}(e_i^n - e_i^{n-1}) = K_1\, h^{-\beta} \displaystyle\sum_{k=0}^{i+1} g_k^{(\beta)} e_{i-k+1}^n + K_2\, h^{-\beta} \displaystyle\sum_{k=0}^{M-i+1} g_k^{(\beta)} e_{i+k-1}^n \\ \qquad\qquad + (r_1)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (4.9), the application of Theorem 4.1.2 produces

$$\|e^n\|_\infty \leqslant \tau \sum_{m=1}^{n} \|(r_1)^m\|_\infty \leqslant n\tau c_1(\tau + h) \leqslant c_1 T(\tau + h), \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

4.2 The second-order method based on the WSGL formula for 1D problem

In this section, a difference method of order two in both time and space will be developed for the problem (4.1)–(4.3).

Suppose $u(x,\cdot) \in C^3[0,T]$ and the function $\tilde{u}(x,t)$ defined in Section 4.1 satisfies the condition of Theorem 1.4.3, that is, $\tilde{u}(\cdot,t) \in \mathscr{C}^{2+\beta}(\mathcal{R})$.

4.2.1 Derivation of the difference scheme

Considering equation (4.1) at the point (x_i, t_n), we have

$$u_t(x_i, t_n) = K_1\ {}_0\mathbf{D}_x^\beta u(x_i, t_n) + K_2\ {}_x\mathbf{D}_L^\beta u(x_i, t_n) + f_i^n,$$
$$1 \leqslant i \leqslant M-1,\ 0 \leqslant n \leqslant N.$$

Taking an average on two adjacent time levels gives

$$\begin{aligned}\frac{1}{2}\left[\,u_t(x_i, t_n) + u_t(x_i, t_{n-1})\right] &= \frac{1}{2}K_1\left[{}_0\mathbf{D}_x^\beta u(x_i, t_n) + {}_0\mathbf{D}_x^\beta u(x_i, t_{n-1})\right]\\ &\quad + \frac{1}{2}K_2\left[{}_x\mathbf{D}_L^\beta u(x_i, t_n) + {}_x\mathbf{D}_L^\beta u(x_i, t_{n-1})\right]\\ &\quad + f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \end{aligned}\tag{4.17}$$

For the space-fractional derivatives in (4.17), by the WSGL formula, it follows from Theorem 1.4.3 and Corollary 1.4.2 that

$${}_0\mathbf{D}_x^\beta u(x_i, t_n) = h^{-\beta}\sum_{k=0}^{i+1}\widetilde{w}_k^{(\beta)}U_{i-k+1}^n + O(h^2),\tag{4.18}$$

$${}_x\mathbf{D}_L^\beta u(x_i, t_n) = h^{-\beta}\sum_{k=0}^{M-i+1}\widetilde{w}_k^{(\beta)}U_{i+k-1}^n + O(h^2),\tag{4.19}$$

where the coefficient $\{\widetilde{w}_k^{(\beta)}\}$ is defined by (1.35), which satisfies (1.36).

For the time first-order derivative in (4.17), it follows from the Taylor's formula that

$$\frac{1}{2}\left[u_t(x_i, t_n) + u_t(x_i, t_{n-1})\right] = \frac{1}{\tau}(U_i^n - U_i^{n-1}) + O(\tau^2).\tag{4.20}$$

Substituting (4.18)–(4.20) into (4.17) produces

$$\begin{aligned}\frac{1}{\tau}(U_i^n - U_i^{n-1}) &= K_1\, h^{-\beta}\sum_{k=0}^{i+1}\widetilde{w}_k^{(\beta)}U_{i-k+1}^{n-\frac{1}{2}} + K_2\, h^{-\beta}\sum_{k=0}^{M-i+1}\widetilde{w}_k^{(\beta)}U_{i+k-1}^{n-\frac{1}{2}}\\ &\quad + f_i^{n-\frac{1}{2}} + (r_2)_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N,\end{aligned}\tag{4.21}$$

and there is a positive constant c_2 such that

$$\left|(r_2)_i^{n-\frac{1}{2}}\right| \leqslant c_2(\tau^2 + h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.\tag{4.22}$$

Noticing the initial-boundary value conditions (4.2)–(4.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (4.23)\\ U_0^n = 0,\ U_M^n = 0, \quad 0 \leqslant n \leqslant N. & (4.24)\end{cases}$$

Omitting the small term $(r_2)_i^{n-\frac{1}{2}}$ in (4.21) and replacing the exact solution U_i^n with its numerical one u_i^n produce another difference scheme for solving (4.1)–(4.3) as follows:

$$\begin{cases} \frac{1}{\tau}(u_i^n - u_i^{n-1}) = K_1\, h^{-\beta} \sum_{k=0}^{i+1} \widetilde{w}_k^{(\beta)} u_{i-k+1}^{n-\frac{1}{2}} + K_2\, h^{-\beta} \sum_{k=0}^{M-i+1} \widetilde{w}_k^{(\beta)} u_{i+k-1}^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}}, \\ \qquad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, & (4.25) \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (4.26) \\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N. & (4.27) \end{cases}$$

4.2.2 Solvability of the difference scheme

We start our proof from the following important lemma.

Lemma 4.2.1. *For any mesh function $u = (u_0, u_1, \ldots, u_M) \in \mathring{\mathcal{U}}_h$, it holds*

$$h \sum_{i=1}^{M-1} \left(\sum_{k=0}^{i+1} \widetilde{w}_k^{(\beta)} u_{i-k+1} \right) u_i \leqslant 0, \tag{4.28}$$

$$h \sum_{i=1}^{M-1} \left(\sum_{k=0}^{M-i+1} \widetilde{w}_k^{(\beta)} u_{i+k-1} \right) u_i \leqslant 0. \tag{4.29}$$

Proof. We only prove (4.28) here. The inequality (4.29) can be proved similarly.

Noticing $u_0 = u_M = 0$ and (1.36), it follows

$$\begin{aligned} & h \sum_{i=1}^{M-1} \left(\sum_{k=0}^{i+1} \widetilde{w}_k^{(\beta)} u_{i-k+1} \right) u_i \\ = {} & h \sum_{i=1}^{M-1} \left(\sum_{k=0}^{i} \widetilde{w}_k^{(\beta)} u_{i-k+1} \right) u_i \\ = {} & \widetilde{w}_1^{(\beta)} h \sum_{i=1}^{M-1} (u_i)^2 + (\widetilde{w}_0^{(\beta)} + \widetilde{w}_2^{(\beta)}) h \sum_{i=1}^{M-2} u_i u_{i+1} \\ & + \sum_{k=3}^{M-1} \widetilde{w}_k^{(\beta)} h \sum_{i=k}^{M-1} u_{i-k+1} u_i \\ \leqslant {} & \widetilde{w}_1^{(\beta)} \|u\|^2 + (\widetilde{w}_0^{(\beta)} + \widetilde{w}_2^{(\beta)}) h \sum_{i=1}^{M-2} \frac{(u_i)^2 + (u_{i+1})^2}{2} \\ & + \sum_{k=3}^{M-1} \widetilde{w}_k^{(\beta)} h \sum_{i=k}^{M-1} \frac{(u_i)^2 + (u_{i-k+1})^2}{2} \\ \leqslant {} & \left(\sum_{k=0}^{M-1} \widetilde{w}_k^{(\beta)} \right) \|u\|^2 \leqslant 0, \end{aligned}$$

which implies the truth of (4.28). The proof ends. □

Next, the unique solvability of the difference scheme (4.25)–(4.27) will be proved.

Theorem 4.2.1. *The difference scheme* (4.25)–(4.27) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \dots, u_M^n).$$

The value of u^0 is uniquely determined by (4.26)–(4.27). Now suppose the values of $u^0, u^1, \dots, u^{n-1}$ have been uniquely determined, from (4.25) and (4.27), we can obtain the linear system in u^n. To show its unique solvability, it suffices to show the corresponding homogeneous one

$$\begin{cases} \frac{1}{\tau} u_i^n = \frac{1}{2} K_1\, h^{-\beta} \sum_{k=0}^{i+1} \widetilde{w}_k^{(\beta)} u_{i-k+1}^n + \frac{1}{2} K_2\, h^{-\beta} \sum_{k=0}^{M-i+1} \widetilde{w}_k^{(\beta)} u_{i+k-1}^n, & 1 \leqslant i \leqslant M-1, \quad (4.30) \\ u_0^n = u_M^n = 0 & (4.31) \end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (4.30) with u^n, it follows from Lemma 4.2.1 that

$$\begin{aligned} \frac{1}{\tau} h \sum_{i=1}^{M-1} (u_i^n)^2 &= \frac{1}{2} K_1\, h^{-\beta} h \sum_{i=1}^{M-1} \left(\sum_{k=0}^{i+1} \widetilde{w}_k^{(\beta)} u_{i-k+1}^n \right) u_i^n \\ &\quad + \frac{1}{2} K_2\, h^{-\beta} h \sum_{i=1}^{M-1} \left(\sum_{k=0}^{M-i+1} \widetilde{w}_k^{(\beta)} u_{i+k-1}^n \right) u_i^n \\ &\leqslant 0, \end{aligned}$$

which implies $\|u^n\| = 0$. Combining with (4.31), it follows that $u^n = 0$, namely, the homogeneous system (4.30)–(4.31) has only the trivial solution.

By the principle of induction, the difference scheme (4.25)–(4.27) is uniquely solvable. The proof ends. □

4.2.3 Stability of the difference scheme

Theorem 4.2.2. *The solution of the difference scheme* (4.25)–(4.27) *is stable with respect to both the initial value φ and the right-hand function f. More precisely, it holds*

$$\|u^n\| \leqslant \|u^0\| + \tau \sum_{m=1}^{n} \|f^{m-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N,$$

where

$$\|f^{m-\frac{1}{2}}\| = \sqrt{h \sum_{i=1}^{M-1} (f_i^{m-\frac{1}{2}})^2}.$$

Proof. Taking the inner product on both hand sides of (4.25) with $u^{n-\frac{1}{2}}$ yields

$$\frac{1}{2\tau}\left(\|u^n\|^2 - \|u^{n-1}\|^2\right) = K_1\, h^{-\beta} h \sum_{i=1}^{M-1}\left(\sum_{k=0}^{i+1} \widetilde{w}_k^{(\beta)} u_{i-k+1}^{n-\frac{1}{2}}\right) u_i^{n-\frac{1}{2}}$$
$$+ K_2\, h^{-\beta} h \sum_{i=1}^{M-1}\left(\sum_{k=0}^{M-i+1} \widetilde{w}_k^{(\beta)} u_{i+k-1}^{n-\frac{1}{2}}\right) u_i^{n-\frac{1}{2}}$$
$$+ h \sum_{i=1}^{M-1} u_i^{n-\frac{1}{2}} f_i^{n-\frac{1}{2}}, \quad 1 \leqslant n \leqslant N.$$

It follows from Lemma 4.2.1 that

$$\frac{1}{2\tau}\left(\|u^n\|^2 - \|u^{n-1}\|^2\right) \leqslant h \sum_{i=1}^{M-1} u_i^{n-\frac{1}{2}} f_i^{n-\frac{1}{2}}$$
$$\leqslant \|u^{n-\frac{1}{2}}\| \cdot \|f^{n-\frac{1}{2}}\|$$
$$\leqslant \frac{1}{2}\left(\|u^n\| + \|u^{n-1}\|\right)\|f^{n-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N.$$

Hence,

$$\frac{1}{\tau}\left(\|u^n\| - \|u^{n-1}\|\right) \leqslant \|f^{n-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N,$$

or

$$\|u^n\| \leqslant \|u^{n-1}\| + \tau\|f^{n-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N.$$

Performing the recursive process gives

$$\|u^n\| \leqslant \|u^0\| + \tau \sum_{m=1}^{n} \|f^{m-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

4.2.4 Convergence of the difference scheme

Theorem 4.2.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (4.1)–(4.3) *and the difference scheme* (4.25)–(4.27), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then it holds

$$\|e^n\| \leqslant c_2 T\sqrt{L}(\tau^2 + h^2), \quad 1 \leqslant n \leqslant N.$$

Proof. Subtracting (4.25)–(4.27) from (4.21), (4.23)–(4.24), respectively, yields the system of error equations as follows:

$$\begin{cases} \frac{1}{\tau}(e_i^n - e_i^{n-1}) = K_1\, h^{-\beta} \sum\limits_{k=0}^{i+1} \widetilde{w}_k^{(\beta)} e_{i-k+1}^{n-\frac{1}{2}} + K_2\, h^{-\beta} \sum\limits_{k=0}^{M-i+1} \widetilde{w}_k^{(\beta)} e_{i+k-1}^{n-\frac{1}{2}} \\ \qquad\qquad + (r_2)_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (4.22), it follows from Theorem 4.2.2 that

$$\|e^n\| \leqslant \tau \sum_{m=1}^{n} \|(r_2)^{m-\frac{1}{2}}\| \leqslant n\tau c_2 \sqrt{L}(\tau^2 + h^2) \leqslant c_2 T \sqrt{L}(\tau^2 + h^2), \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

4.3 The fourth-order method based on the WSGL formula for 1D problem

In this section, a difference method of order two in time and order four in space will be derived for (4.1)–(4.3) by the mean of the three-term WSGL formula (1.38) in Theorem 1.4.5.

Suppose $u(x,\cdot) \in C^3[0,T]$ and the function $\tilde{u}(x,t)$ defined in Section 4.1 satisfies the condition of Theorem 1.4.5, that is, $\tilde{u}(\cdot,t) \in \mathscr{C}^{4+\beta}(\mathcal{R})$.

For any mesh function $u = (u_0, u_1, \ldots, u_M) \in \mathcal{U}_h$, define the operator

$$(\mathcal{B}^{(\beta)}u)_i = \begin{cases} c_2^\beta u_{i-1} + (1 - 2c_2^\beta)u_i + c_2^\beta u_{i+1}, & 1 \leqslant i \leqslant M-1, \\ u_i, & i = 0,\ M, \end{cases}$$

where

$$c_2^\beta = \frac{-\beta^2 + \beta + 4}{24} \in \left(\frac{1}{12}, \frac{1}{6}\right), \quad 1 < \beta < 2.$$

In addition, for any mesh function $u \in \mathring{\mathcal{U}}_h$, define

$$\|u\|_B^2 = \|u\|^2 - h^2 c_2^\beta \|\delta_x u\|^2.$$

The following lemma is true.

Lemma 4.3.1. *For any mesh function $u \in \mathring{\mathcal{U}}_h$, it holds*

$$(\mathcal{B}^{(\beta)}u, u) = \|u\|_B^2, \quad \frac{1}{3}\|u\|^2 \leqslant \|u\|_B^2 \leqslant \|u\|^2.$$

Proof. For any mesh function $u \in \mathring{\mathcal{U}}_h$, noticing the definition of the operator $\mathcal{B}^{(\beta)}$, a direct calculation yields

$$\begin{aligned}(\mathcal{B}^{(\beta)}u, u) &= ((\mathcal{I} + c_2^\beta h^2 \delta_x^2)u, u)\\ &= (u, u) + c_2^\beta h^2(\delta_x^2 u, u)\\ &= \|u\|^2 - h^2 c_2^\beta \|\delta_x u\|^2 = \|u\|_B^2.\end{aligned}$$

Besides, noticing $\frac{1}{12} < c_2^\beta < \frac{1}{6}$, it follows from the inverse estimate inequality $\|\delta_x u\| \leqslant \frac{2}{h}\|u\|$ in Lemma 2.1.1 that

$$\begin{aligned}\|u\|_B^2 &\geqslant \|u\|^2 - h^2 c_2^\beta \cdot \frac{4}{h^2}\|u\|^2 = (1 - 4c_2^\beta)\|u\|^2\\ &\geqslant \left(1 - 4 \times \frac{1}{6}\right)\|u\|^2 = \frac{1}{3}\|u\|^2.\end{aligned}$$

It is clear that

$$\|u\|_B^2 = \|u\|^2 - h^2 c_2^\beta \|\delta_x u\|^2 \leqslant \|u\|^2.$$

The proof ends. □

4.3.1 Derivation of the difference scheme

Considering equation (4.1) at the point (x_i, t_n), we have

$$u_t(x_i, t_n) = K_1 \, {}_0\mathbf{D}_x^\beta u(x_i, t_n) + K_2 \, {}_x\mathbf{D}_L^\beta u(x_i, t_n) + f_i^n, \quad 0 \leqslant i \leqslant M, \; 0 \leqslant n \leqslant N.$$

Taking an average on two adjacent time levels and performing the operator $\mathcal{B}^{(\beta)}$ to both hand sides of the resultant equality lead to

$$\begin{aligned}&\frac{1}{2}\mathcal{B}^{(\beta)}[u_t(x_i, t_n) + u_t(x_i, t_{n-1})]\\ &= \frac{1}{2}K_1 \left[\mathcal{B}^{(\beta)} \, {}_0\mathbf{D}_x^\beta u(x_i, t_n) + \mathcal{B}^{(\beta)} \, {}_0\mathbf{D}_x^\beta u(x_i, t_{n-1})\right]\\ &\quad + \frac{1}{2}K_2 \left[\mathcal{B}^{(\beta)} \, {}_x\mathbf{D}_L^\beta u(x_i, t_n) + \mathcal{B}^{(\beta)} \, {}_x\mathbf{D}_L^\beta u(x_i, t_{n-1})\right]\\ &\quad + \mathcal{B}^{(\beta)} f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1, \; 1 \leqslant n \leqslant N.\end{aligned} \tag{4.32}$$

For the space-fractional derivatives in (4.32), it follows from Theorem 1.4.5 that

$$\mathcal{B}^{(\beta)} \, {}_0\mathbf{D}_x^\beta u(x_i, t_n) = h^{-\beta} \sum_{k=0}^{i+1} \hat{w}_k^{(\beta)} U_{i-k+1}^n + O(h^4), \tag{4.33}$$

$$\mathcal{B}^{(\beta)}\ {}_x\mathbf{D}_L^{\beta} u(x_i, t_n) = h^{-\beta} \sum_{k=0}^{M-i+1} \hat{w}_k^{(\beta)} U_{i+k-1}^n + O(h^4), \tag{4.34}$$

where the coefficient $\{\hat{w}_k^{(\beta)}\}$ is defined by (1.45), which satisfies (1.46).

For the time first-order derivative in (4.32), it follows from the Taylor's formula that

$$\frac{1}{2}[u_t(x_i, t_n) + u_t(x_i, t_{n-1})] = \frac{1}{\tau}(U_i^n - U_i^{n-1}) + O(\tau^2). \tag{4.35}$$

The substitution of (4.33)–(4.35) into (4.32) produces

$$\begin{aligned}\mathcal{B}^{(\beta)}\delta_t U_i^{n-\frac{1}{2}} &= K_1\, h^{-\beta} \sum_{k=0}^{i+1} \hat{w}_k^{(\beta)} U_{i-k+1}^{n-\frac{1}{2}} + K_2\, h^{-\beta} \sum_{k=0}^{M-i+1} \hat{w}_k^{(\beta)} U_{i+k-1}^{n-\frac{1}{2}} \\ &\quad + \mathcal{B}^{(\beta)} f_i^{n-\frac{1}{2}} + (r_3)_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N,\end{aligned} \tag{4.36}$$

and there is a positive constant c_3 such that

$$\left|(r_3)_i^{n-\frac{1}{2}}\right| \leqslant c_3(\tau^2 + h^4), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{4.37}$$

Noticing the initial-boundary value conditions (4.2)–(4.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (4.38) \\ U_0^n = 0, \quad U_M^n = 0, \quad 0 \leqslant n \leqslant N. & (4.39) \end{cases}$$

Omitting the small term $(r_3)_i^{n-\frac{1}{2}}$ in (4.36) and replacing the exact solution U_i^n with its numerical one u_i^n, we get another high order difference scheme for solving (4.1)–(4.3) in the form of

$$\begin{cases} \mathcal{B}^{(\beta)}\delta_t u_i^{n-\frac{1}{2}} = K_1\, h^{-\beta} \sum_{k=0}^{i+1} \hat{w}_k^{(\beta)} u_{i-k+1}^{n-\frac{1}{2}} + K_2\, h^{-\beta} \sum_{k=0}^{M-i+1} \hat{w}_k^{(\beta)} u_{i+k-1}^{n-\frac{1}{2}} + \mathcal{B}^{(\beta)} f_i^{n-\frac{1}{2}}, & \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (4.40) \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (4.41) \\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N. & (4.42) \end{cases}$$

4.3.2 Solvability of the difference scheme

At first, there is a lemma similar to Lemma 4.2.1, which will state some properties of the coefficient $\{\hat{w}_k^{(\beta)}\}$.

Lemma 4.3.2. *For any mesh function $u = (u_0, u_1, \ldots, u_M) \in \mathring{\mathcal{U}}_h$, it holds*

$$h \sum_{i=1}^{M-1} \left(\sum_{k=0}^{i+1} \hat{w}_k^{(\beta)} u_{i-k+1} \right) u_i \leqslant 0,$$

$$h \sum_{i=1}^{M-1} \left(\sum_{k=0}^{M-i+1} \hat{w}_k^{(\beta)} u_{i+k-1} \right) u_i \leqslant 0.$$

Next, the unique solvability of the difference scheme (4.40)–(4.42) will be shown.

Theorem 4.3.1. *The difference scheme* (4.40)–(4.42) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The value of u^0 is uniquely determined by (4.41)–(4.42). Now assume that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then we can obtain the linear system in the unknown u^n from (4.40) and (4.42). To show its unique solvability, it is sufficient to verify that the corresponding homogeneous one

$$\begin{cases} \frac{1}{\tau}\mathcal{B}^{(\beta)}u_i^n = \frac{1}{2}K_1\, h^{-\beta}\sum_{k=0}^{i+1}\hat{w}_k^{(\beta)}u_{i-k+1}^n + \frac{1}{2}K_2\, h^{-\beta}\sum_{k=0}^{M-i+1}\hat{w}_k^{(\beta)}u_{i+k-1}^n, \\ \qquad 1 \leqslant i \leqslant M-1, & (4.43) \\ u_0^n = u_M^n = 0 & (4.44) \end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (4.43) with u^n, it follows from Lemma 4.3.2 that

$$\begin{aligned} \frac{1}{\tau}h\sum_{i=1}^{M-1}(\mathcal{B}^{(\beta)}u_i^n)u_i^n &= \frac{1}{2}K_1\, h^{-\beta}h\sum_{i=1}^{M-1}\left(\sum_{k=0}^{i+1}\hat{w}_k^{(\beta)}u_{i-k+1}^n\right)u_i^n \\ &\quad + \frac{1}{2}K_2\, h^{-\beta}h\sum_{i=1}^{M-1}\left(\sum_{k=0}^{M-i+1}\hat{w}_k^{(\beta)}u_{i+k-1}^n\right)u_i^n \\ &\leqslant 0. \end{aligned}$$

Hence, $(\mathcal{B}^{(\beta)}u^n, u^n) \leqslant 0$. According to Lemma 4.3.1, we have

$$(\mathcal{B}^{(\beta)}u^n, u^n) = \|u^n\|_B^2 \geqslant \frac{1}{3}\|u^n\|^2,$$

thus $\|u^n\| = 0$. It follows that $u^n = 0$ with the combination of (4.44), which reveals that the homogeneous system (4.43)–(4.44) has only the trivial solution.

By the principle of induction, the difference scheme (4.40)–(4.42) is uniquely solvable. The proof ends. □

4.3.3 Stability of the difference scheme

Theorem 4.3.2. *The solution of the difference scheme* (4.40)–(4.42) *is stable with respect to both the initial value* φ *and the right-hand function* f. *More precisely, it holds*

$$\|u^n\| \leqslant \sqrt{3}\|u^0\| + 3\tau\sum_{m=1}^{n}\|\mathcal{B}^{(\beta)}f^{m-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N,$$

where

$$\|\mathcal{B}^{(\beta)} f^{m-\frac{1}{2}}\| = \sqrt{h\sum_{i=1}^{M-1}(\mathcal{B}^{(\beta)} f_i^{m-\frac{1}{2}})^2}.$$

Proof. Denote

$$g_i^{n-\frac{1}{2}} = \mathcal{B}^{(\beta)} f_i^{n-\frac{1}{2}}.$$

Taking the inner product on both hand sides of (4.40) with $u^{n-\frac{1}{2}}$, it follows from Lemma 4.3.2 that

$$\begin{aligned}
& h\sum_{i=1}^{M-1}(\mathcal{B}^{(\beta)}\delta_t u_i^{n-\frac{1}{2}})u_i^{n-\frac{1}{2}} \\
&= K_1\, h^{-\beta} h \sum_{i=1}^{M-1}\left(\sum_{k=0}^{i+1} \hat{w}_k^{(\beta)} u_{i-k+1}^{n-\frac{1}{2}}\right) u_i^{n-\frac{1}{2}} \\
&\quad + K_2\, h^{-\beta} h \sum_{i=1}^{M-1}\left(\sum_{k=0}^{M-i+1} \hat{w}_k^{(\beta)} u_{i+k-1}^{n-\frac{1}{2}}\right) u_i^{n-\frac{1}{2}} + h\sum_{i=1}^{M-1} g_i^{n-\frac{1}{2}} u_i^{n-\frac{1}{2}} \\
&\leqslant h\sum_{i=1}^{M-1} g_i^{n-\frac{1}{2}} u_i^{n-\frac{1}{2}} \\
&\leqslant \|g^{n-\frac{1}{2}}\| \cdot \|u^{n-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N.
\end{aligned}$$

Noticing

$$\begin{aligned}
& h\sum_{i=1}^{M-1}(\mathcal{B}^{(\beta)}\delta_t u_i^{n-\frac{1}{2}})u_i^{n-\frac{1}{2}} \\
&= h\sum_{i=1}^{M-1}(\delta_t u_i^{n-\frac{1}{2}} + c_2^{\beta} h^2 \delta_x^2 \delta_t u_i^{n-\frac{1}{2}})u_i^{n-\frac{1}{2}} \\
&= h\sum_{i=1}^{M-1}(u_i^{n-\frac{1}{2}})(\delta_t u_i^{n-\frac{1}{2}}) - c_2^{\beta} h^2 \cdot h\sum_{i=1}^{M}(\delta_t \delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}})(\delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}}) \\
&= \frac{1}{2\tau}(\|u^n\|^2 - \|u^{n-1}\|^2) - c_2^{\beta} h^2 \cdot \frac{1}{2\tau}(\|\delta_x u^n\|^2 - \|\delta_x u^{n-1}\|^2) \\
&= \frac{1}{2\tau}(\|u^n\|_B^2 - \|u^{n-1}\|_B^2), \quad 1 \leqslant n \leqslant N
\end{aligned}$$

and Lemma 4.3.1, further we have

$$\begin{aligned}
\frac{1}{2\tau}(\|u^n\|_B^2 - \|u^{n-1}\|_B^2) &\leqslant \|g^{n-\frac{1}{2}}\| \cdot \|u^{n-\frac{1}{2}}\| \\
&\leqslant \sqrt{3}\|g^{n-\frac{1}{2}}\| \cdot \|u^{n-\frac{1}{2}}\|_B
\end{aligned}$$

$$\leqslant \frac{\sqrt{3}}{2}\|g^{n-\frac{1}{2}}\|(\|u^n\|_B + \|u^{n-1}\|_B), \quad 1 \leqslant n \leqslant N,$$

or

$$\frac{1}{\tau}(\|u^n\|_B - \|u^{n-1}\|_B) \leqslant \sqrt{3}\|g^{n-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N,$$

that is

$$\|u^n\|_B \leqslant \|u^{n-1}\|_B + \sqrt{3}\,\tau\|g^{n-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N.$$

The recursive process will arrive at

$$\|u^n\|_B \leqslant \|u^0\|_B + \sqrt{3}\,\tau\sum_{m=1}^{n}\|g^{m-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N.$$

It is easy from Lemma 4.3.1 to get

$$\frac{1}{\sqrt{3}}\|u^n\| \leqslant \|u^0\| + \sqrt{3}\tau\sum_{m=1}^{n}\|g^{m-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N.$$

Hence the conclusion is true. The proof ends. □

4.3.4 Convergence of the difference scheme

Theorem 4.3.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (4.1)–(4.3) *and the difference scheme* (4.40)–(4.42), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M, \quad 0 \leqslant n \leqslant N,$$

then it holds

$$\|e^n\| \leqslant 3c_3T\sqrt{L}(\tau^2 + h^4), \quad 1 \leqslant n \leqslant N.$$

Proof. The subtraction of (4.40)–(4.42) from (4.36), (4.38)–(4.39), respectively, will produce the system of error equations as follows:

$$\begin{cases} \mathcal{B}^{(\beta)}\delta_t e_i^{n-\frac{1}{2}} = K_1\,h^{-\beta}\sum_{k=0}^{i+1}\hat{w}_k^{(\beta)}e_{i-k+1}^{n-\frac{1}{2}} + K_2\,h^{-\beta}\sum_{k=0}^{M-i+1}\hat{w}_k^{(\beta)}e_{i+k-1}^{n-\frac{1}{2}} \\ \qquad + (r_3)_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (4.37), it follows immediately from Theorem 4.3.2 that

$$\|e^n\| \leqslant 3\tau\sum_{m=1}^{n}\|(r_3)^{m-\frac{1}{2}}\| \leqslant 3n\tau c_3\sqrt{L}(\tau^2 + h^4) \leqslant 3c_3T\sqrt{L}(\tau^2 + h^4), \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

4.4 The fourth-order ADI method based on the WSGL formula for 2D problem

The finite difference method for solving the two-dimensional space-fractional partial differential equations will be considered in this section.

Consider the problem

$$\begin{cases} u_t(x,y,t) = K_1\, {}_0\mathbf{D}_x^{\beta} u(x,y,t) + K_2\, {}_x\mathbf{D}_{L_1}^{\beta} u(x,y,t) \\ \qquad + K_3\, {}_0\mathbf{D}_y^{\gamma} u(x,y,t) + K_4\, {}_y\mathbf{D}_{L_2}^{\gamma} u(x,y,t) + f(x,y,t), \\ \qquad\qquad (x,y) \in \Omega,\ t \in (0,T], & (4.45) \\ u(x,y,0) = \varphi(x,y), \quad (x,y) \in \Omega, & (4.46) \\ u(x,y,t) = 0, \quad (x,y) \in \partial\Omega, \quad t \in [0,T], & (4.47) \end{cases}$$

where $\Omega = (0,L_1) \times (0,L_2)$, $\partial\Omega$ is the boundary of Ω, the constant $K_i (i = 1,2,3,4)$ is nonnegative, $K_1 + K_2 > 0$, $K_3 + K_4 > 0$, $\beta,\gamma \in (1,2)$, the functions f, φ are given and $\varphi(x,y)|_{(x,y)\in\partial\Omega} = 0$.

Introduce the same mesh partition, notations and mesh function spaces $\mathcal{V}_h$, $\mathring{\mathcal{V}}_h$ as those in Section 2.10. Besides, for any mesh function $u \in \mathcal{V}_h$, define the average operators

$$\mathcal{B}_x^{\beta} u_{ij} = \begin{cases} c_2^{\beta} u_{i-1,j} + (1-2c_2^{\beta}) u_{ij} + c_2^{\beta} u_{i+1,j}, & 1 \leqslant i \leqslant M_1 - 1, \\ u_{ij}, & i = 0 \text{ or } i = M_1, \end{cases} \quad 0 \leqslant j \leqslant M_2,$$

$$\mathcal{B}_y^{\gamma} u_{ij} = \begin{cases} c_2^{\gamma} u_{i,j-1} + (1-2c_2^{\gamma}) u_{ij} + c_2^{\gamma} u_{i,j+1}, & 1 \leqslant j \leqslant M_2 - 1, \\ u_{ij}, & j = 0 \text{ or } j = M_2, \end{cases} \quad 0 \leqslant i \leqslant M_1,$$

where

$$c_2^{\beta} = \frac{-\beta^2 + \beta + 4}{24} \in \left(\frac{1}{12}, \frac{1}{6}\right), \quad c_2^{\gamma} = \frac{-\gamma^2 + \gamma + 4}{24} \in \left(\frac{1}{12}, \frac{1}{6}\right).$$

Introduce the following difference operators:

$$\delta_x^{\beta} u_{ij} = K_1 h_1^{-\beta} \sum_{k=0}^{i+1} \hat{w}_k^{(\beta)} u_{i-k+1,j} + K_2 h_1^{-\beta} \sum_{k=0}^{M_1-i+1} \hat{w}_k^{(\beta)} u_{i+k-1,j},$$

$$\delta_y^{\gamma} u_{ij} = K_3 h_2^{-\gamma} \sum_{k=0}^{j+1} \hat{w}_k^{(\gamma)} u_{i,j-k+1} + K_4 h_2^{-\gamma} \sum_{k=0}^{M_2-j+1} \hat{w}_k^{(\gamma)} u_{i,j+k-1}.$$

For any fixed $y \in [0,L_2]$, $t \in [0,T]$, define the function

$$\hat{v}(x,y,t) = \begin{cases} u(x,y,t), & x \in [0,L_1], \\ 0, & x \notin [0,L_1]. \end{cases}$$

For any fixed $x \in [0, L_1]$, $t \in [0, T]$, define the function

$$\hat{w}(x,y,t) = \begin{cases} u(x,y,t), & y \in [0, L_2], \\ 0, & y \notin [0, L_2]. \end{cases}$$

Suppose $\hat{v}(\cdot, y, t) \in \mathscr{C}^{4+\beta}(\mathcal{R})$, $\hat{w}(x, \cdot, t) \in \mathscr{C}^{4+\gamma}(\mathcal{R})$ and $u(x, y, \cdot) \in C^3[0, T]$.

Define the mesh functions

$$U_{ij}^n = u(x_i, y_j, t_n), \quad f_{ij}^n = f(x_i, y_j, t_n), \quad (i,j) \in \bar{\omega}, \ 0 \leqslant n \leqslant N.$$

4.4.1 Derivation of the difference scheme

Denote

$$\mathbf{D}_x^\beta u(x,y,t) = K_1\, {}_0\mathbf{D}_x^\beta u(x,y,t) + K_2\, {}_x\mathbf{D}_{L_1}^\beta u(x,y,t),$$
$$\mathbf{D}_y^\gamma u(x,y,t) = K_3\, {}_0\mathbf{D}_y^\gamma u(x,y,t) + K_4\, {}_y\mathbf{D}_{L_2}^\gamma u(x,y,t).$$

Considering equation (4.45) at the point (x_i, y_j, t_n), we have

$$u_t(x_i, y_j, t_n) = \mathbf{D}_x^\beta u(x_i, y_j, t_n) + \mathbf{D}_y^\gamma u(x_i, y_j, t_n) + f_{ij}^n,$$
$$(i,j) \in \bar{\omega}, \ 0 \leqslant n \leqslant N.$$

Taking an average on two adjacent time levels and performing the operator $\mathcal{B}_x^\beta \mathcal{B}_y^\gamma$ to both hand sides of the resultant equality lead to

$$\begin{aligned} &\frac{1}{2}\mathcal{B}_x^\beta \mathcal{B}_y^\gamma [u_t(x_i, y_j, t_n) + u_t(x_i, y_j, t_{n-1})] \\ &= \frac{1}{2}\mathcal{B}_y^\gamma [\mathcal{B}_x^\beta (\mathbf{D}_x^\beta U_{ij}^n + \mathbf{D}_x^\beta U_{ij}^{n-1})] + \frac{1}{2}\mathcal{B}_x^\beta [\mathcal{B}_y^\gamma (\mathbf{D}_y^\gamma U_{ij}^n + \mathbf{D}_y^\gamma U_{ij}^{n-1})] + \mathcal{B}_x^\beta \mathcal{B}_y^\gamma f_{ij}^{n-\frac{1}{2}}, \\ &\qquad (i,j) \in \omega, \ 1 \leqslant n \leqslant N, \end{aligned} \tag{4.48}$$

with $f_{ij}^{n-\frac{1}{2}} = \frac{1}{2}(f_{ij}^n + f_{ij}^{n-1})$.

For the time first-order derivative in (4.48), it follows from the Taylor's formula that

$$\frac{1}{2}[u_t(x_i, y_j, t_n) + u_t(x_i, y_j, t_{n-1})] = \frac{1}{\tau}(U_{ij}^n - U_{ij}^{n-1}) + O(\tau^2). \tag{4.49}$$

For the space-fractional derivatives in (4.48), it follows from Theorem 1.4.5 that

$$\mathcal{B}_x^\beta \mathbf{D}_x^\beta U_{ij}^n = K_1 h_1^{-\beta} \sum_{k=0}^{i+1} \hat{w}_k^{(\beta)} U_{i-k+1,j}^n + K_2 h_1^{-\beta} \sum_{k=0}^{M_1-i+1} \hat{w}_k^{(\beta)} U_{i+k-1,j}^n + O(h_1^4)$$

$$= \delta_x^\beta U_{ij}^n + O(h_1^4), \tag{4.50}$$

$$\mathcal{B}_y^\gamma \mathbf{D}_y^\gamma U_{ij}^n = K_3 h_2^{-\gamma} \sum_{k=0}^{j+1} \hat{w}_k^{(\gamma)} U_{i,j-k+1}^n + K_4 h_2^{-\gamma} \sum_{k=0}^{M_2-j+1} \hat{w}_k^{(\gamma)} U_{i,j+k-1}^n + O(h_2^4)$$
$$= \delta_y^\gamma U_{ij}^n + O(h_2^4). \tag{4.51}$$

Substituting (4.49)–(4.51) into (4.48), we get

$$\mathcal{B}_x^\beta \mathcal{B}_y^\gamma \delta_t U_{ij}^{n-\frac{1}{2}} = \mathcal{B}_y^\gamma \delta_x^\beta U_{ij}^{n-\frac{1}{2}} + \mathcal{B}_x^\beta \delta_y^\gamma U_{ij}^{n-\frac{1}{2}} + \mathcal{B}_x^\beta \mathcal{B}_y^\gamma f_{ij}^{n-\frac{1}{2}} + (r_4)_{ij}^{n-\frac{1}{2}},$$
$$(i,j) \in \omega,\ 1 \leqslant n \leqslant N, \tag{4.52}$$

and there is a positive constant c_4 such that

$$\left|(r_4)_{ij}^{n-\frac{1}{2}}\right| \leqslant c_4(\tau^2 + h_1^4 + h_2^4), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N.$$

Adding a small term $\frac{1}{4}\tau^2 \delta_x^\beta \delta_y^\gamma \delta_t U_{ij}^{n-\frac{1}{2}}$ to both hand sides of (4.52) arrives at

$$\mathcal{B}_x^\beta \mathcal{B}_y^\gamma \delta_t U_{ij}^{n-\frac{1}{2}} + \frac{\tau^2}{4} \delta_x^\beta \delta_y^\gamma \delta_t U_{ij}^{n-\frac{1}{2}} = \mathcal{B}_y^\gamma \delta_x^\beta U_{ij}^{n-\frac{1}{2}} + \mathcal{B}_x^\beta \delta_y^\gamma U_{ij}^{n-\frac{1}{2}}$$
$$+ \mathcal{B}_x^\beta \mathcal{B}_y^\gamma f_{ij}^{n-\frac{1}{2}} + (r_5)_{ij}^{n-\frac{1}{2}}, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \tag{4.53}$$

where

$$(r_5)_{ij}^{n-\frac{1}{2}} = (r_4)_{ij}^{n-\frac{1}{2}} + \frac{\tau^2}{4} \delta_x^\beta \delta_y^\gamma \delta_t U_{ij}^{n-\frac{1}{2}}$$

and there is a positive constant c_5 such that

$$\left|(r_5)_{ij}^{n-\frac{1}{2}}\right| \leqslant c_5(\tau^2 + h_1^4 + h_2^4), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \tag{4.54}$$

Noticing the initial-boundary value conditions (4.46)–(4.47), we have

$$\begin{cases} U_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (4.55) \\ U_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (4.56) \end{cases}$$

Neglecting the small term $(r_5)_{ij}^{n-\frac{1}{2}}$ in (4.53) and replacing the exact solution U_{ij}^n with its numerical one u_{ij}^n, we get a difference scheme for solving (4.45)–(4.47) as follows:

$$\begin{cases} \mathcal{B}_x^\beta \mathcal{B}_y^\gamma \delta_t u_{ij}^{n-\frac{1}{2}} + \frac{\tau^2}{4} \delta_x^\beta \delta_y^\gamma \delta_t u_{ij}^{n-\frac{1}{2}} = \mathcal{B}_y^\gamma \delta_x^\beta u_{ij}^{n-\frac{1}{2}} + \mathcal{B}_x^\beta \delta_y^\gamma u_{ij}^{n-\frac{1}{2}} + \mathcal{B}_x^\beta \mathcal{B}_y^\gamma f_{ij}^{n-\frac{1}{2}}, & \\ \qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (4.57) \\ u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (4.58) \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega, \quad 0 \leqslant n \leqslant N. & (4.59) \end{cases}$$

Reformulate equation (4.57) as

$$\begin{aligned}&\left(\mathcal{B}_x^{\beta}-\frac{\tau}{2}\delta_x^{\beta}\right)\left(\mathcal{B}_y^{\gamma}-\frac{\tau}{2}\delta_y^{\gamma}\right)u_{ij}^{n}\\&=\left(\mathcal{B}_x^{\beta}+\frac{\tau}{2}\delta_x^{\beta}\right)\left(\mathcal{B}_y^{\gamma}+\frac{\tau}{2}\delta_y^{\gamma}\right)u_{ij}^{n-1}+\tau\mathcal{B}_x^{\beta}\mathcal{B}_y^{\gamma}f_{ij}^{n-\frac{1}{2}}.\end{aligned}\tag{4.60}$$

Let

$$u_{ij}^{*}=\left(\mathcal{B}_y^{\gamma}-\frac{\tau}{2}\delta_y^{\gamma}\right)u_{ij}^{n},$$

then the difference scheme (4.57)–(4.59) can be decomposed into the following ADI form:

On the time level $t=t_n$ $(1\leqslant n\leqslant N)$, at first, for any fixed j $(1\leqslant j\leqslant M_2-1)$, solve the one-dimensional problem with respect to $\{u_{ij}^{*}\mid 0\leqslant i\leqslant M_1\}$ in x direction

$$\begin{cases}\left(\mathcal{B}_x^{\beta}-\frac{\tau}{2}\delta_x^{\beta}\right)u_{ij}^{*}=\left(\mathcal{B}_x^{\beta}+\frac{\tau}{2}\delta_x^{\beta}\right)\left(\mathcal{B}_y^{\gamma}+\frac{\tau}{2}\delta_y^{\gamma}\right)u_{ij}^{n-1}+\tau\mathcal{B}_x^{\beta}\mathcal{B}_y^{\gamma}f_{ij}^{n-\frac{1}{2}}, & 1\leqslant i\leqslant M_1-1,\\ u_{0j}^{*}=\left(\mathcal{B}_y^{\gamma}-\frac{\tau}{2}\delta_y^{\gamma}\right)u_{0j}^{n},\quad u_{M_1,j}^{*}=\left(\mathcal{B}_y^{\gamma}-\frac{\tau}{2}\delta_y^{\gamma}\right)u_{M_1,j}^{n}\end{cases}$$

to get the value of

$$\{u_{ij}^{*}\mid 1\leqslant i\leqslant M_1-1\};$$

Then, for any fixed i $(1\leqslant i\leqslant M_1-1)$, solve the one-dimensional problem with respect to $\{u_{ij}^{n}\mid 0\leqslant j\leqslant M_2\}$ in y direction

$$\begin{cases}\left(\mathcal{B}_y^{\gamma}-\frac{\tau}{2}\delta_y^{\gamma}\right)u_{ij}^{n}=u_{ij}^{*}, & 1\leqslant j\leqslant M_2-1,\\ u_{i0}^{n}=0,\quad u_{i,M_2}^{n}=0\end{cases}$$

to get the value of

$$\{u_{ij}^{n}\mid 1\leqslant j\leqslant M_2-1\}.$$

4.4.2 Three lemmas

In this subsection, three important lemmas will be listed so as to facilitate the analysis on the difference scheme presented above.

In view of the self-adjoint and positive definite operators $\mathcal{B}_x^{\beta}$ and $\mathcal{B}_y^{\gamma}$, they can be performed the square root decomposition, that is, there are two operators $\mathcal{Q}_x^{\beta}$ and $\mathcal{Q}_y^{\gamma}$, such that

$$\mathcal{B}_x^{\beta}=(\mathcal{Q}_x^{\beta})^2,\quad \mathcal{B}_y^{\gamma}=(\mathcal{Q}_y^{\gamma})^2.$$

It is easy to see that $\mathcal{B}_x^\beta$ and $\mathcal{B}_y^\gamma$ are commutable and so are $\mathcal{Q}_x^\beta$ and $\mathcal{Q}_y^\gamma$.

Next, three useful lemmas will be introduced.

Lemma 4.4.1. *For any mesh function $v \in \mathring{\mathcal{V}}_h$, it holds*

$$\frac{1}{3}\|v\| \leqslant \|\mathcal{Q}_x^\beta \mathcal{Q}_y^\gamma v\|, \quad \|(\mathcal{Q}_x^\beta)^{-1}(\mathcal{Q}_y^\gamma)^{-1} v\| \leqslant 3\|v\|.$$

Proof. It follows from the definitions of the operators $\mathcal{B}_x^\beta$, $\mathcal{B}_y^\gamma$ and Lemma 4.3.1 that

$$\frac{1}{3}\|v\|^2 \leqslant (\mathcal{B}_x^\beta v, v) \leqslant \|v\|^2, \quad \frac{1}{3}\|v\|^2 \leqslant (\mathcal{B}_y^\gamma v, v) \leqslant \|v\|^2.$$

On the one hand, we have

$$\begin{aligned}(\mathcal{B}_x^\beta \mathcal{B}_y^\gamma v, v) &= ((\mathcal{Q}_x^\beta)^2 \mathcal{B}_y^\gamma v, v) = (\mathcal{Q}_x^\beta \mathcal{B}_y^\gamma v, \mathcal{Q}_x^\beta v) \\ &= (\mathcal{B}_y^\gamma \mathcal{Q}_x^\beta v, \mathcal{Q}_x^\beta v) \geqslant \frac{1}{3}(\mathcal{Q}_x^\beta v, \mathcal{Q}_x^\beta v) = \frac{1}{3}(\mathcal{B}_x^\beta v, v) \geqslant \frac{1}{9}\|v\|^2.\end{aligned} \tag{4.61}$$

On the other hand, it holds

$$\begin{aligned}(\mathcal{B}_x^\beta \mathcal{B}_y^\gamma v, v) &= ((\mathcal{Q}_x^\beta)^2 (\mathcal{Q}_y^\gamma)^2 v, v) \\ &= (\mathcal{Q}_x^\beta \mathcal{Q}_y^\gamma v, \mathcal{Q}_x^\beta \mathcal{Q}_y^\gamma v) = \|\mathcal{Q}_x^\beta \mathcal{Q}_y^\gamma v\|^2.\end{aligned} \tag{4.62}$$

The combination of (4.61) and (4.62) gives

$$\frac{1}{3}\|v\| \leqslant \|\mathcal{Q}_x^\beta \mathcal{Q}_y^\gamma v\|. \tag{4.63}$$

Moreover, it follows from (4.63) that

$$\frac{1}{3}\|(\mathcal{Q}_x^\beta)^{-1}(\mathcal{Q}_y^\gamma)^{-1} v\| \leqslant \|\mathcal{Q}_x^\beta \mathcal{Q}_y^\gamma (\mathcal{Q}_x^\beta)^{-1}(\mathcal{Q}_y^\gamma)^{-1} v\| = \|v\|,$$

which implies the second inequality in this lemma. The proof ends. □

Lemma 4.4.2. *For any mesh function $v \in \mathring{\mathcal{V}}_h$, it holds*

$$(v, \delta_x^\beta v) \leqslant 0, \quad (v, \delta_y^\gamma v) \leqslant 0.$$

The proof can be proceeded similarly to that for Lemma 4.2.1 and the details are omitted here.

Lemma 4.4.3. *For any mesh function $v \in \mathring{\mathcal{V}}_h$, it holds*

$$\left\|\left(\mathcal{Q}_x^\beta - \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\right)v\right\|^2 \geqslant \|\mathcal{Q}_x^\beta v\|^2 + \frac{\tau^2}{4}\|(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta v\|^2,$$

$$\left\|\left(\mathcal{Q}_x^\beta + \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\right)v\right\|^2 \leqslant \|\mathcal{Q}_x^\beta v\|^2 + \frac{\tau^2}{4}\|(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta v\|^2,$$

$$\left\|\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)v\right\|^2 \geqslant \|\mathcal{Q}_y^\gamma v\|^2 + \frac{\tau^2}{4}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma v\|^2,$$
$$\left\|\left(\mathcal{Q}_y^\gamma + \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)v\right\|^2 \leqslant \|\mathcal{Q}_y^\gamma v\|^2 + \frac{\tau^2}{4}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma v\|^2.$$

Proof. By Lemma 4.4.2, some calculations give

$$\begin{aligned}
&\left\|\left(\mathcal{Q}_x^\beta - \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\right)v\right\|^2 \\
&= \left(\left(\mathcal{Q}_x^\beta - \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\right)v, \left(\mathcal{Q}_x^\beta - \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\right)v\right) \\
&= (\mathcal{Q}_x^\beta v, \mathcal{Q}_x^\beta v) + \frac{\tau^2}{4}((\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta v, (\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta v) - \tau(\mathcal{Q}_x^\beta v, (\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta v) \\
&= \|\mathcal{Q}_x^\beta v\|^2 + \frac{\tau^2}{4}\|(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta v\|^2 - \tau(v, \delta_x^\beta v) \\
&\geqslant \|\mathcal{Q}_x^\beta v\|^2 + \frac{\tau^2}{4}\|(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta v\|^2.
\end{aligned}$$

Other three inequalities in this lemma can be proved similarly. The proof ends. □

The following part will focus on the analysis for the difference scheme (4.57)–(4.59).

4.4.3 Solvability of the difference scheme

Theorem 4.4.1. *The difference scheme* (4.57)–(4.59) *is uniquely solvable.*

Proof. Let

$$u^n = \{u_{ij}^n \mid (i,j) \in \bar{\omega}\}.$$

The value of u^0 is obviously determined by (4.58)–(4.59).

Now assume that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then we can obtain the linear system in the unknown u^n from (4.57) and (4.59). To show its unique solvability, it is sufficient to verify that the corresponding homogeneous one

$$\begin{cases} \left(\mathcal{B}_x^\beta - \frac{\tau}{2}\delta_x^\beta\right)\left(\mathcal{B}_y^\gamma - \frac{\tau}{2}\delta_y^\gamma\right)u_{ij}^n = 0, \quad (i,j) \in \omega, & (4.64) \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega & (4.65) \end{cases}$$

has only the trivial solution.

To this end, performing the operator $(\mathcal{Q}_x^\beta)^{-1}(\mathcal{Q}_y^\gamma)^{-1}$ to both hand sides of (4.64) yields

$$\left(\mathcal{Q}_x^\beta - \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\right)\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u_{ij}^n = 0, \quad (i,j) \in \omega.$$

From Lemmas 4.4.1, 4.4.2 and 4.4.3, we obtain

$$\begin{aligned}
0 &= \left\|\left(\mathcal{Q}_x^\beta - \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\right)\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u^n\right\|^2 \\
&= \left\|\mathcal{Q}_x^\beta\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u^n\right\|^2 + \frac{\tau^2}{4}\left\|(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u^n\right\|^2 \\
&\quad - \tau\left(\mathcal{Q}_x^\beta\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u^n, (\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u^n\right) \\
&= \left\|\mathcal{Q}_x^\beta\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u^n\right\|^2 + \frac{\tau^2}{4}\left\|(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u^n\right\|^2 \\
&\quad - \tau\left(\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u^n, \delta_x^\beta\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u^n\right) \\
&\geqslant \left\|\mathcal{Q}_x^\beta\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u^n\right\|^2 + \frac{\tau^2}{4}\left\|(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)u^n\right\|^2 \\
&= \left\|\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)\mathcal{Q}_x^\beta u^n\right\|^2 + \frac{\tau^2}{4}\left\|\left(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\right)(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^n\right\|^2 \\
&\geqslant \|\mathcal{Q}_y^\gamma\mathcal{Q}_x^\beta u^n\|^2 + \frac{\tau^2}{4}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\mathcal{Q}_x^\beta u^n\|^2 \\
&\quad + \frac{\tau^2}{4}\left(\|\mathcal{Q}_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^n\|^2 + \frac{\tau^2}{4}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^n\|^2\right) \qquad (4.66) \\
&\geqslant \|\mathcal{Q}_y^\gamma\mathcal{Q}_x^\beta u^n\|^2 \geqslant \frac{1}{9}\|u^n\|^2,
\end{aligned}$$

thus, $\|u^n\| = 0$. We conclude that $u^n = 0$ from (4.65).

By the principle of induction, the difference scheme (4.57)–(4.59) is uniquely solvable. The proof ends. □

4.4.4 Stability of the difference scheme

Theorem 4.4.2. *Suppose $\{u_{ij}^n \mid (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N\}$ is the solution of the difference scheme (4.57)–(4.59) and denote $s_{ij}^{n-\frac{1}{2}} = \mathcal{B}_x^\beta\mathcal{B}_y^\gamma f_{ij}^{n-\frac{1}{2}}$, then it holds*

$$\begin{aligned}
\|u^n\| \leqslant 3\Big(&\|\mathcal{Q}_y^\gamma\mathcal{Q}_x^\beta u^0\| + \frac{\tau}{2}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\mathcal{Q}_x^\beta u^0\| + \frac{\tau}{2}\|\mathcal{Q}_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^0\| \\
&+ \frac{\tau^2}{4}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^0\|\Big) + 9\tau\sum_{m=1}^{n}\|s^{m-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N,
\end{aligned}$$

where

$$\|s^{m-\frac{1}{2}}\| = \sqrt{h\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(s_{ij}^{m-\frac{1}{2}})^2}.$$

Proof. Noticing that equation (4.57) can be rewritten as (4.60) and performing the operator $(\mathcal{Q}_x^\beta)^{-1}(\mathcal{Q}_y^\gamma)^{-1}$ to both hand sides of (4.60) arrive at

$$\begin{aligned}&\Big(\mathcal{Q}_x^\beta - \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\Big)\Big(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\Big)u_{ij}^n\\ &= \Big(\mathcal{Q}_x^\beta + \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\Big)\Big(\mathcal{Q}_y^\gamma + \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\Big)u_{ij}^{n-1}\\ &\quad + \tau(\mathcal{Q}_x^\beta)^{-1}(\mathcal{Q}_y^\gamma)^{-1}s_{ij}^{n-\frac{1}{2}}, \quad (i,j)\in\omega,\ 1\leqslant n\leqslant N.\end{aligned} \tag{4.67}$$

Taking the norm on both hand sides of (4.67), it follows from the triangle inequality that

$$\begin{aligned}&\Big\|\Big(\mathcal{Q}_x^\beta - \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\Big)\Big(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\Big)u^n\Big\|\\ &\leqslant \Big\|\Big(\mathcal{Q}_x^\beta + \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\Big)\Big(\mathcal{Q}_y^\gamma + \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\Big)u^{n-1}\Big\|\\ &\quad + \tau\big\|(\mathcal{Q}_x^\beta)^{-1}(\mathcal{Q}_y^\gamma)^{-1}s^{n-\frac{1}{2}}\big\|, \quad 1\leqslant n\leqslant N.\end{aligned} \tag{4.68}$$

In analogy to the derivation of (4.66), we have

$$\begin{aligned}&\Big\|\Big(\mathcal{Q}_x^\beta - \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\Big)\Big(\mathcal{Q}_y^\gamma - \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\Big)u^n\Big\|^2\\ &\geqslant \|\mathcal{Q}_y^\gamma\mathcal{Q}_x^\beta u^n\|^2 + \frac{\tau^2}{4}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\mathcal{Q}_x^\beta u^n\|^2 + \frac{\tau^2}{4}\|\mathcal{Q}_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^n\|^2\\ &\quad + \frac{\tau^4}{16}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^n\|^2\end{aligned} \tag{4.69}$$

and

$$\begin{aligned}&\Big\|\Big(\mathcal{Q}_x^\beta + \frac{\tau}{2}(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta\Big)\Big(\mathcal{Q}_y^\gamma + \frac{\tau}{2}(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\Big)u^{n-1}\Big\|^2\\ &\leqslant \|\mathcal{Q}_y^\gamma\mathcal{Q}_x^\beta u^{n-1}\|^2 + \frac{\tau^2}{4}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\mathcal{Q}_x^\beta u^{n-1}\|^2 + \frac{\tau^2}{4}\|\mathcal{Q}_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^{n-1}\|^2\\ &\quad + \frac{\tau^4}{16}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^{n-1}\|^2.\end{aligned} \tag{4.70}$$

Denote

$$\begin{aligned}E^n = \Big(&\|\mathcal{Q}_y^\gamma\mathcal{Q}_x^\beta u^n\|^2 + \frac{\tau^2}{4}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma\mathcal{Q}_x^\beta u^n\|^2 + \frac{\tau^2}{4}\|\mathcal{Q}_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^n\|^2\\ &+ \frac{\tau^4}{16}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^n\|^2\Big)^{1/2}, \quad 0\leqslant n\leqslant N.\end{aligned}$$

It follows from (4.68)–(4.70) and Lemma 4.4.1 that

$$E^n \leqslant E^{n-1} + \tau\big\|(\mathcal{Q}_x^\beta)^{-1}(\mathcal{Q}_y^\gamma)^{-1}s^{n-\frac{1}{2}}\big\| \leqslant E^{n-1} + 3\tau\|s^{n-\frac{1}{2}}\|, \quad 1\leqslant n\leqslant N.$$

Applying the recursive process leads to

$$E^n \leqslant E^0 + 3\tau \sum_{m=1}^{n} \|s^{m-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N.$$

Noticing

$$E^n \geqslant \|\mathcal{Q}_y^\gamma \mathcal{Q}_x^\beta u^n\| \geqslant \frac{1}{3}\|u^n\|,$$

further it follows

$$\begin{aligned}\|u^n\| &\leqslant 3E^0 + 9\tau \sum_{m=1}^{n} \|s^{m-\frac{1}{2}}\| \\ &\leqslant 3\Big(\|\mathcal{Q}_y^\gamma \mathcal{Q}_x^\beta u^0\| + \frac{\tau}{2}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma \mathcal{Q}_x^\beta u^0\| + \frac{\tau}{2}\|\mathcal{Q}_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^0\| \\ &\quad + \frac{\tau^2}{4}\|(\mathcal{Q}_y^\gamma)^{-1}\delta_y^\gamma(\mathcal{Q}_x^\beta)^{-1}\delta_x^\beta u^0\|\Big) + 9\tau \sum_{m=1}^{n} \|s^{m-\frac{1}{2}}\|, \quad 1 \leqslant n \leqslant N,\end{aligned}$$

which is exactly the desired result. The proof ends. □

4.4.5 Convergence of the difference scheme

Theorem 4.4.3. *Suppose* $\{U_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *and* $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (4.45)–(4.47) *and the difference scheme* (4.57)–(4.59), *respectively. Let*

$$e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N,$$

then it holds

$$\|e^n\| \leqslant 9T\sqrt{L_1L_2}\, c_5(\tau^2 + h_1^4 + h_2^4), \quad 1 \leqslant n \leqslant N.$$

Proof. Subtracting (4.57)–(4.59) from (4.53), (4.55)–(4.56), respectively, will arrive at the system of error equations as follows:

$$\begin{cases} \mathcal{B}_x^\beta \mathcal{B}_y^\gamma \delta_t e_{ij}^{n-\frac{1}{2}} + \dfrac{\tau^2}{4}\delta_x^\beta \delta_y^\gamma \delta_t e_{ij}^{n-\frac{1}{2}} = \mathcal{B}_y^\gamma \delta_x^\beta e_{ij}^{n-\frac{1}{2}} + \mathcal{B}_x^\beta \delta_y^\gamma e_{ij}^{n-\frac{1}{2}} + (r_5)_{ij}^{n-\frac{1}{2}}, \\ \hfill (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ e_{ij}^0 = 0, \quad (i,j) \in \omega, \\ e_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (4.54), the application of Theorem 4.4.2 immediately yields

$$\|e^n\| \leqslant 9\tau \sum_{m=1}^{n} \|(r_5)^{m-\frac{1}{2}}\| \leqslant 9T\sqrt{L_1L_2}\, c_5(\tau^2 + h_1^4 + h_2^4), \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

4.5 Supplementary remarks and discussions

1. This chapter mainly focused on the finite difference methods for solving the initial-boundary value problems of space-fractional partial differential equations in R-L type. For 1D problem (4.1)–(4.3), three difference schemes of order one in both time and space, order two in both time and space, together with order two in time and four in space, respectively, were built. The unique solvability, stability and convergence of each scheme were analyzed. For 2D problem (4.45)–(4.47), an ADI difference scheme of order two in time and four in space was mainly addressed along with the analysis on its unique solvability, stability and convergence.

2. For the R-L fractional derivative in the space-fractional partial differential equations, Meerschaert and Tadjeran pointed out that the difference scheme using the standard G-L formula to approximate the R-L fractional derivative was unstable in [58, 85], and proposed the shifted G-L formula for the approximation of the R-L fractional derivative to derive the difference scheme.

3. For the R-L fractional derivative in the space-fractional differential equations, Tian et al.[88] presented a two-term WSGL formula to establish the difference method of order two in both time and space. Besides, a three-term WSGL formula was also derived. Based on the work[88], a series of works have been done by this group; please see [7, 117] and so on.

4. The numerical solution of the following one-dimensional space-fractional partial differential equation:

$$u_t(x,t) = \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + f(x,t)$$

was studied in [4], where a Crank–Nicolson difference scheme of order two in both time and space was developed with the help of the central difference quotient formula (Lemma 1.5.1) to approximate the Riesz fractional derivative. In [64, 102], the numerical methods for solving the diffusion and advection-diffusion equations with Riesz fractional derivatives were also discussed.

5. A fourth-order numerical differentiation formula (1.54) to approximate the weighted value of Riesz fractional derivatives at three points was proposed by Zhao et al.[116]. On this basis, a higher-order difference method was investigated for solving the nonlinear space-fractional Schrödinger equation in the Riesz derivative type. Ding et al.[14, 15] derived a different higher-order numerical differentiation formula to approximate the Riesz fractional derivative and applied it to the numerical solution of the space-fractional differential equation in the Riesz derivative type.

6. For the difference schemes to solve the space-fractional differential equations, Wang et al.[91, 95] and Lei et al.[43] developed some fast methods in view of the special structures of difference schemes.

7. Wang et al.[97, 98] presented implicit conservative difference schemes for the space fractional nonlinear Schrödinger equations. Wang and Huang[92, 93], Hao and

Sun[34], He and Pan[36] have studied the difference methods for the fractional Ginzburg–Landau equation.

Exercises 4

4.1 For the problem (4.1)–(4.3), construct the following explicit difference scheme:

$$\begin{cases} \dfrac{1}{\tau}(u_i^{n+1} - u_i^n) = K_1\, h^{-\beta} \displaystyle\sum_{k=0}^{i+1} g_k^{(\beta)} u_{i-k+1}^n + K_2\, h^{-\beta} \sum_{k=0}^{M-i+1} g_k^{(\beta)} u_{i+k-1}^n + f_i^n, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 0 \leqslant n \leqslant N-1, \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\tilde{u}(x,t)$ like that in Section 4.1 and suppose $\tilde{u}(\cdot,t) \in \mathscr{C}^{1+\beta}(\mathcal{R})$. For this difference scheme, try to

(1) analyze the truncation error;
(2) show the stability with respect to the initial value φ and the function f when $(K_1 + K_2)\frac{\beta\tau}{h^\beta} \leqslant 1$;
(3) show the convergence when $(K_1 + K_2)\frac{\beta\tau}{h^\beta} \leqslant 1$, and derive the error expression.

4.2 For the problem (4.1)–(4.3), construct the following implicit difference scheme:

$$\begin{cases} \dfrac{1}{\tau}(u_i^n - u_i^{n-1}) = K_1\, h^{-\beta} \displaystyle\sum_{k=0}^{i+1} \widetilde{w}_k^{(\beta)} u_{i-k+1}^n + K_2\, h^{-\beta} \sum_{k=0}^{M-i+1} \widetilde{w}_k^{(\beta)} u_{i+k-1}^n + f_i^n, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\tilde{u}(x,t)$ like that in Section 4.1 and suppose $\tilde{u}(\cdot,t) \in \mathscr{C}^{2+\beta}(\mathcal{R})$. For this difference scheme, try to

(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the initial value φ and the function f;
(4) show the convergence and derive the error expression.

4.3 For the problem (4.1)–(4.3), construct the following difference scheme:

$$\begin{cases} \mathcal{B}^{(\beta)} \dfrac{u_i^n - u_i^{n-1}}{\tau} = K_1\, h^{-\beta} \displaystyle\sum_{k=0}^{i+1} \hat{w}_k^{(\beta)} u_{i-k+1}^n + K_2\, h^{-\beta} \sum_{k=0}^{M-i+1} \hat{w}_k^{(\beta)} u_{i+k-1}^n + \mathcal{B}^{(\beta)} f_i^n, \\ \qquad\qquad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = 0,\ u_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\tilde{u}(x,t)$ like that in Section 4.1 and suppose $\tilde{u}(\cdot,t) \in \mathscr{C}^{4+\beta}(\mathcal{R})$.

For this difference scheme, try to
(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the initial value φ and the function f;
(4) show the convergence and derive the error expression.

4.4 For the problem (4.45)–(4.47), construct the following difference scheme:

$$\begin{cases} \delta_t u_{ij}^{n-\frac{1}{2}} + \dfrac{\tau^2}{4}\delta_x^\beta \delta_y^\gamma \delta_t u_{ij}^{n-\frac{1}{2}} = \delta_x^\beta u_{ij}^{n-\frac{1}{2}} + \delta_y^\gamma u_{ij}^{n-\frac{1}{2}} + f_{ij}^{n-\frac{1}{2}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

Define the functions $\hat{v}(x,y,t)$, $\hat{w}(x,y,t)$ like those in Section 4.4 and suppose $\hat{v}(\cdot,y,t) \in \mathscr{C}^{2+\beta}(\mathcal{R})$, $\hat{w}(x,\cdot,t) \in \mathscr{C}^{2+\gamma}(\mathcal{R})$.
For this difference scheme, try to
(1) analyze the truncation error;
(2) show the unique solvability;
(3) write the ADI form;
(4) show the stability with respect to the initial value φ and the function f;
(5) show the convergence and derive the error expression.

4.5 For the problem

$$\begin{cases} u_t(x,t) = \dfrac{\partial^\beta u(x,t)}{\partial |x|^\beta} + f(x,t), \quad 0 < x < L,\ 0 < t \leqslant T, & (4.71) \\ u(x,0) = \varphi(x), \quad 0 < x < L, & (4.72) \\ u(0,t) = 0, \quad u(L,t) = 0, \quad 0 \leqslant t \leqslant T, & (4.73) \end{cases}$$

where

$$\frac{\partial^\beta u(x,t)}{\partial |x|^\beta} = -\Psi_\beta\big({}_0\mathbf{D}_x^\beta u(x,t) + {}_x\mathbf{D}_L^\beta u(x,t)\big), \quad \Psi_\beta = \frac{1}{2\cos(\frac{\beta\pi}{2})},$$

$${}_0\mathbf{D}_x^\beta u(x,t) = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_0^x \frac{u(\xi,t)}{(x-\xi)^{\beta-1}}\mathrm{d}\xi,$$

$${}_x\mathbf{D}_L^\beta u(x,t) = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_x^L \frac{u(\xi,t)}{(\xi-x)^{\beta-1}}\mathrm{d}\xi,$$

the functions f, φ are given and $\varphi(0) = \varphi(L) = 0$.

Construct the following difference scheme:

$$\begin{cases} \dfrac{u_i^n - u_i^{n-1}}{\tau} = -h^{-\beta} \displaystyle\sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} u_{i-k}^n + f_i^n, & 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N, \end{cases}$$

where the coefficient $\{\hat{g}_k^{(\beta)}\}$ is defined by (1.48).

Define the function $\tilde{u}(x,t)$ like that in Section 4.1 and suppose $\tilde{u}(\cdot,t) \in \mathscr{C}^{2+\beta}(\mathcal{R})$.

For this difference scheme, try to

(1) analyze the truncation error;

(2) show the unique solvability;

(3) show the stability with respect to the initial value φ and the function f;

(4) show the convergence and derive the error expression.

4.6 For the problem (4.71)–(4.73), construct the following difference scheme:

$$\begin{cases} \dfrac{u_i^n - u_i^{n-1}}{\tau} = -h^{-\beta} \displaystyle\sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} u_{i-k}^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}}, & 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N, \end{cases}$$

where the coefficient $\{\hat{g}_k^{(\beta)}\}$ is defined by (1.48).

Define the function $\tilde{u}(x,t)$ like that in Section 4.1 and suppose $\tilde{u}(\cdot,t) \in \mathscr{C}^{2+\beta}(\mathcal{R})$.

For this difference scheme, try to

(1) analyze the truncation error;

(2) show the unique solvability;

(3) show the stability with respect to the initial value φ and the function f:

(4) show the convergence and derive the error expression.

5 Difference methods for the time-space-fractional differential equations

In this chapter, the finite difference methods for a class of time-space-fractional differential equations (Bloch–Torrey equations) will be considered. Many applications for this kind of equations can be found. It has been successfully applied to describe the diffusion image of human brain tissues and provides new insights into further investigations of tissue structures and the microenvironment. The Bloch–Torrey equation consists of both the time-fractional derivative (Caputo derivative) and the space-fractional derivative (Riesz derivative). In this chapter, for 1D problem, the method of order two in both time and space and another method of order two in time and four in space will be successively discussed. For 2D problem, the method of order two in both time and space and another method of order two in time and four in space will be developed in sequel. The unique solvability, stability and convergence of each scheme will be analyzed. The whole chapter is divided into 5 sections.

5.1 The second-order method in space for 1D problem

In this section, the following 1D initial-boundary value problem of time-space-fractional differential equation

$$\begin{cases} {}_0^C D_t^{\alpha} u(x,t) = \dfrac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + f(x,t), \quad 0 < x < L,\ 0 < t \leqslant T, & (5.1) \\ u(x,0) = \varphi(x), \quad 0 < x < L, & (5.2) \\ u(0,t) = 0, \quad u(L,t) = 0, \quad 0 \leqslant t \leqslant T & (5.3) \end{cases}$$

will be considered, where $\alpha \in (0,1)$, $\beta \in (1,2)$, ${}_0^C D_t^{\alpha} u(x,t)$ is the α-th order Caputo fractional derivative, $\frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}}$ is the β-th order Riesz fractional derivative, that is,

$$\frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} = -\Psi_{\beta}\big({}_0\mathbf{D}_x^{\beta} u(x,t) + {}_x\mathbf{D}_L^{\beta} u(x,t)\big), \quad \Psi_{\beta} = \frac{1}{2\cos(\frac{\beta\pi}{2})},$$

$${}_0\mathbf{D}_x^{\beta} u(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_0^x \frac{u(\xi,t)}{(x-\xi)^{\beta-1}} \mathrm{d}\xi,$$

$${}_x\mathbf{D}_L^{\beta} u(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_x^L \frac{u(\xi,t)}{(\xi-x)^{\beta-1}} \mathrm{d}\xi,$$

the functions f, φ are given and $\varphi(0) = \varphi(L) = 0$.

Take the same mesh partition and notations as those in Section 2.1. In addition, let $\sigma = 1 - \frac{\alpha}{2}$, $t_{n-1+\sigma} = t_{n-1} + \sigma\tau$, $s = \tau^{\alpha}\Gamma(2-\alpha)$.

https://doi.org/10.1515/9783110616064-005

Define the mesh functions

$$U_i^n = u(x_i, t_n), \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N;$$
$$f_i^{n-1+\sigma} = f(x_i, t_{n-1+\sigma}), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.$$

For any fixed $t \in [0, T]$, define the function

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & 0 \leqslant x \leqslant L, \\ 0, & x \notin [0, L]. \end{cases}$$

Suppose $u(x,\cdot) \in C^3[0,T]$ and $\tilde{u}(\cdot,t) \in \mathscr{C}^{2+\beta}(\mathcal{R})$.

5.1.1 Derivation of the difference scheme

Considering equation (5.1) at the point $(x_i, t_{n-1+\sigma})$, we have

$${}_0^C D_t^\alpha u(x_i, t_{n-1+\sigma}) = \frac{\partial^\beta u(x_i, t_{n-1+\sigma})}{\partial |x|^\beta} + f_i^{n-1+\sigma}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{5.4}$$

For the Caputo fractional derivative in equation (5.4), using L2-1_σ approximation (1.81), it follows from Theorem 1.6.4 that

$${}_0^C D_t^\alpha u(x_i, t_{n-1+\sigma}) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (U_i^{n-k} - U_i^{n-k-1}) + O(\tau^{3-\alpha}). \tag{5.5}$$

For the Riesz fractional derivative in equation (5.4), the result of a linear interpolation approximation reads

$$\frac{\partial^\beta u(x_i, t_{n-1+\sigma})}{\partial |x|^\beta} = \sigma \frac{\partial^\beta u(x_i, t_n)}{\partial |x|^\beta} + (1-\sigma) \frac{\partial^\beta u(x_i, t_{n-1})}{\partial |x|^\beta} + O(\tau^2). \tag{5.6}$$

Moreover, it follows from Theorem 1.5.1 that

$$\frac{\partial^\beta u(x_i, t_n)}{\partial |x|^\beta} = -h^{-\beta} \sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} U_{i-k}^n + O(h^2). \tag{5.7}$$

The combination of (5.6) and (5.7) arrives at

$$\frac{\partial^\beta u(x_i, t_{n-1+\sigma})}{\partial |x|^\beta} = -h^{-\beta} \sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} [\sigma U_{i-k}^n + (1-\sigma) U_{i-k}^{n-1}] + O(\tau^2 + h^2). \tag{5.8}$$

Substituting (5.5) and (5.8) into (5.4), we obtain

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (U_i^{n-k} - U_i^{n-k-1})$$

$$= -h^{-\beta} \sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} [\sigma U_{i-k}^n + (1-\sigma) U_{i-k}^{n-1}] + f_i^{n-1+\sigma} + (r_1)_i^n,$$

$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{5.9}$$

and there is a positive constant c_1 such that

$$|(r_1)_i^n| \leqslant c_1(\tau^2 + h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{5.10}$$

Noticing the initial-boundary value conditions (5.2)–(5.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (5.11) \\ U_0^n = 0, \quad U_M^n = 0, \quad 0 \leqslant n \leqslant N. & (5.12) \end{cases}$$

Omitting the small term $(r_1)_i^n$ in (5.9) and replacing the exact solution U_i^n with its numerical one u_i^n, a difference scheme for solving (5.1)–(5.3) can be produced as

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (u_i^{n-k} - u_i^{n-k-1}) \\ = -h^{-\beta} \sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} [\sigma u_{i-k}^n + (1-\sigma) u_{i-k}^{n-1}] + f_i^{n-1+\sigma}, \\ \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (5.13) \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (5.14) \\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N. & (5.15) \end{cases}$$

Next, some analyses on the difference scheme (5.13)–(5.15) will be carried out.

5.1.2 Solvability of the difference scheme

Theorem 5.1.1. *The difference scheme* (5.13)–(5.15) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The value of u^0 is obviously determined by (5.14)–(5.15).

Now assume that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n can be obtained from (5.13) and (5.15). To show its unique solvability, it suffices to verify that the corresponding homogeneous one

$$\begin{cases} \dfrac{1}{s} c_0^{(n,\alpha)} u_i^n = -\sigma h^{-\beta} \sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} u_{i-k}^n, \quad 1 \leqslant i \leqslant M-1, & (5.16) \\ u_0^n = 0, \quad u_M^n = 0 & (5.17) \end{cases}$$

has only the trivial solution.

Rewrite (5.16) as

$$\left[\frac{1}{s}c_0^{(n,\alpha)} + \sigma h^{-\beta}\hat{g}_0^{(\beta)}\right]u_i^n = \sigma h^{-\beta}\sum_{\substack{k=i-M\\k\neq 0}}^{i}(-\hat{g}_k^{(\beta)})u_{i-k}^n, \quad 1 \leqslant i \leqslant M-1. \tag{5.18}$$

Suppose $\|u^n\|_\infty = |u_{i_n}^n|$, where $i_n \in \{1, 2, \ldots, M-1\}$. Letting $i = i_n$ in (5.18) and taking the absolute value on both hand sides of the equality, it follows from Lemma 1.5.2 that

$$\begin{aligned}
&\left[\frac{1}{s}c_0^{(n,\alpha)} + \sigma h^{-\beta}\hat{g}_0^{(\beta)}\right]\|u^n\|_\infty \\
&\leqslant \sigma h^{-\beta}\sum_{\substack{k=i_n-M\\k\neq 0}}^{i_n}(-\hat{g}_k^{(\beta)})|u_{i_n-k}^n| \\
&\leqslant \sigma h^{-\beta}\sum_{\substack{k=i_n-M\\k\neq 0}}^{i_n}(-\hat{g}_k^{(\beta)})\|u^n\|_\infty \\
&\leqslant \sigma h^{-\beta}\hat{g}_0^{(\beta)}\|u^n\|_\infty.
\end{aligned}$$

Hence, $\|u^n\|_\infty = 0$, which implies that (5.16)–(5.17) has only the trivial solution.

By the principle of induction, the theorem is true. The proof ends. □

5.1.3 An important lemma

In this subsection, we present an important lemma.

Lemma 5.1.1. *For any mesh function* $v = (v_0, v_1, \ldots, v_M) \in \mathring{\mathcal{U}}_h$, *it holds*

$$-h^{-\beta}h\sum_{i=1}^{M-1}\left(\sum_{k=i-M}^{i}\hat{g}_k^{(\beta)}v_{i-k}\right)v_i \leqslant -c_*^{(\beta)}(2L)^{-\beta}h\sum_{i=1}^{M-1}v_i^2,$$

where $c_*^{(\beta)}$ *is defined by* (1.55), $1 < \beta < 2$.

Proof. From Lemma 1.5.2 and Lemma 1.5.3, it is easy to see that

$$\hat{g}_k^{(\beta)} < 0, \quad |k| \geqslant 1; \qquad \sum_{k=-\infty}^{\infty}\hat{g}_k^{(\beta)} = 0; \qquad \sum_{|k|=M}^{\infty}(-\hat{g}_k^{(\beta)}) \geqslant \frac{c_*^{(\beta)}}{(M+1)^\beta}, \quad M \geqslant 1,$$

therefore,

$$A \equiv -h^{-\beta}h\sum_{i=1}^{M-1}\left(\sum_{k=i-M}^{i}\hat{g}_k^{(\beta)}v_{i-k}\right)v_i$$

$$
\begin{aligned}
&= h^{-\beta}\left[h\sum_{i=1}^{M-1}(-\hat{g}_0^{(\beta)})v_i^2 + h\sum_{i=1}^{M-1}\sum_{\substack{k=i-M\\k\neq 0}}^{i}(-\hat{g}_k^{(\beta)})v_{i-k}v_i\right]\\
&\leqslant h^{-\beta}\left[h\sum_{i=1}^{M-1}(-\hat{g}_0^{(\beta)})v_i^2 + \frac{1}{2}h\sum_{i=1}^{M-1}\sum_{\substack{k=i-M\\k\neq 0}}^{i}(-\hat{g}_k^{(\beta)})(v_{i-k}^2+v_i^2)\right]\\
&= h^{-\beta}\left[h\sum_{i=1}^{M-1}(-\hat{g}_0^{(\beta)})v_i^2 + \frac{1}{2}h\sum_{i=1}^{M-1}\sum_{\substack{k=i-M\\k\neq 0}}^{i}(-\hat{g}_k^{(\beta)})v_i^2 + \frac{1}{2}h\sum_{i=1}^{M-1}\sum_{\substack{k=i-M\\k\neq 0}}^{i}(-\hat{g}_k^{(\beta)})v_{i-k}^2\right].
\end{aligned}
\tag{5.19}
$$

For the second term on the right-hand side in (5.19), we have

$$
\frac{1}{2}h\sum_{i=1}^{M-1}\sum_{\substack{k=i-M\\k\neq 0}}^{i}(-\hat{g}_k^{(\beta)})v_i^2 \leqslant \frac{1}{2}h\sum_{i=1}^{M-1}\sum_{\substack{k=1-M\\k\neq 0}}^{M-1}(-\hat{g}_k^{(\beta)})v_i^2 = \frac{1}{2}\sum_{|k|=1}^{M-1}(-\hat{g}_k^{(\beta)})\cdot h\sum_{i=1}^{M-1}v_i^2. \tag{5.20}
$$

For the third term on the right-hand side in (5.19), we have

$$
\begin{aligned}
&\frac{1}{2}h\sum_{i=1}^{M-1}\sum_{\substack{k=i-M\\k\neq 0}}^{i}(-\hat{g}_k^{(\beta)})v_{i-k}^2\\
&= \frac{1}{2}h\left[\sum_{k=1-M}^{-1}(-\hat{g}_k^{(\beta)})\sum_{i=1}^{k+M}v_{i-k}^2 + \sum_{k=1}^{M-1}(-\hat{g}_k^{(\beta)})\sum_{i=k}^{M-1}v_{i-k}^2\right]\\
&\leqslant \frac{1}{2}\left[\sum_{k=1-M}^{-1}(-\hat{g}_k^{(\beta)}) + \sum_{k=1}^{M-1}(-\hat{g}_k^{(\beta)})\right]\cdot h\sum_{i=1}^{M-1}v_i^2\\
&= \frac{1}{2}\sum_{|k|=1}^{M-1}(-\hat{g}_k^{(\beta)})\cdot h\sum_{i=1}^{M-1}v_i^2.
\end{aligned}
\tag{5.21}
$$

Substituting (5.20) and (5.21) into (5.19), it follows from Lemma 1.5.2 and Lemma 1.5.3 that

$$
\begin{aligned}
A &\leqslant h^{-\beta}\left[(-\hat{g}_0^{(\beta)})h\sum_{i=1}^{M-1}v_i^2 + \sum_{|k|=1}^{M-1}(-\hat{g}_k^{(\beta)})\cdot h\sum_{i=1}^{M-1}v_i^2\right]\\
&= h^{-\beta}\sum_{k=1-M}^{M-1}(-\hat{g}_k^{(\beta)})\cdot h\sum_{i=1}^{M-1}v_i^2\\
&= h^{-\beta}\left(\sum_{|k|\geqslant M}\hat{g}_k^{(\beta)}\right)\cdot h\sum_{i=1}^{M-1}v_i^2\\
&\leqslant -h^{-\beta}\frac{c_*^{(\beta)}}{(M+1)^{\beta}}\cdot h\sum_{i=1}^{M-1}v_i^2\\
&= -(Mh)^{-\beta}c_*^{(\beta)}\left(\frac{M}{M+1}\right)^{\beta}\cdot h\sum_{i=1}^{M-1}v_i^2
\end{aligned}
$$

$$\leqslant -(Mh)^{-\beta} c_*^{(\beta)} \left(\frac{1}{2}\right)^{\beta} \cdot h \sum_{i=1}^{M-1} v_i^2$$
$$= -c_*^{(\beta)} (2L)^{-\beta} \cdot h \sum_{i=1}^{M-1} v_i^2.$$

The proof ends. □

Similarly, the following conclusion can be proved (the only difference is to replace $v_{i-k}v_i$ and v_i^2 with $(\mathbf{v}_{i-k}, \mathbf{v}_i)$ and $\|\mathbf{v}_i\|^2$, resp.).

Corollary 5.1.1. *Suppose $\mathcal{V}$ is an inner product space, $(\cdot,\cdot)$ is an inner product in $\mathcal{V}$ and $\|\cdot\|$ is the induced norm; For any mesh functions $\boldsymbol{v}^0, \boldsymbol{v}^1, \ldots, \boldsymbol{v}^M \in \mathcal{V}$ satisfying $\boldsymbol{v}^0 = 0$, $\boldsymbol{v}^M = 0$, it holds*

$$-h^{-\beta} h \sum_{i=1}^{M-1} \left(\sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} \boldsymbol{v}^{i-k}, \boldsymbol{v}^i \right) \leqslant -c_*^{(\beta)} (2L)^{-\beta} h \sum_{i=1}^{M-1} \|\boldsymbol{v}^i\|^2,$$

where $c_*^{(\beta)}$ is defined by (1.55), $1 < \beta < 2$.

5.1.4 Stability of the difference scheme

Theorem 5.1.2. *Suppose $\{u_i^n \mid 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N\}$ is the solution of the difference scheme* (5.13)–(5.15), *then it holds*

$$\|u^n\|^2 \leqslant \|u^0\|^2 + \frac{(2L)^{\beta} \Gamma(1-\alpha)}{2c_*^{(\beta)}} \max_{1 \leqslant m \leqslant n} \{t_m^{\alpha} \|f^{m-1+\sigma}\|^2\}, \quad 1 \leqslant n \leqslant N,$$

where

$$\|f^{m-1+\sigma}\|^2 = h \sum_{i=1}^{M-1} (f_i^{m-1+\sigma})^2.$$

Proof. Taking the inner product on both hand sides of (5.13) with $\sigma u^n + (1-\sigma) u^{n-1}$ yields

$$\begin{aligned} &\frac{1}{s} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} h \sum_{i=1}^{M-1} (u_i^{n-k} - u_i^{n-k-1}) [\sigma u_i^n + (1-\sigma) u_i^{n-1}] \\ =& -h^{-\beta} h \sum_{i=1}^{M-1} \sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} [\sigma u_{i-k}^n + (1-\sigma) u_{i-k}^{n-1}] \cdot [\sigma u_i^n + (1-\sigma) u_i^{n-1}] \\ &+ h \sum_{i=1}^{M-1} f_i^{n-1+\sigma} [\sigma u_i^n + (1-\sigma) u_i^{n-1}]. \end{aligned} \tag{5.22}$$

Now each term in (5.22) will be estimated.

For the left-hand side, it follows from Lemma 2.6.1 that

$$
\begin{aligned}
&\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)} h \sum_{i=1}^{M-1} (u_i^{n-k} - u_i^{n-k-1})[\sigma u_i^n + (1-\sigma)u_i^{n-1}] \\
&= \frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)} (u^{n-k} - u^{n-k-1}, \sigma u^n + (1-\sigma)u^{n-1}) \\
&\geqslant \frac{1}{2s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)} (\|u^{n-k}\|^2 - \|u^{n-k-1}\|^2). \qquad (5.23)
\end{aligned}
$$

For the first term on the right-hand side, using Lemma 5.1.1, we have

$$
\begin{aligned}
&-h^{-\beta} h \sum_{i=1}^{M-1} \sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} [\sigma u_{i-k}^n + (1-\sigma)u_{i-k}^{n-1}] \cdot [\sigma u_i^n + (1-\sigma)u_i^{n-1}] \\
&\leqslant -c_*^{(\beta)} (2L)^{-\beta} h \sum_{i=1}^{M-1} [\sigma u_i^n + (1-\sigma)u_i^{n-1}]^2 \\
&= -c_*^{(\beta)} (2L)^{-\beta} \|\sigma u^n + (1-\sigma)u^{n-1}\|^2. \qquad (5.24)
\end{aligned}
$$

For the second term on the right-hand side, with the aid of the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
&h \sum_{i=1}^{M-1} f_i^{n-1+\sigma} [\sigma u_i^n + (1-\sigma)u_i^{n-1}] \\
&\leqslant \|f^{n-1+\sigma}\| \cdot \|\sigma u^n + (1-\sigma)u^{n-1}\| \\
&\leqslant c_*^{(\beta)} (2L)^{-\beta} \|\sigma u^n + (1-\sigma)u^{n-1}\|^2 + \frac{(2L)^\beta}{4c_*^{(\beta)}} \|f^{n-1+\sigma}\|^2. \qquad (5.25)
\end{aligned}
$$

Substituting (5.23), (5.24) and (5.25) into (5.22) arrives at

$$
\frac{1}{2s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)} (\|u^{n-k}\|^2 - \|u^{n-k-1}\|^2) \leqslant \frac{(2L)^\beta}{4c_*^{(\beta)}} \|f^{n-1+\sigma}\|^2, \quad 1 \leqslant n \leqslant N. \qquad (5.26)
$$

It follows from Lemma 1.6.3 that

$$
\frac{s}{c_{n-1}^{(n,\alpha)}} = \frac{\tau^\alpha \Gamma(2-\alpha)}{c_{n-1}^{(n,\alpha)}} < \frac{1}{1-\alpha} n^\alpha \tau^\alpha \Gamma(2-\alpha) = t_n^\alpha \Gamma(1-\alpha).
$$

Reformulate (5.26) as

$$
\begin{aligned}
c_0^{(n,\alpha)} \|u^n\|^2 &\leqslant \sum_{k=1}^{n-1} (c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)}) \|u^{n-k}\|^2 + c_{n-1}^{(n,\alpha)} \|u^0\|^2 + \frac{s(2L)^\beta}{2c_*^{(\beta)}} \|f^{n-1+\sigma}\|^2 \\
&\leqslant \sum_{k=1}^{n-1} (c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)}) \|u^{n-k}\|^2
\end{aligned}
$$

$$+ c_{n-1}^{(n,\alpha)}\left[\|u^0\|^2 + \frac{(2L)^\beta t_n^\alpha \Gamma(1-\alpha)}{2c_*^{(\beta)}}\|f^{n-1+\sigma}\|^2\right], \quad 1 \leqslant n \leqslant N.$$

The inductive process will lead to

$$\|u^n\|^2 \leqslant \|u^0\|^2 + \frac{(2L)^\beta \Gamma(1-\alpha)}{2c_*^{(\beta)}} \max_{1\leqslant m\leqslant n}\{t_m^\alpha \|f^{m-1+\sigma}\|^2\}, \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

5.1.5 Convergence of the difference scheme

Theorem 5.1.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (5.1)–(5.3) *and the difference scheme* (5.13)–(5.15), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then it holds

$$\|e^n\| \leqslant \sqrt{\frac{2^{\beta-1}L^{1+\beta}T^\alpha \Gamma(1-\alpha)}{c_*^{(\beta)}}}\, c_1(\tau^2 + h^2), \quad 1 \leqslant n \leqslant N. \tag{5.27}$$

Proof. The subtraction of (5.13)–(5.15) from (5.9), (5.11)–(5.12), respectively, will produce the system of error equations as follows:

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \displaystyle\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(e_i^{n-k} - e_i^{n-k-1}) \\ = -h^{-\beta} \displaystyle\sum_{k=i-M}^{i} \hat{g}_k^{(\beta)}(\sigma e_{i-k}^n + (1-\sigma)e_{i-k}^{n-1}) + (r_1)_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (5.10), the application of Theorem 5.1.2 yields

$$\begin{aligned} \|e^n\|^2 &\leqslant \frac{(2L)^\beta \Gamma(1-\alpha)}{2c_*^{(\beta)}} \max_{1\leqslant m\leqslant n}\{t_m^\alpha \|(r_1)^m\|^2\} \\ &\leqslant \frac{2^{\beta-1}L^{1+\beta}t_n^\alpha \Gamma(1-\alpha)}{c_*^{(\beta)}} c_1^2(\tau^2 + h^2)^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Taking the square root on both hand sides of the inequality above will lead to (5.27). The proof ends. □

5.2 The fourth-order method in space for 1D problem

In this section, another difference scheme of order two in time and four in space for solving the problem (5.1)–(5.3) will be developed.

Suppose $u = (u_0, u_1, \ldots, u_M) \in \mathcal{U}_h$, define the average operator

$$\mathcal{A}_h^\beta u_i = \begin{cases} \frac{\beta}{24} u_{i-1} + (1 - \frac{\beta}{12}) u_i + \frac{\beta}{24} u_{i+1}, & 1 \leqslant i \leqslant M-1, \\ u_i, & i = 0,\ M. \end{cases}$$

It is apparent that

$$\mathcal{A}_h^\beta u_i = \left(\mathcal{I} + \frac{\beta}{24} h^2 \delta_x^2\right) u_i, \quad 1 \leqslant i \leqslant M-1.$$

Suppose $u, v \in \mathring{\mathcal{U}}_h$. Noticing

$$h \sum_{i=1}^{M-1} (\mathcal{A}_h^\beta u_i) v_i = h \sum_{i=1}^{M-1} \left(u_i + \frac{\beta}{24} h^2 \delta_x^2 u_i\right) v_i = (u, v) - \frac{\beta}{24} h^2 (\delta_x u, \delta_x v),$$

define an inner product and the induced norm as follows:

$$(u, v)_A = h \sum_{i=1}^{M-1} (\mathcal{A}_h^\beta u_i) v_i, \quad \|u\|_A = \sqrt{(u, u)_A}\,.$$

It follows by noticing Lemma 2.1.1 and $\beta \in (1, 2)$ that

$$h \sum_{i=1}^{M-1} (\mathcal{A}_h^\beta u_i) u_i = \|u\|^2 - \frac{\beta}{24} h^2 \|\delta_x u\|^2 \geqslant \left(1 - \frac{\beta}{6}\right) \|u\|^2 \geqslant \frac{2}{3} \|u\|^2. \tag{5.28}$$

It is easy to know that

$$\frac{2}{3} \|u\|^2 \leqslant \|u\|_A^2 \leqslant \|u\|^2. \tag{5.29}$$

Define the function $\tilde{u}(x, t)$ like that in Section 5.1. Suppose $\tilde{u}(\cdot, t) \in \mathscr{C}^{4+\beta}(\mathcal{R})$ and $u(x, \cdot) \in C^3[0, T]$.

5.2.1 Derivation of the difference scheme

Considering equation (5.1) at the point $(x_i, t_{n-1+\sigma})$, we have

$${}_0^C D_t^\alpha u(x_i, t_{n-1+\sigma}) = \frac{\partial^\beta u(x_i, t_{n-1+\sigma})}{\partial |x|^\beta} + f_i^{n-1+\sigma}, \quad 0 \leqslant i \leqslant M,\ 1 \leqslant n \leqslant N.$$

Using (3.85), it follows that

$$\begin{aligned}{}_0^C D_t^{\alpha} u(x_i, t_{n-1+\sigma}) = & \left[\sigma \frac{\partial^{\beta} u(x_i, t_n)}{\partial |x|^{\beta}} + (1-\sigma) \frac{\partial^{\beta} u(x_i, t_{n-1})}{\partial |x|^{\beta}}\right] \\ & + f_i^{n-1+\sigma} + O(\tau^2), \quad 0 \leqslant i \leqslant M,\ 1 \leqslant n \leqslant N. \end{aligned} \tag{5.30}$$

Performing the operator $\mathcal{A}_h^{\beta}$ to both hand sides of (5.30) gives

$$\begin{aligned}\mathcal{A}_h^{\beta}{}_0^C D_t^{\alpha} u(x_i, t_{n-1+\sigma}) = & \left[\sigma \mathcal{A}_h^{\beta} \frac{\partial^{\beta} u(x_i, t_n)}{\partial |x|^{\beta}} + (1-\sigma) \mathcal{A}_h^{\beta} \frac{\partial^{\beta} u(x_i, t_{n-1})}{\partial |x|^{\beta}}\right] \\ & + \mathcal{A}_h^{\beta} f_i^{n-1+\sigma} + O(\tau^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \end{aligned} \tag{5.31}$$

For the Caputo derivative in (5.31), using L2-1_{σ} approximation (1.81), it follows from Theorem 1.6.4 that

$${}_0^C D_t^{\alpha} u(x_i, t_{n-1+\sigma}) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (U_i^{n-k} - U_i^{n-k-1}) + O(\tau^{3-\alpha}). \tag{5.32}$$

For the Riesz derivative in (5.31), it follows from Theorem 1.5.2 that

$$\mathcal{A}_h^{\beta} \frac{\partial^{\beta} u(x_i, t_n)}{\partial |x|^{\beta}} = -h^{-\beta} \sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} U_{i-k}^n + O(h^4). \tag{5.33}$$

Substituting (5.32) and (5.33) into (5.31) arrives at

$$\begin{aligned} & \mathcal{A}_h^{\beta} \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (U_i^{n-k} - U_i^{n-k-1}) \\ & = -h^{-\beta} \sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} [\sigma U_{i-k}^n + (1-\sigma) U_{i-k}^{n-1}] + \mathcal{A}_h^{\beta} f_i^{n-1+\sigma} + (r_2)_i^n, \\ & \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \end{aligned} \tag{5.34}$$

and there is a positive constant c_2 such that

$$\left|(r_2)_i^n\right| \leqslant c_2(\tau^2 + h^4), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{5.35}$$

Noticing the initial-boundary value conditions (5.2)–(5.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (5.36) \\ U_0^n = 0, \quad U_M^n = 0, \quad 0 \leqslant n \leqslant N. & (5.37) \end{cases}$$

Neglecting the small term $(r_2)_i^n$ in (5.34) and replacing the exact solution U_i^n with its numerical one u_i^n, another difference scheme for solving (5.1)–(5.3) can be obtained as

$$\begin{cases} \mathcal{A}_h^\beta \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \displaystyle\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u_i^{n-k} - u_i^{n-k-1}) \\ = -h^{-\beta} \displaystyle\sum_{k=i-M}^{i} \hat{g}_k^{(\beta)}[\sigma u_{i-k}^n + (1-\sigma)u_{i-k}^{n-1}] + \mathcal{A}_h^\beta f_i^{n-1+\sigma}, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (5.38) \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, & (5.39) \\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N. & (5.40) \end{cases}$$

Next, some analyses on the difference scheme (5.38)–(5.40) will be made.

5.2.2 Solvability of the difference scheme

Theorem 5.2.1. *The difference scheme* (5.38)–(5.40) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The value of u^0 is uniquely determined by (5.39)–(5.40).

Now suppose that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n can be obtained from (5.38) and (5.40). To show its unique solvability, it suffices to verify that the corresponding homogeneous one

$$\begin{cases} \dfrac{1}{s} c_0^{(n,\alpha)} \mathcal{A}_h^\beta u_i^n = -\sigma h^{-\beta} \displaystyle\sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} u_{i-k}^n, \quad 1 \leqslant i \leqslant M-1, & (5.41) \\ u_0^n = 0, \quad u_M^n = 0 & (5.42) \end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (5.41) with u^n arrives at

$$\frac{1}{s} c_0^{(n,\alpha)} h \sum_{i=1}^{M-1} (\mathcal{A}_h^\beta u_i^n) u_i^n = -\sigma h^{-\beta} h \sum_{i=1}^{M-1} \left(\sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} u_{i-k}^n \right) u_i^n. \tag{5.43}$$

It follows from Lemma 5.1.1 that

$$-h^{-\beta} h \sum_{i=1}^{M-1} \left[\sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} u_{i-k}^n \right] u_i^n \leqslant -c_*^{(\beta)} (2L)^{-\beta} \|u^n\|^2 \leqslant 0. \tag{5.44}$$

The substitution of (5.44) and (5.28) into (5.43) will give $\|u^n\| = 0$. Then it can be concluded that $u^n = 0$ from (5.42).

By the principle of induction, the difference scheme (5.38)–(5.40) has a unique solution. The proof ends. □

5.2.3 Stability of the difference scheme

Theorem 5.2.2. *Suppose* $\{u_i^n \mid 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme* (5.38)–(5.40), *then it holds*

$$\|u^n\|^2 \leqslant \frac{3}{2}\left[\|u^0\|^2 + \frac{(2L)^\beta \Gamma(1-\alpha)}{2c_*^{(\beta)}} \max_{1\leqslant m\leqslant n}\{t_m^\alpha \|\mathcal{A}_h^\beta f^{m-1+\sigma}\|^2\}\right], \quad 1 \leqslant n \leqslant N,$$

where

$$\|\mathcal{A}_h^\beta f^{m-1+\sigma}\|^2 = h\sum_{i=1}^{M-1}(\mathcal{A}_h^\beta f_i^{m-1+\sigma})^2.$$

Proof. Taking the inner product on both hand sides of (5.38) with $\sigma u^n + (1-\sigma)u^{n-1}$ will produce

$$\begin{aligned}
&\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)} h\sum_{i=1}^{M-1}[\mathcal{A}_h^\beta(u_i^{n-k} - u_i^{n-k-1})][\sigma u_i^n + (1-\sigma)u_i^{n-1}] \\
&= -h^{-\beta} h\sum_{i=1}^{M-1}\sum_{k=i-M}^{i} \hat{g}_k^{(\beta)}[\sigma u_{i-k}^n + (1-\sigma)u_{i-k}^{n-1}][\sigma u_i^n + (1-\sigma)u_i^{n-1}] \\
&\quad + h\sum_{i=1}^{M-1}(\mathcal{A}_h^\beta f_i^{n-1+\sigma})[\sigma u_i^n + (1-\sigma)u_i^{n-1}] \\
&\equiv C + D.
\end{aligned} \tag{5.45}$$

Now each term in (5.45) will be estimated.

For the left-hand side of (5.45), we have

$$\begin{aligned}
B &\equiv \frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)} h\sum_{i=1}^{M-1}[\mathcal{A}_h^\beta(u_i^{n-k} - u_i^{n-k-1})][\sigma u_i^n + (1-\sigma)u_i^{n-1}] \\
&= \frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(u^{n-k} - u^{n-k-1}, \sigma u^n + (1-\sigma)u^{n-1})_A.
\end{aligned}$$

By Lemma 2.6.1, we have

$$B \geqslant \frac{1}{2s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(\|u^{n-k}\|_A^2 - \|u^{n-k-1}\|_A^2). \tag{5.46}$$

It follows from Lemma 5.1.1 that the first term on the right-hand side

$$C \leqslant -c_*^{(\beta)}(2L)^{-\beta}\|\sigma u^n + (1-\sigma)u^{n-1}\|^2. \tag{5.47}$$

By the Cauchy–Schwarz inequality, we know that the second term on the right-hand side

$$D \leqslant \|\mathcal{A}_h^\beta f^{n-1+\sigma}\| \cdot \|\sigma u^n + (1-\sigma)u^{n-1}\|$$

$$\leqslant c_*^{(\beta)}(2L)^{-\beta}\|\sigma u^n+(1-\sigma)u^{n-1}\|^2+\frac{(2L)^\beta}{4c_*^{(\beta)}}\|\mathcal{A}_h^\beta f^{n-1+\sigma}\|^2. \tag{5.48}$$

Substituting (5.46)–(5.48) into (5.45) gives

$$\frac{1}{2s}\sum_{k=0}^{n-1}c_k^{(n,\alpha)}(\|u^{n-k}\|_A^2-\|u^{n-k-1}\|_A^2)\leqslant\frac{(2L)^\beta}{4c_*^{(\beta)}}\|\mathcal{A}_h^\beta f^{n-1+\sigma}\|^2,\quad 1\leqslant n\leqslant N.$$

Noticing

$$\frac{s}{c_{n-1}^{(n,\alpha)}}\leqslant t_n^\alpha\Gamma(1-\alpha),$$

it follows from the inequality above that

$$\begin{aligned}
&c_0^{(n,\alpha)}\|u^n\|_A^2\\
&\leqslant\sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)}-c_k^{(n,\alpha)})\|u^{n-k}\|_A^2+c_{n-1}^{(n,\alpha)}\|u^0\|_A^2+\frac{s(2L)^\beta}{2c_*^{(\beta)}}\|\mathcal{A}_h^\beta f^{n-1+\sigma}\|^2\\
&\leqslant\sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)}-c_k^{(n,\alpha)})\|u^{n-k}\|_A^2\\
&\quad+c_{n-1}^{(n,\alpha)}\left[\|u^0\|_A^2+\frac{(2L)^\beta t_n^\alpha\Gamma(1-\alpha)}{2c_*^{(\beta)}}\|\mathcal{A}_h^\beta f^{n-1+\sigma}\|^2\right],\quad 1\leqslant n\leqslant N.
\end{aligned}$$

The inductive process will lead to

$$\|u^n\|_A^2\leqslant\|u^0\|_A^2+\frac{(2L)^\beta\Gamma(1-\alpha)}{2c_*^{(\beta)}}\max_{1\leqslant m\leqslant n}\{t_m^\alpha\|\mathcal{A}_h^\beta f^{m-1+\sigma}\|^2\},\quad 1\leqslant n\leqslant N.$$

By noticing (5.29), further we have

$$\|u^n\|^2\leqslant\frac{3}{2}\left[\|u^0\|^2+\frac{(2L)^\beta\Gamma(1-\alpha)}{2c_*^{(\beta)}}\max_{1\leqslant m\leqslant n}\{t_m^\alpha\|\mathcal{A}_h^\beta f^{m-1+\sigma}\|^2\}\right],\quad 1\leqslant n\leqslant N.$$

The proof ends. □

5.2.4 Convergence of the difference scheme

Theorem 5.2.3. *Suppose* $\{U_i^n\mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *and* $\{u_i^n\mid 0\leqslant i\leqslant M, 0\leqslant n\leqslant N\}$ *are solutions of the problem* (5.1)–(5.3) *and the difference scheme* (5.38)–(5.40), *respectively. Let*

$$e_i^n=U_i^n-u_i^n,\quad 0\leqslant i\leqslant M,\ 0\leqslant n\leqslant N,$$

then it holds

$$\|e^n\|\leqslant\sqrt{\frac{3}{4}\cdot\frac{2^\beta L^{1+\beta}T^\alpha\Gamma(1-\alpha)}{c_*^{(\beta)}}}\,c_2(\tau^2+h^4),\quad 1\leqslant n\leqslant N. \tag{5.49}$$

Proof. Subtracting (5.38)–(5.40) from (5.34), (5.36)–(5.37), respectively, gives the system of error equations as follows:

$$\begin{cases} \mathcal{A}_h^{\beta} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \displaystyle\sum_{k=0}^{n-1} c_k^{(n,\alpha)} (e_i^{n-k} - e_i^{n-k-1}) \\ = -h^{-\beta} \displaystyle\sum_{k=i-M}^{i} \hat{g}_k^{(\beta)} (\sigma e_{i-k}^n + (1-\sigma) e_{i-k}^{n-1}) + (r_2)_i^n, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0,\ e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (5.35), the application of Theorem 5.2.2 immediately yields

$$\begin{aligned} \|e^n\|^2 &\leqslant \frac{3}{2} \cdot \frac{(2L)^{\beta} t_n^{\alpha} \Gamma(1-\alpha)}{2c_*^{(\beta)}} \max_{1 \leqslant m \leqslant n} \|(r_2)^m\|^2 \\ &\leqslant \frac{3}{2} \cdot \frac{2^{\beta} L^{1+\beta} T^{\alpha} \Gamma(1-\alpha)}{2c_*^{(\beta)}} c_2^2 (\tau^2 + h^4)^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Taking the square root on both hand sides of the inequality above will arrive at (5.49). The proof ends. □

5.3 The second-order method in space for 2D problem

In this section, consider the following 2D initial-boundary value problem:

$$\begin{cases} {}_0^C D_t^{\alpha} u(x,y,t) = K_1 \dfrac{\partial^{\beta} u(x,y,t)}{\partial |x|^{\beta}} + K_2 \dfrac{\partial^{\gamma} u(x,y,t)}{\partial |y|^{\gamma}} + f(x,y,t), \\ \qquad\qquad (x,y) \in \Omega,\ 0 < t \leqslant T, & (5.50) \\ u(x,y,0) = \varphi(x,y), \quad (x,y) \in \Omega, & (5.51) \\ u(x,y,t) = 0, \quad (x,y) \in \partial\Omega, \quad 0 \leqslant t \leqslant T, & (5.52) \end{cases}$$

where $\alpha \in (0,1)$, $\beta \in (1,2)$, $\gamma \in (1,2)$, $K_1 > 0$, $K_2 > 0$, $\Omega = (0,L_1) \times (0,L_2)$ and $\varphi(x,y)|_{(x,y) \in \partial\Omega} = 0$.

Take the same mesh partition and notations as those in Section 2.10. In addition, define $\sigma = 1 - \frac{\alpha}{2}$, $t_{n-1+\sigma} = t_{n-1} + \sigma\tau$, $s = \tau^{\alpha}\Gamma(2-\alpha)$.

Similar to Section 4.4, define functions $\hat{v}(x,y,t)$ and $\hat{w}(x,y,t)$. Suppose $\hat{v}(\cdot,y,t) \in \mathscr{C}^{2+\beta}(\mathcal{R})$, $\hat{w}(x,\cdot,t) \in \mathscr{C}^{2+\gamma}(\mathcal{R})$ and $u(x,y,\cdot) \in C^3[0,T]$.

Define the mesh functions

$$\begin{aligned} U_{ij}^n &= u(x_i, y_j, t_n), \quad (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N; \\ f_{ij}^{n-1+\sigma} &= f(x_i, y_j, t_{n-1+\sigma}), \quad (i,j) \in \bar{\omega},\ 1 \leqslant n \leqslant N. \end{aligned}$$

5.3.1 Derivation of the difference scheme

Considering equation (5.50) at the point $(x_i, y_j, t_{n-1+\sigma})$, we have

$$ {}_0^C D_t^\alpha u(x_i, y_j, t_{n-1+\sigma}) = K_1 \frac{\partial^\beta u(x_i, y_j, t_{n-1+\sigma})}{\partial |x|^\beta} + K_2 \frac{\partial^\gamma u(x_i, y_j, t_{n-1+\sigma})}{\partial |y|^\gamma} + f_{ij}^{n-1+\sigma}, $$
$$ (i,j) \in \omega, \ 1 \leqslant n \leqslant N. \tag{5.53} $$

For the Caputo derivative in (5.53), using L2-1$_\sigma$ approximation (1.81), it follows from Theorem 1.6.4 that

$$ {}_0^C D_t^\alpha u(x_i, y_j, t_{n-1+\sigma}) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (U_{ij}^{n-k} - U_{ij}^{n-k-1}) + O(\tau^{3-\alpha}). \tag{5.54} $$

For the Riesz derivatives in (5.53), it follows from the linear interpolation and Theorem 1.5.1 that

$$ \begin{aligned} &\frac{\partial^\beta u(x_i, y_j, t_{n-1+\sigma})}{\partial |x|^\beta} \\ &= \sigma \frac{\partial^\beta u(x_i, y_j, t_n)}{\partial |x|^\beta} + (1-\sigma) \frac{\partial^\beta u(x_i, y_j, t_{n-1})}{\partial |x|^\beta} + O(\tau^2) \\ &= \sigma \left[-h_1^{-\beta} \sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} U_{i-k,j}^n \right] + (1-\sigma) \left[-h_1^{-\beta} \sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} U_{i-k,j}^{n-1} \right] + O(\tau^2 + h_1^2) \\ &= -h_1^{-\beta} \sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} [\sigma U_{i-k,j}^n + (1-\sigma) U_{i-k,j}^{n-1}] + O(\tau^2 + h_1^2). \end{aligned} \tag{5.55} $$

Similarly, we have

$$ \frac{\partial^\gamma u(x_i, y_j, t_{n-1+\sigma})}{\partial |y|^\gamma} = -h_2^{-\gamma} \sum_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} [\sigma U_{i,j-k}^n + (1-\sigma) U_{i,j-k}^{n-1}] + O(\tau^2 + h_2^2). \tag{5.56} $$

Substituting (5.54)–(5.56) into (5.53) yields

$$ \begin{aligned} &\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (U_{ij}^{n-k} - U_{ij}^{n-k-1}) \\ &= -K_1 h_1^{-\beta} \sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} [\sigma U_{i-k,j}^n + (1-\sigma) U_{i-k,j}^{n-1}] \\ &\quad - K_2 h_2^{-\gamma} \sum_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} [\sigma U_{i,j-k}^n + (1-\sigma) U_{i,j-k}^{n-1}] \\ &\quad + f_{ij}^{n-1+\sigma} + (r_3)_{ij}^n, \quad (i,j) \in \omega, \ 1 \leqslant n \leqslant N, \end{aligned} \tag{5.57} $$

and there is a positive constant c_3 such that

$$|(r_3)_{ij}^n| \leqslant c_3(\tau^2 + h_1^2 + h_2^2), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \tag{5.58}$$

Noticing the initial-boundary value conditions (5.51)–(5.52), we have

$$\begin{cases} U_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (5.59)\\ U_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (5.60)\end{cases}$$

Omitting the small term $(r_3)_{ij}^n$ in (5.57) and replacing the exact solution U_{ij}^n with its numerical one u_{ij}^n, we get a difference scheme for solving (5.50)–(5.52) in the form of

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum\limits_{k=0}^{n-1} c_k^{(n,\alpha)} (u_{ij}^{n-k} - u_{ij}^{n-k-1}) \\ = -K_1 h_1^{-\beta} \sum\limits_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} [\sigma u_{i-k,j}^n + (1-\sigma) u_{i-k,j}^{n-1}] \\ \quad - K_2 h_2^{-\gamma} \sum\limits_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} [\sigma u_{i,j-k}^n + (1-\sigma) u_{i,j-k}^{n-1}] + f_{ij}^{n-1+\sigma}, \\ \qquad\qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (5.61)\\ u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (5.62)\\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (5.63)\end{cases}$$

5.3.2 Solvability of the difference scheme

Theorem 5.3.1. *The difference scheme* (5.61)–(5.63) *is uniquely solvable.*

Proof. Let

$$u^n = \{u_{ij}^n \mid (i,j) \in \bar{\omega}\}.$$

The value of u^0 is determined by (5.62)–(5.63).

Now assume that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then we can obtain the linear system in the unknown u^n from (5.61) and (5.63). To show its unique solvability, it is sufficient to verify that the corresponding homogeneous one

$$\begin{cases} \dfrac{1}{s} c_0^{(n,\alpha)} u_{ij}^n = -K_1 \sigma h_1^{-\beta} \sum\limits_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} u_{i-k,j}^n - K_2 \sigma h_2^{-\gamma} \sum\limits_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} u_{i,j-k}^n, \quad (i,j) \in \omega, & (5.64)\\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega & (5.65)\end{cases}$$

has only the trivial solution.

Rewrite (5.64) as

$$\left[\frac{1}{s}c_0^{(n,\alpha)} + K_1\sigma h_1^{-\beta}\hat{g}_0^{(\beta)} + K_2\sigma h_2^{-\gamma}\hat{g}_0^{(\gamma)}\right]u_{ij}^n$$
$$= K_1\sigma h_1^{-\beta}\sum_{\substack{k=i-M_1\\k\neq 0}}^{i}(-\hat{g}_k^{(\beta)})u_{i-k,j}^n + K_2\sigma h_2^{-\gamma}\sum_{\substack{k=j-M_2\\k\neq 0}}^{j}(-\hat{g}_k^{(\gamma)})u_{i,j-k}^n, \quad (i,j)\in\omega. \tag{5.66}$$

Suppose $\|u^n\|_\infty = |u_{i_n,j_n}^n|$, where $(i_n,j_n)\in\omega$. In (5.66), letting $(i,j)=(i_n,j_n)$ and taking the absolute value of both hand sides, the application of the triangle inequality gives

$$\left[\frac{1}{s}c_0^{(n,\alpha)} + K_1\sigma h_1^{-\beta}\hat{g}_0^{(\beta)} + K_2\sigma h_2^{-\gamma}\hat{g}_0^{(\gamma)}\right]\|u^n\|_\infty$$
$$\leqslant K_1\sigma h_1^{-\beta}\sum_{\substack{k=i_n-M_1\\k\neq 0}}^{i_n}(-\hat{g}_k^{(\beta)})\|u^n\|_\infty + K_2\sigma h_2^{-\gamma}\sum_{\substack{k=j_n-M_2\\k\neq 0}}^{j_n}(-\hat{g}_k^{(\gamma)})\|u^n\|_\infty$$
$$\leqslant [K_1\sigma h_1^{-\beta}\hat{g}_0^{(\beta)} + K_2\sigma h_2^{-\gamma}\hat{g}_0^{(\gamma)}]\|u^n\|_\infty,$$

which implies $\|u^n\|_\infty = 0$. The combination with (5.65) will reveal that (5.64)–(5.65) has only the trivial solution.

By the principle of induction, the difference scheme (5.61)–(5.63) is uniquely solvable. The proof ends. □

5.3.3 Stability of the difference scheme

Theorem 5.3.2. *Suppose* $\{u_{ij}^n \mid (i,j)\in\bar{\omega},\ 0\leqslant n\leqslant N\}$ *is the solution of* (5.61)–(5.63)*, then it holds*

$$\|u^n\|^2 \leqslant \|u^0\|^2 + \frac{\Gamma(1-\alpha)}{8}\left[\frac{(2L_1)^\beta}{K_1c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2c_*^{(\gamma)}}\right]\max_{1\leqslant m\leqslant n}\{t_m^\alpha\|f^{m-1+\sigma}\|^2\}, \quad 1\leqslant n\leqslant N,$$

where

$$\|f^{m-1+\sigma}\|^2 = h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(f_{ij}^{m-1+\sigma})^2.$$

Proof. Taking the inner product on both hand sides of (5.61) with $\sigma u^n + (1-\sigma)u^{n-1}$ will arrive at

$$\frac{1}{s}\sum_{k=0}^{n-1}c_k^{(n,\alpha)}h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(u_{ij}^{n-k}-u_{ij}^{n-k-1})[\sigma u_{ij}^n + (1-\sigma)u_{ij}^{n-1}]$$

$$= K_1 h_2 \sum_{j=1}^{M_2-1} \left\{ -h_1^{-\beta} h_1 \sum_{i=1}^{M_1-1} \sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} [\sigma u_{i-k,j}^n + (1-\sigma) u_{i-k,j}^{n-1}][\sigma u_{ij}^n + (1-\sigma) u_{ij}^{n-1}] \right\}$$
$$+ K_2 h_1 \sum_{i=1}^{M_1-1} \left\{ -h_2^{-\gamma} h_2 \sum_{j=1}^{M_2-1} \sum_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} [\sigma u_{i,j-k}^n + (1-\sigma) u_{i,j-k}^{n-1}][\sigma u_{ij}^n + (1-\sigma) u_{ij}^{n-1}] \right\}$$
$$+ h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} f_{ij}^{n-1+\sigma} [\sigma u_{ij}^n + (1-\sigma) u_{ij}^{n-1}]. \tag{5.67}$$

Now each term in (5.67) will be estimated.

By Lemma 2.6.1, we have

$$\sum_{k=0}^{n-1} c_k^{(n,\alpha)} h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (u_{ij}^{n-k} - u_{ij}^{n-k-1})[\sigma u_{ij}^n + (1-\sigma) u_{ij}^{n-1}]$$
$$\geqslant \frac{1}{2} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (\|u^{n-k}\|^2 - \|u^{n-k-1}\|^2). \tag{5.68}$$

It follows from Lemma 5.1.1 that

$$-h_1^{-\beta} h_1 \sum_{i=1}^{M_1-1} \sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} [\sigma u_{i-k,j}^n + (1-\sigma) u_{i-k,j}^{n-1}][\sigma u_{ij}^n + (1-\sigma) u_{ij}^{n-1}]$$
$$\leqslant -c_*^{(\beta)} (2L_1)^{-\beta} h_1 \sum_{i=1}^{M_1-1} [\sigma u_{ij}^n + (1-\sigma) u_{ij}^{n-1}]^2 \tag{5.69}$$

and

$$-h_2^{-\gamma} h_2 \sum_{j=1}^{M_2-1} \sum_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} [\sigma u_{i,j-k}^n + (1-\sigma) u_{i,j-k}^{n-1}][\sigma u_{ij}^n + (1-\sigma) u_{ij}^{n-1}]$$
$$\leqslant -c_*^{(\gamma)} (2L_2)^{-\gamma} h_2 \sum_{j=1}^{M_2-1} [\sigma u_{ij}^n + (1-\sigma) u_{ij}^{n-1}]^2. \tag{5.70}$$

Substituting (5.68)–(5.70) into (5.67) gives

$$\frac{1}{2s} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (\|u^{n-k}\|^2 - \|u^{n-k-1}\|^2)$$
$$\leqslant -K_1 c_*^{(\beta)} (2L_1)^{-\beta} h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} [\sigma u_{ij}^n + (1-\sigma) u_{ij}^{n-1}]^2$$
$$- K_2 c_*^{(\gamma)} (2L_2)^{-\gamma} h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} [\sigma u_{ij}^n + (1-\sigma) u_{ij}^{n-1}]^2$$
$$+ h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} f_{ij}^{n-1+\sigma} [\sigma u_{ij}^n + (1-\sigma) u_{ij}^{n-1}]$$

$$\begin{aligned}
&\leqslant -K_1 c_*^{(\beta)}(2L_1)^{-\beta}\|\sigma u^n + (1-\sigma)u^{n-1}\|^2 \\
&\quad - K_2 c_*^{(\gamma)}(2L_2)^{-\gamma}\|\sigma u^n + (1-\sigma)u^{n-1}\|^2 \\
&\quad + \|f^{n-1+\sigma}\| \cdot \|\sigma u^n + (1-\sigma)u^{n-1}\| \\
&\leqslant \frac{1}{16}\left[\frac{(2L_1)^\beta}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2 c_*^{(\gamma)}}\right]\|f^{n-1+\sigma}\|^2, \quad 1 \leqslant n \leqslant N.
\end{aligned} \tag{5.71}$$

By Lemma 1.6.3, we have

$$\frac{s}{c_{n-1}^{(n,\alpha)}} \leqslant t_n^\alpha \Gamma(1-\alpha).$$

Then it follows from (5.71) that

$$\begin{aligned}
c_0^{(n,\alpha)}\|u^n\|^2 &\leqslant \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})\|u^{n-k}\|^2 + c_{n-1}^{(n,\alpha)}\|u^0\|^2 \\
&\quad + \frac{1}{8}\left[\frac{(2L_1)^\beta}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2 c_*^{(\gamma)}}\right] s\|f^{n-1+\sigma}\|^2 \\
&\leqslant \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})\|u^{n-k}\|^2 \\
&\quad + c_{n-1}^{(n,\alpha)}\left\{\|u^0\|^2 + \frac{1}{8}\left[\frac{(2L_1)^\beta}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2 c_*^{(\gamma)}}\right] t_n^\alpha \Gamma(1-\alpha)\|f^{n-1+\sigma}\|^2\right\}, \quad 1 \leqslant n \leqslant N.
\end{aligned}$$

The inductive process will lead to

$$\|u^n\|^2 \leqslant \|u^0\|^2 + \frac{1}{8}\left[\frac{(2L_1)^\beta}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2 c_*^{(\gamma)}}\right]\Gamma(1-\alpha)\max_{1\leqslant m\leqslant n}\{t_m^\alpha\|f^{m-1+\sigma}\|^2\}, \quad 1 \leqslant n \leqslant N.$$

The proof ends. □

5.3.4 Convergence of the difference scheme

Theorem 5.3.3. *Suppose* $\{U_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *and* $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (5.50)–(5.52) *and the difference scheme* (5.61)–(5.63), *respectively. Let*

$$e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N,$$

then it holds

$$\|e^n\| \leqslant \sqrt{\frac{1}{8}\left[\frac{(2L_1)^\beta}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2 c_*^{(\gamma)}}\right]\Gamma(1-\alpha)T^\alpha L_1 L_2\, c_3(\tau^2 + h_1^2 + h_2^2)}, \quad 1 \leqslant n \leqslant N. \tag{5.72}$$

Proof. The system of error equations can be obtained by the subtraction of (5.61)–(5.63) from (5.57), (5.59)–(5.60), respectively, as follows:

$$\begin{cases} \frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}(e_{ij}^{n-k} - e_{ij}^{n-k-1}) \\ = -K_1 h_1^{-\beta} \sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)}[\sigma e_{i-k,j}^n + (1-\sigma)e_{i-k,j}^{n-1}] \\ \quad - K_2 h_2^{-\gamma} \sum_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)}[\sigma e_{i,j-k}^n + (1-\sigma)e_{i,j-k}^{n-1}] + (r_3)_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ e_{ij}^0 = 0, \quad (i,j) \in \omega, \\ e_{ij}^n = 0, \quad (i,j) \in \partial\omega, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (5.58), the application of Theorem 5.3.2 yields

$$\begin{aligned} \|e^n\|^2 &\leqslant \frac{1}{8}\left[\frac{(2L_1)^\beta}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2 c_*^{(\gamma)}}\right]\Gamma(1-\alpha) \max_{1\leqslant m\leqslant n}\{t_m^\alpha \|(r_3)^m\|^2\} \\ &\leqslant \frac{1}{8}\left[\frac{(2L_1)^\beta}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2 c_*^{(\gamma)}}\right]\Gamma(1-\alpha) T^\alpha L_1 L_2 c_3^2(\tau^2 + h_1^2 + h_2^2)^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Taking the square root on both hand sides of the inequality above will produce (5.72). The proof ends. □

5.4 The fourth-order method in space for 2D problem

In this section, another higher-order difference scheme of order two in time and four in space for solving (5.50)–(5.52) will be developed.

For any mesh function $v = \{v_{ij} \mid (i,j) \in \bar{\omega}\} \in \mathcal{V}_h$, define the following average operators:

$$\mathcal{H}_x^\beta v_{ij} = \begin{cases} \frac{\beta}{24} v_{i-1,j} + (1 - \frac{\beta}{12}) v_{ij} + \frac{\beta}{24} v_{i+1,j}, \ 1 \leqslant i \leqslant M_1 - 1, \\ v_{ij}, \ i = 0,\ M_1, \end{cases} \quad 0 \leqslant j \leqslant M_2;$$

$$\mathcal{H}_y^\gamma v_{ij} = \begin{cases} \frac{\gamma}{24} v_{i,j-1} + (1 - \frac{\gamma}{12}) v_{ij} + \frac{\gamma}{24} v_{i,j+1}, \ 1 \leqslant j \leqslant M_2 - 1, \\ v_{ij}, \ j = 0,\ M_2, \end{cases} \quad 0 \leqslant i \leqslant M_1.$$

Obviously,

$$\mathcal{H}_x^\beta v_{ij} = v_{ij} + \frac{\beta}{24} h_1^2 \delta_x^2 v_{ij}, \quad 1 \leqslant i \leqslant M_1 - 1,\ 0 \leqslant j \leqslant M_2;$$

$$\mathcal{H}_y^\gamma v_{ij} = v_{ij} + \frac{\gamma}{24} h_2^2 \delta_y^2 v_{ij}, \quad 1 \leqslant j \leqslant M_2 - 1,\ 0 \leqslant i \leqslant M_1.$$

Define the functions $\hat{v}(x,y,t)$ and $\hat{w}(x,y,t)$ like those in Section 4.4. Suppose $\hat{v}(\cdot,y,t) \in \mathscr{C}^{4+\beta}(\mathcal{R})$, $\hat{w}(x,\cdot,t) \in \mathscr{C}^{4+\gamma}(\mathcal{R})$ and $u(x,y,\cdot) \in C^3[0,T]$.

5.4.1 Derivation of the difference scheme

Considering equation (5.50) at the point $(x_i, y_j, t_{n-1+\sigma})$, by the mean of the linear interpolation, we have

$$\begin{aligned}
&{}_0^C D_t^\alpha u(x_i, y_j, t_{n-1+\sigma}) \\
&= K_1 \frac{\partial^\beta u(x_i, y_j, t_{n-1+\sigma})}{\partial |x|^\beta} + K_2 \frac{\partial^\gamma u(x_i, y_j, t_{n-1+\sigma})}{\partial |y|^\gamma} + f_{ij}^{n-1+\sigma} \\
&= K_1 \left[\sigma \frac{\partial^\beta u(x_i, y_j, t_n)}{\partial |x|^\beta} + (1-\sigma) \frac{\partial^\beta u(x_i, y_j, t_{n-1})}{\partial |x|^\beta} \right] \\
&\quad + K_2 \left[\sigma \frac{\partial^\gamma u(x_i, y_j, t_n)}{\partial |y|^\gamma} + (1-\sigma) \frac{\partial^\gamma u(x_i, y_j, t_{n-1})}{\partial |y|^\gamma} \right] + f_{ij}^{n-1+\sigma} + O(\tau^2), \\
&\qquad\qquad (i,j) \in \bar{\omega},\ 1 \leqslant n \leqslant N.
\end{aligned}$$

Performing the operator $\mathcal{H}_x^\beta \mathcal{H}_y^\gamma$ to both hand sides of the equality above arrives at

$$\begin{aligned}
&\mathcal{H}_x^\beta \mathcal{H}_y^\gamma {}_0^C D_t^\alpha u(x_i, y_j, t_{n-1+\sigma}) \\
&= K_1 \mathcal{H}_y^\gamma \left[\sigma \mathcal{H}_x^\beta \frac{\partial^\beta u(x_i, y_j, t_n)}{\partial |x|^\beta} + (1-\sigma) \mathcal{H}_x^\beta \frac{\partial^\beta u(x_i, y_j, t_{n-1})}{\partial |x|^\beta} \right] \\
&\quad + K_2 \mathcal{H}_x^\beta \left[\sigma \mathcal{H}_y^\gamma \frac{\partial^\gamma u(x_i, y_j, t_n)}{\partial |y|^\gamma} + (1-\sigma) \mathcal{H}_y^\gamma \frac{\partial^\gamma u(x_i, y_j, t_{n-1})}{\partial |y|^\gamma} \right] \\
&\quad + \mathcal{H}_x^\beta \mathcal{H}_y^\gamma f_{ij}^{n-1+\sigma} + O(\tau^2), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \qquad (5.73)
\end{aligned}$$

For the Caputo derivative in (5.73), using L2-1_σ approximation (1.81), it follows from Theorem 1.6.4 that

$${}_0^C D_t^\alpha u(x_i, y_j, t_{n-1+\sigma}) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (U_{ij}^{n-k} - U_{ij}^{n-k-1}) + O(\tau^{3-\alpha}). \qquad (5.74)$$

For the Riesz derivatives in (5.73), it follows from Theorem 1.5.2 that

$$\mathcal{H}_x^\beta \frac{\partial^\beta u(x_i, y_j, t_n)}{\partial |x|^\beta} = -h_1^{-\beta} \sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} U_{i-k,j}^n + O(h_1^4), \qquad (5.75)$$

$$\mathcal{H}_y^\gamma \frac{\partial^\gamma u(x_i, y_j, t_n)}{\partial |y|^\gamma} = -h_2^{-\gamma} \sum_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} U_{i,j-k}^n + O(h_2^4). \qquad (5.76)$$

Substituting (5.74)–(5.76) into (5.73) gives

$$\mathcal{H}_x^\beta \mathcal{H}_y^\gamma \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (U_{ij}^{n-k} - U_{ij}^{n-k-1})$$

$$
\begin{aligned}
&= K_1 \mathcal{H}_y^\gamma(-h_1^{-\beta}) \sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} [\sigma U_{i-k,j}^n + (1-\sigma) U_{i-k,j}^{n-1}] \\
&\quad + K_2 \mathcal{H}_x^\beta(-h_2^{-\gamma}) \sum_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} [\sigma U_{i,j-k}^n + (1-\sigma) U_{i,j-k}^{n-1}] \\
&\quad + \mathcal{H}_x^\beta \mathcal{H}_y^\gamma f_{ij}^{n-1+\sigma} + (r_4)_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \qquad (5.77)
\end{aligned}
$$

and there is a positive constant c_4 such that

$$
|(r_4)_{ij}^n| \leqslant c_4(\tau^2 + h_1^4 + h_2^4), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \qquad (5.78)
$$

Noticing the initial-boundary value conditions (5.51)–(5.52), we have

$$
\begin{cases}
U_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (5.79)\\
U_{ij}^n = 0, \quad (i,j) \in \partial\omega, \quad 0 \leqslant n \leqslant N. & (5.80)
\end{cases}
$$

Neglecting the small term $(r_4)_{ij}^n$ in (5.77) and replacing the exact solution U_{ij}^n with its numerical one u_{ij}^n, we get another difference scheme for solving (5.50)–(5.52) as follows:

$$
\begin{cases}
\mathcal{H}_x^\beta \mathcal{H}_y^\gamma \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum\limits_{k=0}^{n-1} c_k^{(n,\alpha)} (u_{ij}^{n-k} - u_{ij}^{n-k-1}) \\
= K_1 \mathcal{H}_y^\gamma(-h_1^{-\beta}) \sum\limits_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} [\sigma u_{i-k,j}^n + (1-\sigma) u_{i-k,j}^{n-1}] \\
\quad + K_2 \mathcal{H}_x^\beta(-h_2^{-\gamma}) \sum\limits_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} [\sigma u_{i,j-k}^n + (1-\sigma) u_{i,j-k}^{n-1}] \\
\quad + \mathcal{H}_x^\beta \mathcal{H}_y^\gamma f_{ij}^{n-1+\sigma}, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (5.81)\\
u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, & (5.82)\\
u_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (5.83)
\end{cases}
$$

In the subsequent part, the theoretical analyses on this scheme will be implemented.

5.4.2 Solvability of the difference scheme

Theorem 5.4.1. *The difference scheme* (5.81)–(5.83) *is uniquely solvable.*

Proof. Let

$$
u^n = \{u_{ij}^n \mid (i,j) \in \bar{\omega}\}.
$$

The value of u^0 is determined by (5.82)–(5.83).

Now suppose the values of $u^0, u^1, \dots, u^{n-1}$ have been obtained, then the linear system in u^n can be determined by (5.81) and (5.83). To show its unique solvability, it suffices to prove that the corresponding homogeneous one

$$\begin{cases} \frac{1}{s} c_0^{(n,\alpha)} \mathcal{H}_x^\beta \mathcal{H}_y^\gamma u_{ij}^n = -\sigma K_1 h_1^{-\beta} \mathcal{H}_y^\gamma \sum\limits_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} u_{i-k,j}^n \\ \qquad\qquad - \sigma K_2 h_2^{-\gamma} \mathcal{H}_x^\beta \sum\limits_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} u_{i,j-k}^n, \quad (i,j) \in \omega, & (5.84) \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega & (5.85) \end{cases}$$

has only the trivial solution.

Taking the inner product on both hand sides of (5.84) with u^n, we have

$$\begin{aligned} &\frac{1}{s} c_0^{(n,\alpha)} h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (\mathcal{H}_x^\beta \mathcal{H}_y^\gamma u_{ij}^n) u_{ij}^n \\ &= -K_1 \sigma h_1^{-\beta} h_1 \sum_{i=1}^{M_1-1} \sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} \left[h_2 \sum_{j=1}^{M_2-1} (\mathcal{H}_y^\gamma u_{i-k,j}^n) u_{ij}^n \right] \\ &\quad - K_2 \sigma h_2^{-\gamma} h_2 \sum_{j=1}^{M_2-1} \sum_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} \left[h_1 \sum_{i=1}^{M_1-1} (\mathcal{H}_x^\beta u_{i,j-k}^n) u_{ij}^n \right]. \end{aligned} \tag{5.86}$$

Denote $\boldsymbol{v}_i = (0, u_{i1}^n, u_{i2}^n, \dots, u_{i,M_2-1}^n, 0)^{\mathrm{T}}$, then the term $h_2 \sum_{j=1}^{M_2-1} (\mathcal{H}_y^\gamma u_{i-k,j}^n) \cdot u_{ij}^n$ can be taken as the inner product of $\boldsymbol{v}_{i-k}$ with $\boldsymbol{v}_i$. This inner product is similar to $(\cdot,\cdot)_A$ defined in Section 5.2.

Similarly, denote $\boldsymbol{w}_j = (0, u_{1j}^n, u_{2j}^n, \dots, u_{M_1-1,j}^n, 0)^{\mathrm{T}}$, then the term $h_1 \sum_{i=1}^{M_1-1} (\mathcal{H}_x^\beta u_{i,j-k}^n) u_{ij}^n$ can be taken as the inner product of $\boldsymbol{w}_{j-k}$ with $\boldsymbol{w}_j$. This inner product is also similar to $(\cdot,\cdot)_A$ defined in Section 5.2.

It follows from Corollary 5.1.1 and (5.29) that

$$\begin{aligned} &- h_1^{-\beta} h_1 \sum_{i=1}^{M_1-1} \sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} \left[h_2 \sum_{j=1}^{M_2-1} (\mathcal{H}_y^\gamma u_{i-k,j}^n) u_{ij}^n \right] \\ &\leqslant -c_*^{(\beta)} (2L_1)^{-\beta} h_1 \sum_{i=1}^{M_1-1} \left[h_2 \sum_{j=1}^{M_2-1} (\mathcal{H}_y^\gamma u_{ij}^n) u_{ij}^n \right] \\ &\leqslant -\frac{2}{3} c_*^{(\beta)} (2L_1)^{-\beta} \|u^n\|^2 \end{aligned} \tag{5.87}$$

and

$$- h_2^{-\gamma} h_2 \sum_{j=1}^{M_2-1} \sum_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} \left[h_1 \sum_{i=1}^{M_1-1} (\mathcal{H}_x^\beta u_{i,j-k}^n) u_{ij}^n \right]$$

$$\leqslant -c_*^{(\gamma)}(2L_2)^{-\gamma}h_2 \sum_{j=1}^{M_2-1}\left[h_1\sum_{i=1}^{M_1-1}(\mathcal{H}_x^\beta u_{ij}^n)u_{ij}^n\right]$$
$$\leqslant -\frac{2}{3}c_*^{(\gamma)}(2L_2)^{-\gamma}\|u^n\|^2. \tag{5.88}$$

On the other hand, we have

$$h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(\mathcal{H}_x^\beta\mathcal{H}_y^\gamma u_{ij}^n)u_{ij}^n \geqslant \frac{1}{3}\|u^n\|^2. \tag{5.89}$$

Substituting (5.87)–(5.89) into (5.86) produces $\|u^n\| = 0$. Then $u^n = 0$ is followed by combining with (5.85).

By the principle of induction, the difference scheme (5.81)–(5.83) is uniquely solvable. The proof ends. □

5.4.3 Stability of the difference scheme

Theorem 5.4.2. *Suppose* $\{u_{ij}^n \mid (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme* (5.81)–(5.83)*, then it holds*

$$\|u^n\|^2 \leqslant 3\|u^0\|^2 + \frac{9}{16}\left[\frac{(2L_1)^\beta}{K_1c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2c_*^{(\gamma)}}\right]\Gamma(1-\alpha)\max_{1\leqslant m\leqslant n}\{t_m^\alpha\|\mathcal{H}_x^\beta\mathcal{H}_y^\gamma f^{m-1+\sigma}\|^2\},$$
$$1 \leqslant n \leqslant N, \tag{5.90}$$

where

$$\|\mathcal{H}_x^\beta\mathcal{H}_y^\gamma f^{m-1+\sigma}\|^2 = h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(\mathcal{H}_x^\beta\mathcal{H}_y^\gamma f_{ij}^{m-1+\sigma})^2.$$

Proof. Taking the inner product on both hand sides of (5.81) with $\sigma u^n + (1-\sigma)u^{n-1}$, we have

$$\frac{1}{s}\sum_{k=0}^{n-1}c_k^{(n,\alpha)}h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}[\mathcal{H}_x^\beta\mathcal{H}_y^\gamma(u_{ij}^{n-k}-u_{ij}^{n-k-1})][\sigma u_{ij}^n + (1-\sigma)u_{ij}^{n-1}]$$
$$= K_1h_2\sum_{j=1}^{M_2-1}\Bigg\{-h_1^{-\beta}h_1\sum_{i=1}^{M_1-1}\sum_{k=i-M_1}^{i}\hat{g}_k^{(\beta)}[\mathcal{H}_y^\gamma(\sigma u_{i-k,j}^n + (1-\sigma)u_{i-k,j}^{n-1})]$$
$$\cdot[\sigma u_{ij}^n + (1-\sigma)u_{ij}^{n-1}]\Bigg\}$$
$$+ K_2h_1\sum_{i=1}^{M_1-1}\Bigg\{(-h_2^{-\gamma})h_2\sum_{j=1}^{M_2-1}\sum_{k=j-M_2}^{j}\hat{g}_k^{(\gamma)}[\mathcal{H}_x^\beta(\sigma u_{i,j-k}^n + (1-\sigma)u_{i,j-k}^{n-1})]$$

$$\cdot\left[\sigma u_{ij}^{n} + (1-\sigma)u_{ij}^{n-1}\right]\Bigg\}$$

$$+ h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} (\mathcal{H}_x^{\beta}\mathcal{H}_y^{\gamma} f_{ij}^{n-1+\sigma})\left[\sigma u_{ij}^{n} + (1-\sigma)u_{ij}^{n-1}\right]. \tag{5.91}$$

Now each term in (5.91) will be estimated.

Suppose $u, v \in \mathring{\mathcal{V}}_h$. Define the inner product

$$(u,v)_H = h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} (\mathcal{H}_x^{\beta}\mathcal{H}_y^{\gamma} u_{ij}) v_{ij}.$$

It follows from Lemma 2.6.1 that

$$\begin{aligned}
&\sum_{k=0}^{n-1} c_k^{(n,\alpha)} h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \left[\mathcal{H}_x^{\beta}\mathcal{H}_y^{\gamma}(u_{ij}^{n-k} - u_{ij}^{n-k-1})\right]\left[\sigma u_{ij}^{n} + (1-\sigma)u_{ij}^{n-1}\right]\\
&= \sum_{k=0}^{n-1} c_k^{(n,\alpha)} (u^{n-k} - u^{n-k-1}, \sigma u^{n} + (1-\sigma)u^{n-1})_H\\
&\geqslant \frac{1}{2}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}\left[(u^{n-k}, u^{n-k})_H - (u^{n-k-1}, u^{n-k-1})_H\right].
\end{aligned} \tag{5.92}$$

For the former two terms on the right-hand side of (5.91), by Corollary 5.1.1, similar to (5.87) and (5.88), we have

$$\begin{aligned}
&h_2 \sum_{j=1}^{M_2-1}\Bigg\{-h_1^{-\beta} h_1 \sum_{i=1}^{M_1-1}\sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)}\left[\mathcal{H}_y^{\gamma}(\sigma u_{i-k,j}^{n} + (1-\sigma)u_{i-k,j}^{n-1})\right]\\
&\quad\cdot\left[\sigma u_{ij}^{n} + (1-\sigma)u_{ij}^{n-1}\right]\Bigg\}\\
&\leqslant -\frac{2}{3}c_*^{(\beta)}(2L_1)^{-\beta}\left\|\sigma u^{n} + (1-\sigma)u^{n-1}\right\|^2
\end{aligned} \tag{5.93}$$

and

$$\begin{aligned}
&h_1 \sum_{i=1}^{M_1-1}\Bigg\{(-h_2^{-\gamma}) h_2 \sum_{j=1}^{M_2-1}\sum_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)}\left[\mathcal{H}_x^{\beta}(\sigma u_{i,j-k}^{n} + (1-\sigma)u_{i,j-k}^{n-1})\right]\\
&\quad\cdot\left[\sigma u_{ij}^{n} + (1-\sigma)u_{ij}^{n-1}\right]\Bigg\}\\
&\leqslant -\frac{2}{3}c_*^{(\gamma)}(2L_2)^{-\gamma}\left\|\sigma u^{n} + (1-\sigma)u^{n-1}\right\|^2.
\end{aligned} \tag{5.94}$$

For the last term on the right-hand side of (5.91), it follows from the Cauchy–Schwarz inequality that

$$h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} (\mathcal{H}_x^{\beta}\mathcal{H}_y^{\gamma} f_{ij}^{n-1+\sigma})\left[\sigma u_{ij}^{n} + (1-\sigma)u_{ij}^{n-1}\right]$$

$$\leqslant \|\mathcal{H}_x^{\beta}\mathcal{H}_y^{\gamma} f^{n-1+\sigma}\| \cdot \|\sigma u^n + (1-\sigma)u^{n-1}\|$$
$$\leqslant \left[\frac{2}{3}K_1 c_*^{(\beta)}(2L_1)^{-\beta} + \frac{2}{3}K_2 c_*^{(\gamma)}(2L_2)^{-\gamma}\right]\|\sigma u^n + (1-\sigma)u^{n-1}\|^2$$
$$+ \frac{3}{32}\left[\frac{(2L_1)^{\beta}}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^{\gamma}}{K_2 c_*^{(\gamma)}}\right]\|\mathcal{H}_x^{\beta}\mathcal{H}_y^{\gamma} f^{n-1+\sigma}\|^2. \tag{5.95}$$

Substituting (5.92)–(5.95) into (5.91) arrives at

$$\frac{1}{s}\sum_{k=0}^{n-1} c_k^{(n,\alpha)}[(u^{n-k}, u^{n-k})_H - (u^{n-k-1}, u^{n-k-1})_H]$$
$$\leqslant \frac{3}{16}\left[\frac{(2L_1)^{\beta}}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^{\gamma}}{K_2 c_*^{(\gamma)}}\right]\|\mathcal{H}_x^{\beta}\mathcal{H}_y^{\gamma} f^{n-1+\sigma}\|^2, \quad 1 \leqslant n \leqslant N. \tag{5.96}$$

By Lemma 1.6.3, we have

$$\frac{s}{c_{n-1}^{(n,\alpha)}} \leqslant t_n^{\alpha}\Gamma(1-\alpha).$$

Then it follows from (5.96) that

$$c_0^{(n,\alpha)}(u^n, u^n)_H$$
$$\leqslant \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})(u^{n-k}, u^{n-k})_H + c_{n-1}^{(n,\alpha)}(u^0, u^0)_H$$
$$+ \frac{3}{16}s\left[\frac{(2L_1)^{\beta}}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^{\gamma}}{K_2 c_*^{(\gamma)}}\right]\|\mathcal{H}_x^{\beta}\mathcal{H}_y^{\gamma} f^{n-1+\sigma}\|^2$$
$$\leqslant \sum_{k=1}^{n-1}(c_{k-1}^{(n,\alpha)} - c_k^{(n,\alpha)})(u^{n-k}, u^{n-k})_H + c_{n-1}^{(n,\alpha)}\Big\{(u^0, u^0)_H$$
$$+ \frac{3}{16}\left[\frac{(2L_1)^{\beta}}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^{\gamma}}{K_2 c_*^{(\gamma)}}\right]\Gamma(1-\alpha)t_n^{\alpha}\|\mathcal{H}_x^{\beta}\mathcal{H}_y^{\gamma} f^{n-1+\sigma}\|^2\Big\}, \quad 1 \leqslant n \leqslant N.$$

The application of the inductive process can yield

$$(u^n, u^n)_H \leqslant (u^0, u^0)_H + \frac{3}{16}\left[\frac{(2L_1)^{\beta}}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^{\gamma}}{K_2 c_*^{(\gamma)}}\right]\Gamma(1-\alpha)$$
$$\cdot \max_{1\leqslant m\leqslant n}\{t_m^{\alpha}\|\mathcal{H}_x^{\beta}\mathcal{H}_y^{\gamma} f^{m-1+\sigma}\|^2\}, \quad 1 \leqslant n \leqslant N. \tag{5.97}$$

Noticing

$$\frac{1}{3}\|u^n\|^2 \leqslant (u^n, u^n)_H \leqslant \|u^n\|^2,$$

the inequality (5.90) can be obtained from (5.97). The proof ends. □

5.4.4 Convergence of the difference scheme

Theorem 5.4.3. *Suppose* $\{U_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *and* $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (5.50)–(5.52) *and the difference scheme* (5.81)–(5.83), *respectively. Let*

$$e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N,$$

then it holds

$$\|e^n\| \leqslant \frac{3}{4}\sqrt{\left[\frac{(2L_1)^\beta}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2 c_*^{(\gamma)}}\right]\Gamma(1-\alpha)T^\alpha L_1 L_2\, c_4(\tau^2 + h_1^4 + h_2^4)}, \quad 1 \leqslant n \leqslant N. \tag{5.98}$$

Proof. Subtracting (5.81)–(5.83) from (5.77), (5.79)–(5.80), respectively, produces the system of error equations as follows:

$$\begin{cases} \dfrac{1}{s}\displaystyle\sum_{k=0}^{n-1} c_k^{(n,\alpha)} \mathcal{H}_x^\beta \mathcal{H}_y^\gamma (e_{ij}^{n-k} - e_{ij}^{n-k-1}) \\ = K_1 \mathcal{H}_y^\gamma (-h_1^{-\beta}) \displaystyle\sum_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} [\sigma e_{i-k,j}^n + (1-\sigma) e_{i-k,j}^{n-1}] \\ \quad + K_2 \mathcal{H}_x^\beta (-h_2^{-\gamma}) \displaystyle\sum_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} [\sigma e_{i,j-k}^n + (1-\sigma) e_{i,j-k}^{n-1}] + (r_4)_{ij}^n, \\ \qquad\qquad\qquad\qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ e_{ij}^0 = 0, \quad (i,j) \in \omega, \\ e_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (5.78), the application of Theorem 5.4.2 immediately yields

$$\begin{aligned} \|e^n\|^2 &\leqslant \frac{9}{16}\left[\frac{(2L_1)^\beta}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2 c_*^{(\gamma)}}\right]\Gamma(1-\alpha) \max_{1\leqslant m\leqslant n}\{t_m^\alpha \|(r_4)^m\|^2\} \\ &\leqslant \frac{9}{16}\left[\frac{(2L_1)^\beta}{K_1 c_*^{(\beta)}} + \frac{(2L_2)^\gamma}{K_2 c_*^{(\gamma)}}\right]\Gamma(1-\alpha) T^\alpha L_1 L_2 c_4^2 (\tau^2 + h_1^4 + h_2^4)^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Taking the square root on both hand sides of the inequality above will reach (5.98). The proof ends. □

5.5 Supplementary remarks and discussions

1. The finite difference methods for 1D and 2D time-space-fractional Bloch–Torrey equations were discussed in this chapter. The time Caputo fractional derivative was

handled by L2-1_σ approximation of order $3-\alpha$, and the space Riesz fractional derivatives or the weighted values of space Riesz derivatives at three points were approximated by the fractional central difference quotient formula (Theorem 1.5.1 or Theorem 1.5.2). Several difference schemes were derived and for each of them, the unique solvability, stability and convergence in L^2 norm were proved[80].

2. In [110], the following fourth-order numerical differentiation formula to approximate the Riesz derivatives was established:

$$\left(-\frac{\beta}{24}\right)\left[-\frac{\Delta_h^\beta f(x-h)}{h^\beta}\right]+\left(1+\frac{\beta}{12}\right)\left[-\frac{\Delta_h^\beta f(x)}{h^\beta}\right]$$
$$+\left(-\frac{\beta}{24}\right)\left[-\frac{\Delta_h^\beta f(x+h)}{h^\beta}\right]=\frac{\partial^\beta f(x)}{\partial|x|^\beta}+O(h^4). \quad (5.99)$$

Comparing (1.54) with (5.99), the former one is to use the fractional central difference quotient formula to approximate the weighted value of Riesz derivatives at three points, whereas, the latter one is to use the weighted value of the fractional central difference quotient formula to approximate the Riesz derivative at one point.

3. In [106], Yu et al. investigated the numerical solutions of 3D time-space-fractional Bloch–Torrey equations, where the time Caputo derivative was discretized by the L1 formula and the space Riesz derivatives were approximated by the shifted G-L formula (1.47) and a positive-type difference scheme was derived. The unconditional stability and convergence of the resultant scheme were proved by the maximum principle and the convergence order in the maximum norm was $O(\tau^{2-\alpha}+h_x+h_y+h_z)$. In [107], the authors studied the numerical solutions of 2D time-space-fractional Bloch–Torrey equations, where the time Caputo derivative was discretized by the L1 formula and the space Riesz derivatives were approximated by the fractional central difference quotient formula (Theorem 1.5.1, or [4]) and a positive-type difference scheme of order $O(\tau^{2-\alpha}+h_x^2+h_y^2)$ was proposed. The unconditional stability and convergence of the resultant scheme were proved by the maximum principle and the convergence order in the maximum norm was $O(\tau^{2-\alpha}+h_x^2+h_y^2)$. The method in [107] can also be used to solve the 3D time-space-fractional Bloch–Torrey equations by the finite difference method[73].

4. Ran and Zhang[65] derived a two-level Crank–Nicolson difference scheme and a three-level linearized difference scheme for the nonlinear time-space-fractional Schrödinger equations.

5. Xu and Sun[100] developed a fast second-order difference scheme for the time-space fractional equation.

Exercises 5

5.1 For the problem (5.1)–(5.3), construct the following difference scheme:

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)}u_i^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_i^k - a_{n-1}^{(\alpha)}u_i^0\right] \\ = -h^{-\beta}\sum\limits_{k=i-M}^{i}\hat{g}_k^{(\beta)}u_{i-k}^n + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\tilde{u}(x,t)$ like that in Section 5.1 and suppose $\tilde{u}(\cdot,t) \in \mathscr{C}^{2+\beta}(\mathcal{R})$. For this difference scheme, try to
(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the initial value φ and the function f;
(4) show the convergence and derive the error expression.

5.2 For the problem (5.1)–(5.3), construct the following difference scheme:

$$\begin{cases} \mathcal{A}_h^{\beta}\dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)}u_i^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_i^k - a_{n-1}^{(\alpha)}u_i^0\right] \\ = -h^{-\beta}\sum\limits_{k=i-M}^{i}\hat{g}_k^{(\beta)}u_{i-k}^n + \mathcal{A}_h^{\beta}f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\tilde{u}(x,t)$ like that in Section 5.1 and suppose $\tilde{u}(\cdot,t) \in \mathscr{C}^{4+\beta}(\mathcal{R})$. For this difference scheme, try to
(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the initial value φ and the function f;
(4) show the convergence and derive the error expression.

5.3 For the problem (5.50)–(5.52), construct the following difference scheme:

$$\begin{cases} \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)}u_{ij}^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_{ij}^k - a_{n-1}^{(\alpha)}u_{ij}^0\right] \\ = K_1(-h_1^{-\beta})\sum\limits_{k=i-M_1}^{i}\hat{g}_k^{(\beta)}u_{i-k,j}^n + K_2(-h_2^{-\gamma})\sum\limits_{k=j-M_2}^{j}\hat{g}_k^{(\gamma)}u_{i,j-k}^n + f_{ij}^n, \\ \qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

Define the functions $\hat{v}(x, y, t)$ and $\hat{w}(x, y, t)$ like those in Section 4.4 and suppose $\hat{v}(\cdot, y, t) \in \mathscr{C}^{2+\beta}(\mathcal{R})$, $\hat{w}(x, \cdot, t) \in \mathscr{C}^{2+\gamma}(\mathcal{R})$.
For this difference scheme, try to
(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the initial value φ and the function f;
(4) show the convergence and derive the error expression.

5.4 For the problem (5.50)–(5.52), construct the following difference scheme:

$$\begin{cases} \mathcal{H}_x^\beta \mathcal{H}_y^\gamma \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)} u_{ij}^n - \sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_{ij}^k - a_{n-1}^{(\alpha)} u_{ij}^0\right] \\ = K_1(-h_1^{-\beta})\mathcal{H}_y^\gamma \sum\limits_{k=i-M_1}^{i} \hat{g}_k^{(\beta)} u_{i-k,j}^n + K_2(-h_2^{-\gamma})\mathcal{H}_x^\beta \sum\limits_{k=j-M_2}^{j} \hat{g}_k^{(\gamma)} u_{i,j-k}^n + \mathcal{H}_x^\beta \mathcal{H}_y^\gamma f_{ij}^n, \\ \qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

Define the functions $\hat{v}(x, y, t)$ and $\hat{w}(x, y, t)$ like those in Section 4.4 and suppose $\hat{v}(\cdot, y, t) \in \mathscr{C}^{4+\beta}(\mathcal{R})$, $\hat{w}(x, \cdot, t) \in \mathscr{C}^{4+\gamma}(\mathcal{R})$.
For this difference scheme, try to
(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the initial value φ and the function f;
(4) show the convergence and derive the error expression.

6 Difference methods for the time distributed-order subdiffusion equations

In the previous chapters, numerical solutions of the multiterm time-fractional differential equations have been discussed. When the number of terms in the time-fractional derivatives tends to the infinity, the time distributed-order differential equation is derived. It can be used to model the complex processes with the varying diffusion exponents as the time, such as the retarding subdiffusion, superslow diffusion, accelerating superdiffusion and so on. Numerous applications in polymer physics, kinetics of particles moving in the quenched random force fields, iterated map models, etc. have been found. In this chapter, the finite difference methods for solving a class of time distributed-order subdiffusion equations will be concerned. For each scheme, the unique solvability, stability and convergence will be investigated. The entire chapter consists of 7 sections.

6.1 The second-order method in both space and distributed order for 1D problem

Consider the following 1D initial-boundary value problem of time distributed-order subdiffusion equation

$$\begin{cases} \mathcal{D}_t^w u(x,t) = u_{xx}(x,t) + f(x,t), \quad 0 < x < L,\ 0 < t \leqslant T, & (6.1)\\ u(x,0) = 0, \quad 0 < x < L, & (6.2)\\ u(0,t) = \varphi_1(t), \quad u(L,t) = \varphi_2(t), \quad 0 \leqslant t \leqslant T, & (6.3)\end{cases}$$

where $\varphi_1(0) = 0$, $\varphi_2(0) = 0$ and

$$\mathcal{D}_t^w u(x,t) = \int_0^1 w(\alpha)\, {}_0^C D_t^\alpha u(x,t) \mathrm{d}\alpha,$$

$${}_0^C D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} u_\xi(x,\xi)\mathrm{d}\xi, & 0 \leqslant \alpha < 1,\\ u_t(x,t), & \alpha = 1,\end{cases}$$

$w(\alpha) \geqslant 0$, $\int_0^1 w(\alpha)\mathrm{d}\alpha = c_0 > 0$, the functions f, φ_1 and φ_2 are given.

Define the function $\hat{u}(x,t)$ like that in Section 2.1. Suppose $\hat{u}(x,\cdot) \in \mathscr{C}^{2+1}(\mathcal{R})$ and $u(\cdot,t) \in C^4[0,L]$.

https://doi.org/10.1515/9783110616064-006

6.1.1 Derivation of the difference scheme

Take three positive integers J, M, N and denote $\Delta\alpha = \frac{1}{2J}$, $h = \frac{L}{M}$, $\tau = \frac{T}{N}$; $\alpha_l = l\Delta\alpha$ $(0 \leqslant l \leqslant 2J)$; $x_i = ih$ $(0 \leqslant i \leqslant M)$; $t_n = n\tau$ $(0 \leqslant n \leqslant N)$. Introduce the same mesh function spaces and notations like those in Section 2.1.

The distributed-order integral is firstly discretized with proper quadrature formulae. The composite trapezoid formula is listed below.

Lemma 6.1.1. (Composite trapezoid formula) *Suppose function $s \in C^2[0,1]$, then it holds*

$$\int_0^1 s(\alpha)\mathrm{d}\alpha = \Delta\alpha \sum_{l=0}^{2J} c_l s(\alpha_l) - \frac{\Delta\alpha^2}{12} s''(\xi), \quad \xi \in (0,1),$$

where

$$c_l = \begin{cases} \frac{1}{2}, & l = 0,\ 2J, \\ 1, & 1 \leqslant l \leqslant 2J-1. \end{cases}$$

Define the mesh functions

$$U_i^n = u(x_i, t_n), \quad f_i^n = f(x_i, t_n), \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N.$$

Considering equation (6.1) at the point (x_i, t_n), we have

$$\mathcal{D}_t^w u(x_i, t_n) = u_{xx}(x_i, t_n) + f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{6.4}$$

Let

$$s(\alpha, x_i, t_n) = w(\alpha)\ {}_0^C D_t^\alpha u(x_i, t_n).$$

Suppose function $s(\cdot, x_i, t_n) \in C^2[0,1]$. It follows from Lemma 6.1.1 that

$$\begin{aligned} \mathcal{D}_t^w u(x_i, t_n) &= \int_0^1 s(\alpha, x_i, t_n)\mathrm{d}\alpha \\ &= \Delta\alpha \sum_{l=0}^{2J} c_l s(\alpha_l, x_i, t_n) - \frac{\Delta\alpha^2}{12} \frac{\partial^2 s(\alpha, x_i, t_n)}{\partial\alpha^2}\bigg|_{\alpha=\xi_i^n} \\ &= \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\ {}_0^C D_t^{\alpha_l} u(x_i, t_n) + O(\Delta\alpha^2), \end{aligned} \tag{6.5}$$

where $\xi_i^n \in (0,1)$.

By Corollary 1.4.1, associated with (6.2), we have

$${}_0^C D_t^\alpha u(x_i, t_n) = {}_0\mathbf{D}_t^\alpha u(x_i, t_n)$$

$$= \tau^{-\alpha} \sum_{k=0}^{n} w_k^{(\alpha)} U_i^{n-k} + O(\tau^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{6.6}$$

where $\{w_k^{(\alpha)}\}$ is defined by (1.33)–(1.34). Substituting (6.6) into (6.5) produces

$$\mathcal{D}_t^w u(x_i, t_n) = \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} U_i^{n-k} + O(\tau^2 + \Delta\alpha^2),$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{6.7}$$

Notice

$$u_{xx}(x_i, t_n) = \delta_x^2 U_i^n + O(h^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{6.8}$$

Substituting (6.7) and (6.8) into (6.4) arrives at

$$\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} U_i^{n-k} = \delta_x^2 U_i^n + f_i^n + (r_1)_i^n,$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{6.9}$$

and there is a positive constant c_1 such that

$$|(r_1)_i^n| \leqslant c_1(\tau^2 + h^2 + \Delta\alpha^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{6.10}$$

Noticing the initial-boundary value conditions (6.2)–(6.3), we have

$$\begin{cases} U_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, & (6.11) \\ U_0^n = \varphi_1(t_n), \quad U_M^n = \varphi_2(t_n), \quad 0 \leqslant n \leqslant N. & (6.12) \end{cases}$$

Neglecting the small term $(r_1)_i^n$ in (6.9) and replacing the exact solution U_i^n with its numerical one u_i^n, we get a difference scheme for solving (6.1)–(6.3) as follows:

$$\begin{cases} \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} u_i^{n-k} = \delta_x^2 u_i^n + f_i^n, & \\ \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (6.13) \\ u_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, & (6.14) \\ u_0^n = \varphi_1(t_n), \quad u_M^n = \varphi_2(t_n), \quad 0 \leqslant n \leqslant N. & (6.15) \end{cases}$$

In the subsequent part, denote

$$\mu = \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \tau^{-\alpha_l} w_0^{(\alpha_l)}. \tag{6.16}$$

6.1.2 Solvability of the difference scheme

Theorem 6.1.1. *The difference scheme* (6.13)–(6.15) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The value of u^0 is determined by (6.14)–(6.15).

Now assume that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n can be obtained from (6.13) and (6.15). To show its unique solvability, it suffices to verify that the corresponding homogeneous one

$$\begin{cases} \mu\, u_i^n = \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (6.17) \\ u_0^n = u_M^n = 0 & (6.18) \end{cases}$$

has only the trivial solution.

Reformulate (6.17) as

$$\left(\mu + \frac{2}{h^2}\right) u_i^n = \frac{1}{h^2}(u_{i-1}^n + u_{i+1}^n), \quad 1 \leqslant i \leqslant M-1.$$

Suppose $\|u^n\|_\infty = |u_{i_n}^n|$, where $i_n \in \{1, 2, \ldots, M-1\}$. Letting $i = i_n$ in the equality above and taking the absolute value of both hand sides, the application of the triangle inequality produces

$$\left(\mu + \frac{2}{h^2}\right)\|u^n\|_\infty \leqslant \frac{2}{h^2}\|u^n\|_\infty,$$

which implies $\|u^n\|_\infty = 0$, hence $u^n = 0$.

By the principle of induction, the difference scheme (6.13)–(6.15) is uniquely solvable. The proof ends. □

For the analyses on the stability and convergence of the difference scheme (6.13)–(6.15), three useful lemmas are essential.

6.1.3 Three lemmas

Definition 6.1.1. The matrix

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{3-n} & t_{2-n} \\ t_2 & t_1 & t_0 & \cdots & t_{4-n} & t_{3-n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_{n-2} & t_{n-3} & t_{n-4} & \cdots & t_0 & t_{-1} \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 & t_0 \end{bmatrix} \tag{6.19}$$

is called the **Toeplitz** matrix. That is, the entries of an $n \times n$ Toeplitz matrix T_n are constant along each diagonal.

Definition 6.1.2. [6] If the entries $\{t_k\}_{k=1-n}^{n-1}$ of the Toeplitz matrix (6.19) are Fourier coefficients of function $f(x)$, that is,

$$t_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,$$

then the function $f(x)$ is called the generating function of matrix T_n.

Lemma 6.1.2. [5, 6] (Grenander–Szegö theorem) *For the* Toeplitz *matrix T_n given by* (6.19), *if its generating function $f(x)$ defined on $[-\pi,\pi]$ is continuous and real-valued, then*

$$f_{\min} \leqslant \lambda_{\min}(T_n) \leqslant \lambda_{\max}(T_n) \leqslant f_{\max},$$

where $f_{\min}$ and $f_{\max}$ denote the minimum and maximum values of $f(x)$, respectively, $\lambda_{\min}(T_n)$ and $\lambda_{\max}(T_n)$ denote the smallest and largest eigenvalues of T_n, respectively.

Moreover, if $f_{\min} < f_{\max}$, then for $n \geqslant 1$, any eigenvalue $\lambda(T_n)$ of T_n satisfies

$$f_{\min} < \lambda(T_n) < f_{\max}.$$

In particular, T_n is positive definite when $f_{\min} > 0$.

Lemma 6.1.3. [96] *Let the coefficient $\{w_k^{(\alpha)}\}$ be defined by* (1.33)–(1.34). *Then, for any vector $(v_0, v_1, \ldots, v_m)^{\mathrm{T}} \in \mathcal{R}^{m+1}$, it holds*

$$\sum_{n=0}^{m}\left(\sum_{k=0}^{n} w_k^{(\alpha)} v_{n-k}\right)v_n \geqslant 0.$$

Proof. Notice the fact that the quadratic form

$$\sum_{n=0}^{m}\left(\sum_{k=0}^{n} w_k^{(\alpha)} v_{n-k}\right)v_n$$

is nonnegative is equivalent to that the symmetric Toeplitz matrix

$$W = \begin{bmatrix} w_0^{(\alpha)} & \frac{1}{2}w_1^{(\alpha)} & \frac{1}{2}w_2^{(\alpha)} & \cdots & \frac{1}{2}w_m^{(\alpha)} \\ \frac{1}{2}w_1^{(\alpha)} & w_0^{(\alpha)} & \frac{1}{2}w_1^{(\alpha)} & \cdots & \frac{1}{2}w_{m-1}^{(\alpha)} \\ \frac{1}{2}w_2^{(\alpha)} & \frac{1}{2}w_1^{(\alpha)} & w_0^{(\alpha)} & \cdots & \frac{1}{2}w_{m-2}^{(\alpha)} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{2}w_m^{(\alpha)} & \frac{1}{2}w_{m-1}^{(\alpha)} & \frac{1}{2}w_{m-2}^{(\alpha)} & \cdots & w_0^{(\alpha)} \end{bmatrix}$$

is positive semidefinite.

Next, we try to show that the Toeplitz matrix W is positive semidefinite.
The generating function of W is given by

$$\begin{aligned}
f(\alpha,x) &= w_0^{(\alpha)} + \frac{1}{2}\sum_{k=1}^{\infty} w_k^{(\alpha)} \mathrm{e}^{\mathrm{i}kx} + \frac{1}{2}\sum_{k=1}^{\infty} w_k^{(\alpha)} \mathrm{e}^{-\mathrm{i}kx} \\
&= \left(1+\frac{\alpha}{2}\right) g_0^{(\alpha)} + \frac{1}{2}\sum_{k=1}^{\infty}\left[\left(1+\frac{\alpha}{2}\right) g_k^{(\alpha)} - \frac{\alpha}{2} g_{k-1}^{(\alpha)}\right]\mathrm{e}^{\mathrm{i}kx} \\
&\quad + \frac{1}{2}\sum_{k=1}^{\infty}\left[\left(1+\frac{\alpha}{2}\right) g_k^{(\alpha)} - \frac{\alpha}{2} g_{k-1}^{(\alpha)}\right]\mathrm{e}^{-\mathrm{i}kx} \\
&= \frac{1}{2}\left(1+\frac{\alpha}{2}\right)\sum_{k=0}^{\infty} g_k^{(\alpha)} \mathrm{e}^{\mathrm{i}kx} + \frac{1}{2}\left(1+\frac{\alpha}{2}\right)\sum_{k=0}^{\infty} g_k^{(\alpha)} \mathrm{e}^{-\mathrm{i}kx} \\
&\quad - \frac{\alpha}{4}\sum_{k=0}^{\infty} g_k^{(\alpha)} \mathrm{e}^{\mathrm{i}(k+1)x} - \frac{\alpha}{4}\sum_{k=0}^{\infty} g_k^{(\alpha)} \mathrm{e}^{-\mathrm{i}(k+1)x} \\
&= \frac{1}{2}\left(1+\frac{\alpha}{2}\right)(1-\mathrm{e}^{\mathrm{i}x})^{\alpha} + \frac{1}{2}\left(1+\frac{\alpha}{2}\right)(1-\mathrm{e}^{-\mathrm{i}x})^{\alpha} \\
&\quad - \frac{\alpha}{4}(1-\mathrm{e}^{\mathrm{i}x})^{\alpha}\mathrm{e}^{\mathrm{i}x} - \frac{\alpha}{4}(1-\mathrm{e}^{-\mathrm{i}x})^{\alpha}\mathrm{e}^{-\mathrm{i}x}.
\end{aligned}$$

As we see, the function $f(\alpha,x)$ is an even function with respect to x with the period 2π, hence we only need to consider function $f(\alpha,x)$ for $x \in [0,\pi]$. Reformulate $f(\alpha,x)$ as

$$\begin{aligned}
f(\alpha,x) &= \frac{1}{2}\left(1+\frac{\alpha}{2}\right)[(\mathrm{e}^{-\frac{\mathrm{i}}{2}x} - \mathrm{e}^{\frac{\mathrm{i}}{2}x})\mathrm{e}^{\frac{\mathrm{i}}{2}x}]^{\alpha} \\
&\quad + \frac{1}{2}\left(1+\frac{\alpha}{2}\right)[(\mathrm{e}^{\frac{\mathrm{i}}{2}x} - \mathrm{e}^{-\frac{\mathrm{i}}{2}x})\mathrm{e}^{-\frac{\mathrm{i}}{2}x}]^{\alpha} \\
&\quad - \frac{\alpha}{4}[(\mathrm{e}^{-\frac{\mathrm{i}}{2}x} - \mathrm{e}^{\frac{\mathrm{i}}{2}x})\mathrm{e}^{\frac{\mathrm{i}}{2}x}]^{\alpha}\mathrm{e}^{\mathrm{i}x} - \frac{\alpha}{4}[(\mathrm{e}^{\frac{\mathrm{i}}{2}x} - \mathrm{e}^{-\frac{\mathrm{i}}{2}x})\mathrm{e}^{-\frac{\mathrm{i}}{2}x}]^{\alpha}\mathrm{e}^{-\mathrm{i}x} \\
&= \frac{1}{2}\left(1+\frac{\alpha}{2}\right)\left[2\mathrm{i}\sin\left(-\frac{x}{2}\right)\mathrm{e}^{\frac{\mathrm{i}}{2}x}\right]^{\alpha} + \frac{1}{2}\left(1+\frac{\alpha}{2}\right)\left[2\mathrm{i}\sin\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{\mathrm{i}}{2}x}\right]^{\alpha} \\
&\quad - \frac{\alpha}{4}\left[2\mathrm{i}\sin\left(-\frac{x}{2}\right)\mathrm{e}^{\frac{\mathrm{i}}{2}x}\right]^{\alpha}\mathrm{e}^{\mathrm{i}x} - \frac{\alpha}{4}\left[2\mathrm{i}\sin\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{\mathrm{i}}{2}x}\right]^{\alpha}\mathrm{e}^{-\mathrm{i}x} \\
&= \left[2\sin\left(\frac{x}{2}\right)\right]^{\alpha}\left\{\frac{1}{2}\left(1+\frac{\alpha}{2}\right)[\mathrm{e}^{\mathrm{i}(\frac{x}{2}-\frac{\pi}{2})\alpha} + \mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\frac{x}{2})\alpha}]\right. \\
&\quad \left. - \frac{\alpha}{4}[\mathrm{e}^{\mathrm{i}(\frac{x}{2}-\frac{\pi}{2})\alpha}\cdot \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\frac{x}{2})\alpha}\cdot \mathrm{e}^{-\mathrm{i}x}]\right\} \\
&= \left[2\sin\left(\frac{x}{2}\right)\right]^{\alpha}\left\{\left(1+\frac{\alpha}{2}\right)\cos\left[\frac{\alpha}{2}(\pi-x)\right] - \frac{\alpha}{2}\cos\left[\frac{\alpha}{2}(\pi-x)-x\right]\right\}.
\end{aligned}$$

Let

$$h(\alpha,x) = \left(1+\frac{\alpha}{2}\right)\cos\left[\frac{\alpha}{2}(\pi-x)\right] - \frac{\alpha}{2}\cos\left[\frac{\alpha}{2}(\pi-x)-x\right].$$

One can easily check that

$$\begin{aligned} h_x(\alpha,x) &= \frac{\alpha}{2}\left(1+\frac{\alpha}{2}\right)\left\{\sin\left[\frac{\alpha}{2}(\pi-x)\right]-\sin\left[\frac{\alpha}{2}(\pi-x)-x\right]\right\} \\ &= \frac{\alpha}{2}\left(1+\frac{\alpha}{2}\right)\left\{\sin\left[\left(\frac{\alpha}{2}(\pi-x)-\frac{x}{2}\right)+\frac{x}{2}\right]\right. \\ &\qquad \left.-\sin\left[\left(\frac{\alpha}{2}(\pi-x)-\frac{x}{2}\right)-\frac{x}{2}\right]\right\} \\ &= \alpha\left(1+\frac{\alpha}{2}\right)\cos\left[\frac{\alpha}{2}(\pi-x)-\frac{x}{2}\right]\sin\left(\frac{x}{2}\right). \end{aligned}$$

Therefore, $h_x(\alpha,x) \geqslant 0$ when $x \in [0,\pi]$. Hence, $h(\alpha,x) \geqslant h(\alpha,0) = \cos\frac{\alpha\pi}{2} \geqslant 0$, which implies that $f(\alpha,x) \geqslant 0$.

The lemma now follows as a result of Lemma 6.1.2. The proof ends. □

Lemma 6.1.4. *For the constant μ defined by* (6.16), *it holds*

$$\mu\tau = O\left(\frac{1}{|\ln\tau|}\right).$$

Proof.

$$\begin{aligned} \mu\tau &= \Delta\alpha\sum_{l=0}^{2J} c_l w(\alpha_l)\cdot\frac{1}{\tau^{\alpha_l}}\left(1+\frac{\alpha_l}{2}\right)\tau \\ &\sim \int_0^1 w(\alpha)\left(1+\frac{\alpha}{2}\right)\tau^{1-\alpha}\mathrm{d}\alpha \\ &= w(\alpha^*)\left(1+\frac{\alpha^*}{2}\right)\int_0^1 \tau^{1-\alpha}\mathrm{d}\alpha \\ &= w(\alpha^*)\left(1+\frac{\alpha^*}{2}\right)\frac{\tau^{1-\alpha}}{|\ln\tau|}\bigg|_{\alpha=0}^{1} \\ &= w(\alpha^*)\left(1+\frac{\alpha^*}{2}\right)\frac{1-\tau}{|\ln\tau|}, \end{aligned}$$

where $\alpha^* \in (0,1)$. Hence, $\mu\tau = O(\frac{1}{|\ln\tau|})$. The proof ends. □

6.1.4 Stability of the difference scheme

Theorem 6.1.2. *Suppose $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ is the solution of the difference scheme*

$$\begin{cases} \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} v_i^{n-k} = \delta_x^2 v_i^n + g_i^n, \\ \qquad\qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ v_i^0 = \varphi_i, \quad 1 \leqslant i \leqslant M-1, \\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases} \tag{6.20}$$

Then it holds

$$\tau \sum_{n=1}^{m} \|\delta_x v^n\|^2 \leqslant 2\mu\tau\|v^0\|^2 + \frac{L^2}{6}\tau \sum_{n=1}^{m}\|g^n\|^2, \quad 1 \leqslant m \leqslant N,$$

where

$$\|g^n\|^2 = h\sum_{i=1}^{M-1}(g_i^n)^2.$$

Proof. Making the inner product on both hand sides of (6.20) with v^n, it follows from Lemma 2.1.1 that

$$\begin{aligned} &\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)}(v^{n-k}, v^n) \\ &= (\delta_x^2 v^n, v^n) + (g^n, v^n) \\ &= -\|\delta_x v^n\|^2 + (g^n, v^n) \\ &\leqslant -\|\delta_x v^n\|^2 + \frac{3}{L^2}\|v^n\|^2 + \frac{L^2}{12}\|g^n\|^2 \\ &\leqslant -\|\delta_x v^n\|^2 + \frac{1}{2}\|\delta_x v^n\|^2 + \frac{L^2}{12}\|g^n\|^2 \\ &= -\frac{1}{2}\|\delta_x v^n\|^2 + \frac{L^2}{12}\|g^n\|^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Summing up for n from 1 to m produces

$$\begin{aligned} &\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum_{n=1}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(v^{n-k}, v^n) \\ &\leqslant -\frac{1}{2}\sum_{n=1}^{m}\|\delta_x v^n\|^2 + \frac{L^2}{12}\sum_{n=1}^{m}\|g^n\|^2, \quad 1 \leqslant m \leqslant N. \end{aligned}$$

Adding the term $\mu(v^0, v^0)$ to both hand sides of the inequality above gives

$$\begin{aligned} &\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(v^{n-k}, v^n) \\ &\leqslant -\frac{1}{2}\sum_{n=1}^{m}\|\delta_x v^n\|^2 + \mu(v^0, v^0) + \frac{L^2}{12}\sum_{n=1}^{m}\|g^n\|^2, \quad 1 \leqslant m \leqslant N. \end{aligned} \tag{6.21}$$

By Lemma 6.1.3, we have

$$\sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(v^{n-k}, v^n) = h\sum_{i=1}^{M-1}\sum_{n=0}^{m}\left(\sum_{k=0}^{n} w_k^{(\alpha_l)} v_i^{n-k}\right)v_i^n \geqslant 0. \tag{6.22}$$

It is clear from (6.21) and (6.22) that

$$\tau\sum_{n=1}^{m}\|\delta_x v^n\|^2 \leqslant 2\mu\tau\|v^0\|^2 + \frac{L^2}{6}\tau\sum_{n=1}^{m}\|g^n\|^2, \quad 1 \leqslant m \leqslant N.$$

The proof ends. □

6.1.5 Convergence of the difference scheme

Theorem 6.1.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (6.1)–(6.3) *and the difference scheme* (6.13)–(6.15), *respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then it holds

$$\tau\sum_{n=1}^{N}\|e^n\|_\infty \leqslant \frac{\sqrt{6}}{12}L^2 T c_1(\tau^2 + h^2 + \Delta\alpha^2).$$

Proof. Subtracting (6.13)–(6.15) from (6.9), (6.11)–(6.12), respectively, we get the system of error equations as follows:

$$\begin{cases} \Delta\alpha\sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l}\sum_{k=0}^{n} w_k^{(\alpha_l)} e_i^{n-k} = \delta_x^2 e_i^n + (r_1)_i^n, \\ \qquad\qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (6.10), the application of Theorem 6.1.2 produces

$$\tau\sum_{n=1}^{N}\|\delta_x e^n\|^2 \leqslant \frac{L^2}{6}\tau\sum_{n=1}^{N}\|(r_1)^n\|^2 \leqslant \frac{L^2}{6}TL[c_1(\tau^2 + h^2 + \Delta\alpha^2)]^2.$$

By the Cauchy–Schwarz inequality, Lemma 2.1.1 and the inequality above, we have

$$\left(\tau\sum_{n=1}^{N}\|e^n\|_\infty\right)^2 \leqslant \left(\tau\sum_{n=1}^{N}1\right)\left(\tau\sum_{n=1}^{N}\|e^n\|_\infty^2\right)$$

$$\leqslant T \cdot \frac{L}{4} \tau \sum_{n=1}^{N} \|\delta_x e^n\|^2$$

$$\leqslant \frac{LT}{4} \cdot \frac{L^3}{6} T [c_1(\tau^2 + h^2 + \Delta\alpha^2)]^2,$$

that is,

$$\tau \sum_{n=1}^{N} \|e^n\|_\infty \leqslant \frac{\sqrt{6}}{12} L^2 T c_1(\tau^2 + h^2 + \Delta\alpha^2).$$

The proof ends. □

6.2 The fourth-order method in both space and distributed order for 1D problem

In this section, we continue to consider the problem (6.1)–(6.3) and want to develop a difference method of order two in time and four in both space and distributed order. The unique solvability, stability and convergence of the proposed difference scheme are also proved.

Define the function $\hat{u}(x,t)$ like that in Section 2.1. Suppose $\hat{u}(x,\cdot) \in \mathscr{C}^{2+1}(\mathcal{R})$ and $u(\cdot,t) \in C^6[0,L]$.

6.2.1 Derivation of the difference scheme

At the beginning, two useful lemmas are presented that will be used later on.

Lemma 6.2.1. (Composite Simpson formula) *Suppose function $s \in C^4[0,1]$, then it holds*

$$\int_0^1 s(\alpha)\mathrm{d}\alpha = \Delta\alpha \sum_{l=0}^{2J} d_l s(\alpha_l) - \frac{\Delta\alpha^4}{180} s^{(4)}(\eta), \quad \eta \in (0,1),$$

where

$$d_l = \begin{cases} \frac{1}{3}, & l = 0,\ 2J, \\ \frac{2}{3}, & l = 2,\ 4,\ \ldots,\ 2J-4,\ 2J-2, \\ \frac{4}{3}, & l = 1,\ 3,\ \ldots,\ 2J-3,\ 2J-1. \end{cases}$$

Denote

$$\nu = \Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \tau^{-\alpha_l} w_0^{(\alpha_l)}. \tag{6.23}$$

Lemma 6.2.2. *For the constant v defined by* (6.23), *it holds*

$$v\tau = O\Big(\frac{1}{|\ln \tau|}\Big).$$

Proof. The proof can be proceeded with the same trick used in the proof for Lemma 6.1.4 and the details are omitted here. □

Now we begin to build the difference scheme for (6.1)–(6.3).

Considering equation (6.1) at the point (x_i, t_n), we have

$$\mathcal{D}_t^w u(x_i, t_n) = u_{xx}(x_i, t_n) + f_i^n, \quad 0 \leqslant i \leqslant M,\ 1 \leqslant n \leqslant N.$$

Performing the operator $\mathcal{A}$ to both hand sides of the equality above yields

$$\mathcal{A}\mathcal{D}_t^w u(x_i, t_n) = \mathcal{A}u_{xx}(x_i, t_n) + \mathcal{A}f_i^n, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{6.24}$$

where the operator $\mathcal{A}$ is defined in Section 2.1.

Let

$$s(\alpha, x_i, t_n) = w(\alpha)\ {}_0^C D_t^\alpha u(x_i, t_n).$$

Suppose function $s(\cdot, x_i, t_n) \in C^4[0,1]$. It follows from Lemma 6.2.1 that

$$\begin{aligned}\mathcal{D}_t^w u(x_i, t_n) &= \int_0^1 s(\alpha, x_i, t_n)\mathrm{d}\alpha \\ &= \Delta\alpha \sum_{l=0}^{2J} d_l s(\alpha_l, x_i, t_n) - \frac{\Delta\alpha^4}{180}\frac{\partial^4 s(\alpha, x_i, t_n)}{\partial \alpha^4}\Big|_{\alpha=\eta_i^n} \\ &= \Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\ {}_0^C D_t^{\alpha_l} u(x_i, t_n) + O(\Delta\alpha^4),\end{aligned} \tag{6.25}$$

where $\eta_i^n \in (0,1)$.

Noticing (6.2), Corollary 1.4.1 and (6.25), we have

$$\mathcal{D}_t^w u(x_i, t_n) = \Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\Big[\tau^{-\alpha_l}\sum_{k=0}^{n} w_k^{(\alpha_l)} U_i^{n-k} + O(\tau^2)\Big] + O(\Delta\alpha^4),$$
$$0 \leqslant i \leqslant M,\ 1 \leqslant n \leqslant N. \tag{6.26}$$

It follows from Lemma 2.1.3 that

$$\mathcal{A}u_{xx}(x_i, t_n) = \delta_x^2 U_i^n + O(h^4), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{6.27}$$

Substituting (6.26) and (6.27) into (6.24) gives

$$\mathcal{A}\Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\left[\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} U_i^{n-k}\right] = \delta_x^2 U_i^n + \mathcal{A} f_i^n + (r_2)_i^n,$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \tag{6.28}$$

and there is a positive constant c_2 such that

$$|(r_2)_i^n| \leqslant c_2(\tau^2 + h^4 + \Delta\alpha^4), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{6.29}$$

Noticing the initial-boundary value conditions (6.2)–(6.3), we have

$$\begin{cases} U_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, & (6.30)\\ U_0^n = \varphi_1(t_n), \quad U_M^n = \varphi_2(t_n), \quad 0 \leqslant n \leqslant N. & (6.31)\end{cases}$$

Omitting the small term $(r_2)_i^n$ in (6.28) and replacing the exact solution U_i^n with its numerical one u_i^n, another difference scheme for solving (6.1)–(6.3) is produced as

$$\begin{cases} \mathcal{A}\Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\left[\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} u_i^{n-k}\right] = \delta_x^2 u_i^n + \mathcal{A} f_i^n, & \\ \qquad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, & (6.32)\\ u_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, & (6.33)\\ u_0^n = \varphi_1(t_n), \quad u_M^n = \varphi_2(t_n), \quad 0 \leqslant n \leqslant N. & (6.34)\end{cases}$$

We now turn to the analyses on this difference scheme.

6.2.2 Solvability of the difference scheme

Theorem 6.2.1. *The difference scheme* (6.32)–(6.34) *is uniquely solvable.*

Proof. Let

$$u^n = (u_0^n, u_1^n, \ldots, u_M^n).$$

The value of u^0 is determined by (6.33)–(6.34).

Now suppose the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the linear system in u^n can be obtained from (6.32) and (6.34). To show its unique solvability, it suffices to verify that the corresponding homogeneous one

$$\begin{cases} \nu \mathcal{A} u_i^n = \delta_x^2 u_i^n, \quad 1 \leqslant i \leqslant M-1, & (6.35)\\ u_0^n = u_M^n = 0 & (6.36)\end{cases}$$

has only the trivial solution.

Making the inner product on both hand sides of (6.35) with u^n arrives at

$$\nu(\mathcal{A}u^n, u^n) = (\delta_x^2 u^n, u^n) = -\|\delta_x u^n\|^2.$$

Noticing

$$(\mathcal{A}u^n, u^n) = \|u^n\|^2 - \frac{h^2}{12}\|\delta_x u^n\|^2 \geqslant \frac{2}{3}\|u^n\|^2,$$

we have

$$\frac{2}{3}\nu\|u^n\|^2 \leqslant -\|\delta_x u^n\|^2 \leqslant 0,$$

thus $\|u^n\| = 0$. Then $u^n = 0$ is followed by noticing (6.36).

By the principle of induction, the theorem is true. The proof ends. □

6.2.3 Stability of the difference scheme

Theorem 6.2.2. *Suppose* $\{v_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \mathcal{A}\Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\left[\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} v_i^{n-k}\right] = \delta_x^2 v_i^n + g_i^n, \\ \qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (6.37) \\ v_i^0 = \varphi_i, \quad 1 \leqslant i \leqslant M-1, & (6.38) \\ v_0^n = 0, \quad v_M^n = 0, \quad 0 \leqslant n \leqslant N. & (6.39) \end{cases}$$

Then it holds

$$\tau \sum_{n=1}^{m} \|\delta_x v^n\|^2 \leqslant 3\nu\tau\|v^0\|^2 + \frac{3L^2}{8}\tau \sum_{n=1}^{m} \|g^n\|^2, \quad 1 \leqslant m \leqslant N,$$

where

$$\|g^n\|^2 = h \sum_{i=1}^{M-1} (g_i^n)^2.$$

Proof. Taking the inner product on both hand sides of (6.37) with $\mathcal{A}v^n$, it follows from Lemma 2.1.1, Lemma 2.1.2 and the Cauchy–Schwarz inequality that

$$\begin{aligned} &\Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} (\mathcal{A}v^{n-k}, \mathcal{A}v^n) \\ &= (\delta_x^2 v^n, \mathcal{A}v^n) + (g^n, \mathcal{A}v^n) \end{aligned}$$

$$
\begin{aligned}
&= -\|\delta_x v^n\|_A^2 + (g^n, \mathcal{A}v^n)\\
&\leqslant -\frac{2}{3}\|\delta_x v^n\|^2 + \|g^n\| \cdot \|\mathcal{A}v^n\|\\
&\leqslant -\frac{2}{3}\|\delta_x v^n\|^2 + \|g^n\| \cdot \|v^n\|\\
&\leqslant -\frac{2}{3}\|\delta_x v^n\|^2 + \frac{2}{L^2}\|v^n\|^2 + \frac{L^2}{8}\|g^n\|^2\\
&\leqslant -\frac{2}{3}\|\delta_x v^n\|^2 + \frac{1}{3}\|\delta_x v^n\|^2 + \frac{L^2}{8}\|g^n\|^2\\
&= -\frac{1}{3}\|\delta_x v^n\|^2 + \frac{L^2}{8}\|g^n\|^2, \quad 1 \leqslant n \leqslant N.
\end{aligned}
$$

Summing up for n from 1 to m leads to

$$
\begin{aligned}
&\Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l} \sum_{n=1}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(\mathcal{A}v^{n-k}, \mathcal{A}v^n)\\
&\leqslant -\frac{1}{3}\sum_{n=1}^{m}\|\delta_x v^n\|^2 + \frac{L^2}{8}\sum_{n=1}^{m}\|g^n\|^2, \quad 1 \leqslant m \leqslant N.
\end{aligned}
$$

Adding the term $\nu(\mathcal{A}v^0, \mathcal{A}v^0)$ to both hand sides of the inequality above yields

$$
\begin{aligned}
&\Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l} \sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(\mathcal{A}v^{n-k}, \mathcal{A}v^n)\\
&\leqslant -\frac{1}{3}\sum_{n=1}^{m}\|\delta_x v^n\|^2 + \nu(\mathcal{A}v^0, \mathcal{A}v^0) + \frac{L^2}{8}\sum_{n=1}^{m}\|g^n\|^2, \quad 1 \leqslant m \leqslant N.
\end{aligned} \tag{6.40}
$$

By Lemma 6.1.3, we have

$$
\sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(\mathcal{A}v^{n-k}, \mathcal{A}v^n) = h\sum_{i=1}^{M-1}\sum_{n=0}^{m}\left[\sum_{k=0}^{n} w_k^{(\alpha_l)}(\mathcal{A}v_i^{n-k})\right](\mathcal{A}v_i^n) \geqslant 0. \tag{6.41}
$$

It follows from (6.40), (6.41) and Lemma 2.1.1 that

$$
\begin{aligned}
\tau\sum_{n=1}^{m}\|\delta_x v^n\|^2 &\leqslant 3\nu\tau(\mathcal{A}v^0, \mathcal{A}v^0) + \frac{3L^2}{8}\tau\sum_{n=1}^{m}\|g^n\|^2\\
&\leqslant 3\nu\tau\|v^0\|^2 + \frac{3L^2}{8}\tau\sum_{n=1}^{m}\|g^n\|^2, \quad 1 \leqslant m \leqslant N.
\end{aligned}
$$

The proof ends. □

6.2.4 Convergence of the difference scheme

Theorem 6.2.3. *Suppose* $\{U_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (6.1)–(6.3) *and the difference scheme* (6.32)–(6.34), *respec-*

tively. Let

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N,$$

then it holds

$$\tau \sum_{n=1}^{N} \|e^n\|_\infty \leqslant \frac{\sqrt{6}}{8} L^2 T c_2 (\tau^2 + h^4 + \Delta\alpha^4).$$

Proof. The subtraction of (6.32)–(6.34) from (6.28), (6.30)–(6.31), respectively, produces the system of error equations as follows:

$$\begin{cases} \mathcal{A}\Delta\alpha \sum\limits_{l=0}^{2J} d_l w(\alpha_l) \left[\tau^{-\alpha_l} \sum\limits_{k=0}^{n} w_k^{(\alpha_l)} e_i^{n-k} \right] = \delta_x^2 e_i^n + (r_2)_i^n, \\ \qquad\qquad\qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ e_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (6.29), the application of Theorem 6.2.2 immediately yields

$$\begin{aligned} \tau \sum_{n=1}^{N} \|\delta_x e^n\|^2 &\leqslant \frac{3L^2}{8} \tau \sum_{n=1}^{N} \|(r_2)^n\|^2 \\ &\leqslant \frac{3L^2}{8} \tau \sum_{n=1}^{N} L[c_2(\tau^2 + h^4 + \Delta\alpha^4)]^2 \\ &\leqslant \frac{3L^3}{8} T [c_2(\tau^2 + h^4 + \Delta\alpha^4)]^2. \end{aligned}$$

It follows from the Cauchy–Schwarz inequality, Lemma 2.1.1 and the inequality above that

$$\begin{aligned} \left(\tau \sum_{n=1}^{N} \|e^n\|_\infty \right)^2 &\leqslant \left(\tau \sum_{n=1}^{N} 1 \right) \left(\tau \sum_{n=1}^{N} \|e^n\|_\infty^2 \right) \\ &\leqslant T \cdot \frac{L}{4} \tau \sum_{n=1}^{N} \|\delta_x e^n\|^2 \\ &\leqslant \frac{LT}{4} \cdot \frac{3L^3}{8} T [c_2(\tau^2 + h^4 + \Delta\alpha^4)]^2. \end{aligned}$$

Taking the square root on both hand sides of the inequality above gives

$$\tau \sum_{n=1}^{N} \|e^n\|_\infty \leqslant \frac{\sqrt{6}}{8} L^2 T c_2 (\tau^2 + h^4 + \Delta\alpha^4).$$

The proof ends. □

6.3 The second-order method in both space and distributed order for 2D problem

Consider the following 2D initial-boundary value problem of time distributed-order subdiffusion equation

$$\begin{cases} \mathcal{D}_t^w u(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) + f(x,y,t), \\ \qquad\qquad\qquad\qquad (x,y) \in \Omega,\ t \in (0,T], & (6.42) \\ u(x,y,0) = 0, \quad (x,y) \in \Omega, & (6.43) \\ u(x,y,t) = \varphi(x,y,t), \quad (x,y) \in \partial\Omega,\ t \in [0,T], & (6.44) \end{cases}$$

where $\Omega = (0,L_1) \times (0,L_2)$, $\partial\Omega$ is the boundary of Ω; When $(x,y) \in \partial\Omega$, $\varphi(x,y,0) = 0$;

$$\mathcal{D}_t^w u(x,y,t) = \int_0^1 w(\alpha)\ {}_0^C D_t^\alpha u(x,y,t)\mathrm{d}\alpha,$$

$${}_0^C D_t^\alpha u(x,y,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} u_\xi(x,y,\xi)\mathrm{d}\xi, & 0 \leqslant \alpha < 1, \\ u_t(x,y,t), & \alpha = 1, \end{cases}$$

$w(\alpha) \geqslant 0$, $\int_0^1 w(\alpha)\mathrm{d}\alpha = c_0 > 0$, the functions f and φ are given.

In this section, a second-order difference scheme for solving (6.42)–(6.44) will be considered and its unique solvability, stability and convergence will also be illustrated.

Take the same mesh partition and notations like those in Section 2.10 in both space and time directions. Besides, for any mesh function $u \in \mathcal{V}_h$, define

$$\Delta_h u_{ij} = \delta_x^2 u_{ij} + \delta_y^2 u_{ij}.$$

For any mesh functions $u, v \in \mathring{\mathcal{V}}_h$, let

$$(\Delta_h u, \Delta_h v) = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (\Delta_h u_{ij})(\Delta_h v_{ij}), \quad \|\Delta_h u\| = \sqrt{(\Delta_h u, \Delta_h u)}.$$

The following lemma is true.

Lemma 6.3.1. [70, 76] *For any mesh function $u \in \mathring{\mathcal{V}}_h$, there is a positive constant c such that*

$$\|u\|_\infty \leqslant c\|\Delta_h u\|.$$

More precisely, we have $c = \sqrt{\frac{1}{32}[\pi(L_1^2 + L_2^2) + \frac{2L_1^3 L_2^3}{(L_1^2+L_2^2)^2}]}$.

Define the function $\hat{u}(x,y,t)$ like that in Section 2.10. Suppose $\hat{u}(x,y,\cdot) \in \mathscr{C}^{2+1}(\mathcal{R})$ and $u(\cdot,\cdot,t) \in C^{(4,4)}(\bar{\Omega})$.

6.3.1 Derivation of the difference scheme

Define the mesh functions

$$U_{ij}^n = u(x_i, y_j, t_n), \quad f_{ij}^n = f(x_i, y_j, t_n), \quad (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N.$$

Considering equation (6.42) at the point (x_i, y_j, t_n), we have

$$\mathcal{D}_t^w u(x_i, y_j, t_n) = u_{xx}(x_i, y_j, t_n) + u_{yy}(x_i, y_j, t_n) + f_{ij}^n,$$
$$(i,j) \in \omega,\ 1 \leqslant n \leqslant N. \tag{6.45}$$

Let

$$s(\alpha, x_i, y_j, t_n) = w(\alpha)\ {}_0^C D_t^\alpha u(x_i, y_j, t_n).$$

Suppose function $s(\cdot, x_i, y_j, t_n) \in C^2[0,1]$. From Lemma 6.1.1, Corollary 1.4.1 and (6.43), we have

$$\begin{aligned} &\mathcal{D}_t^w u(x_i, y_j, t_n) \\ &= \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\ {}_0^C D_t^{\alpha_l} u(x_i, y_j, t_n) + O(\Delta\alpha^2) \\ &= \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \left[\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} U_{ij}^{n-k} + O(\tau^2) \right] + O(\Delta\alpha^2). \end{aligned} \tag{6.46}$$

It follows from Lemma 2.1.3 that

$$u_{xx}(x_i, y_j, t_n) = \delta_x^2 U_{ij}^n + O(h_1^2), \quad u_{yy}(x_i, y_j, t_n) = \delta_y^2 U_{ij}^n + O(h_2^2). \tag{6.47}$$

Substituting (6.46) and (6.47) into (6.45) gives

$$\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \left[\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} U_{ij}^{n-k} \right] = \Delta_h U_{ij}^n + f_{ij}^n + (r_3)_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \tag{6.48}$$

and there is a positive constant c_3 such that

$$\left|(r_3)_{ij}^n\right| \leqslant c_3(\tau^2 + h_1^2 + h_2^2 + \Delta\alpha^2), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \tag{6.49}$$

Noticing the initial-boundary value conditions (6.43)–(6.44), we have

$$\begin{cases} U_{ij}^0 = 0, \quad (i,j) \in \omega, & (6.50) \\ U_{ij}^n = \varphi(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (6.51) \end{cases}$$

Neglecting the small term $(r_3)_{ij}^n$ in (6.48) and replacing the exact solution U_{ij}^n with its numerical one u_{ij}^n, we get a difference scheme for solving (6.42)–(6.44) in the form of

$$\begin{cases} \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\left[\tau^{-\alpha_l}\sum_{k=0}^{n} w_k^{(\alpha_l)} u_{ij}^{n-k}\right] = \Delta_h u_{ij}^n + f_{ij}^n, \\ \qquad\qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (6.52) \\ u_{ij}^0 = 0, \quad (i,j) \in \omega, & (6.53) \\ u_{ij}^n = \varphi(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (6.54) \end{cases}$$

Next, we aim to make some analyses on the difference scheme (6.52)–(6.54).

6.3.2 Solvability of the difference scheme

Theorem 6.3.1. *The difference scheme* (6.52)–(6.54) *is uniquely solvable.*

Proof. Let

$$u^n = \{u_{ij}^n \mid (i,j) \in \bar{\omega}\}.$$

The value of u^0 is determined by (6.53)–(6.54).

Now assume that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then we can obtain the linear system in the unknown u^n from (6.52) and (6.54). To show its unique solvability, it is sufficient to verify that the corresponding homogeneous one

$$\begin{cases} \mu u_{ij}^n = \Delta_h u_{ij}^n, \quad (i,j) \in \omega, & (6.55) \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega & (6.56) \end{cases}$$

has only the trivial solution.

Rewrite (6.55) as follows:

$$\left(\mu + \frac{2}{h_1^2} + \frac{2}{h_2^2}\right)u_{ij}^n = \frac{1}{h_1^2}(u_{i-1,j}^n + u_{i+1,j}^n) + \frac{1}{h_2^2}(u_{i,j-1}^n + u_{i,j+1}^n), \quad (i,j) \in \omega.$$

Suppose $\|u^n\|_\infty = |u_{i_n,j_n}^n|$, where $(i_n, j_n) \in \omega$. Letting $(i,j) = (i_n, j_n)$ in the equality above and taking the absolute value on both hand sides of the result, the application of the triangle inequality gives

$$\left(\mu + \frac{2}{h_1^2} + \frac{2}{h_2^2}\right)\|u^n\|_\infty \leqslant \frac{2}{h_1^2}\|u^n\|_\infty + \frac{2}{h_2^2}\|u^n\|_\infty,$$

therefore, $\|u^n\|_\infty = 0$. Then $u^n = 0$ is concluded from the combination with (6.56).

By the principle of induction, the difference scheme (6.52)–(6.54) is uniquely solvable. The proof ends. □

6.3.3 Stability of the difference scheme

Theorem 6.3.2. *Suppose* $\{v_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} v_{ij}^{n-k} = \Delta_h v_{ij}^n + g_{ij}^n, \\ \qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (6.57) \\ v_{ij}^0 = \varphi_{ij}, \quad (i,j) \in \omega, & (6.58) \\ v_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (6.59) \end{cases}$$

Then it holds

$$\tau \sum_{n=1}^{m} \|\Delta_h v^n\|^2 \leqslant 2\mu\tau \|\nabla_h v^0\|^2 + \tau \sum_{n=1}^{m} \|g^n\|^2, \quad 1 \leqslant m \leqslant N,$$

where

$$\|g^n\|^2 = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (g_{ij}^n)^2.$$

Proof. Making the inner product on both hand sides of (6.57) with $-\Delta_h v^n$, it follows from the Cauchy–Schwarz inequality that

$$\begin{aligned} &\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} (v^{n-k}, -\Delta_h v^n) \\ &= -(\Delta_h v^n, \Delta_h v^n) - (g^n, \Delta_h v^n) \\ &\leqslant -\|\Delta_h v^n\|^2 + \|g^n\| \cdot \|\Delta_h v^n\| \\ &\leqslant -\|\Delta_h v^n\|^2 + \frac{1}{2}\|\Delta_h v^n\|^2 + \frac{1}{2}\|g^n\|^2 \\ &= -\frac{1}{2}\|\Delta_h v^n\|^2 + \frac{1}{2}\|g^n\|^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Summing up for n from 1 to m and adding the term $\mu(v^0, -\Delta_h v^0)$ to both hand sides of the obtained inequality arrive at

$$\begin{aligned} &\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \left[\sum_{n=0}^{m} \sum_{k=0}^{n} w_k^{(\alpha_l)} (v^{n-k}, -\Delta_h v^n) \right] \\ &\leqslant -\frac{1}{2} \sum_{n=1}^{m} \|\Delta_h v^n\|^2 + \mu(v^0, -\Delta_h v^0) + \frac{1}{2} \sum_{n=1}^{m} \|g^n\|^2, \quad 1 \leqslant m \leqslant N. \end{aligned} \tag{6.60}$$

By Lemma 6.1.3, we have

$$\sum_{n=0}^{m} \sum_{k=0}^{n} w_k^{(\alpha_l)} (v^{n-k}, -\Delta_h v^n)$$

$$
\begin{aligned}
&= \sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}[(\delta_x v^{n-k}, \delta_x v^n) + (\delta_y v^{n-k}, \delta_y v^n)] \\
&= h_1 h_2 \sum_{i=1}^{M_1}\sum_{j=1}^{M_2-1}\left[\sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(\delta_x v_{i-\frac{1}{2},j}^{n-k})(\delta_x v_{i-\frac{1}{2},j}^{n})\right] \\
&\quad + h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2}\left[\sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(\delta_y v_{i,j-\frac{1}{2}}^{n-k})(\delta_y v_{i,j-\frac{1}{2}}^{n})\right] \\
&\geqslant 0.
\end{aligned} \tag{6.61}
$$

Combining (6.60) with (6.61) leads to

$$
\begin{aligned}
\tau \sum_{n=1}^{m} \|\Delta_h v^n\|^2 &\leqslant 2\mu\tau(v^0, -\Delta_h v^0) + \tau \sum_{n=1}^{m} \|g^n\|^2 \\
&= 2\mu\tau\|\nabla_h v^0\|^2 + \tau \sum_{n=1}^{m} \|g^n\|^2, \quad 1 \leqslant m \leqslant N.
\end{aligned}
$$

The proof ends. □

6.3.4 Convergence of the difference scheme

Theorem 6.3.3. *Suppose* $\{U_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *and* $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (6.42)–(6.44) *and the difference scheme* (6.52)–(6.54), *respectively. Let*

$$
e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N,
$$

then it holds

$$
\tau \sum_{n=1}^{N} \|e^n\|_\infty \leqslant cT\sqrt{L_1 L_2}\, c_3(\tau^2 + h_1^2 + h_2^2 + \Delta\alpha^2), \tag{6.62}
$$

where the constant c is defined in Lemma 6.3.1.

Proof. Subtracting (6.52)–(6.54) from (6.48), (6.50)–(6.51), respectively, we get the system of error equations as follows:

$$
\begin{cases}
\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\left[\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} e_{ij}^{n-k}\right] = \Delta_h e_{ij}^n + (r_3)_{ij}^n, \\
\qquad\qquad\qquad\qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\
e_{ij}^0 = 0, \quad (i,j) \in \omega, \\
e_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N.
\end{cases}
$$

Noticing (6.49), the application of Theorem 6.3.2 yields

$$\begin{aligned}\tau\sum_{n=1}^{N}\|\Delta_h e^n\|^2 &\leqslant \tau\sum_{n=1}^{N}\|(r_3)^n\|^2\\ &\leqslant \tau\sum_{n=1}^{N}L_1L_2[c_3(\tau^2+h_1^2+h_2^2+\Delta\alpha^2)]^2\\ &\leqslant TL_1L_2[c_3(\tau^2+h_1^2+h_2^2+\Delta\alpha^2)]^2. \qquad (6.63)\end{aligned}$$

It follows from the Cauchy–Schwarz inequality, Lemma 6.3.1 and (6.63) that

$$\begin{aligned}\left(\tau\sum_{n=1}^{N}\|e^n\|_\infty\right)^2 &\leqslant \left(\tau\sum_{n=1}^{N}1\right)\left(\tau\sum_{n=1}^{N}\|e^n\|_\infty^2\right)\\ &\leqslant Tc^2\tau\sum_{n=1}^{N}\|\Delta_h e^n\|^2\\ &\leqslant c^2T^2L_1L_2[c_3(\tau^2+h_1^2+h_2^2+\Delta\alpha^2)]^2,\end{aligned}$$

which implies (6.62). The proof ends. □

6.4 The fourth-order method in both space and distributed order for 2D problem

This section is devoted to a difference method of order two in time and four in both space and distributed order for the problem (6.42)–(6.44).

Define the function $\hat{u}(x,y,t)$ like that in Section 2.10. Suppose $\hat{u}(x,y,\cdot)\in\mathscr{C}^{2+1}(\mathcal{R})$ and $u(\cdot,\cdot,t)\in C^{(6,6)}(\bar{\Omega})$.

6.4.1 Derivation of the difference scheme

Considering equation (6.42) at the point (x_i,y_j,t_n), we have

$$\mathcal{D}_t^w u(x_i,y_j,t_n)=u_{xx}(x_i,y_j,t_n)+u_{yy}(x_i,y_j,t_n)+f_{ij}^n,\quad (i,j)\in\bar{\omega},\ 1\leqslant n\leqslant N.$$

Performing the operator $\mathcal{A}_x\mathcal{A}_y$ to both hand sides of the equality above and noticing Lemma 2.1.3, we have

$$\begin{aligned}&\mathcal{A}_x\mathcal{A}_y\mathcal{D}_t^w u(x_i,y_j,t_n)\\ &=\mathcal{A}_y\Big(\mathcal{A}_x u_{xx}(x_i,y_j,t_n)\Big)+\mathcal{A}_x\Big(\mathcal{A}_y u_{yy}(x_i,y_j,t_n)\Big)+\mathcal{A}_x\mathcal{A}_y f_{ij}^n\end{aligned}$$

$$= \mathcal{A}_y \delta_x^2 U_{ij}^n + \mathcal{A}_x \delta_y^2 U_{ij}^n + \mathcal{A}_x \mathcal{A}_y f_{ij}^n + O(h_1^4 + h_2^4),$$
$$(i,j) \in \omega, \ 1 \leqslant n \leqslant N, \tag{6.64}$$

where the operators $\mathcal{A}_x$ and $\mathcal{A}_y$ are defined in Section 3.10.

Let

$$s(\alpha, x_i, y_j, t_n) = w(\alpha) \, {}_0^C D_t^\alpha u(x_i, y_j, t_n).$$

Suppose function $s(\cdot, x_i, y_j, t_n) \in C^4[0,1]$. By Lemma 6.2.1 and Corollary 1.4.1, we have

$$\begin{aligned}\mathcal{D}_t^w u(x_i, y_j, t_n) &= \Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \ {}_0^C D_t^{\alpha_l} u(x_i, y_j, t_n) + O(\Delta\alpha^4) \\ &= \Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \left[\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} U_{ij}^{n-k} + O(\tau^2) \right] + O(\Delta\alpha^4). \end{aligned} \tag{6.65}$$

Substituting (6.65) into (6.64) yields

$$\begin{aligned}&\mathcal{A}_x \mathcal{A}_y \Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \left[\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} U_{ij}^{n-k} \right] \\ &= \mathcal{A}_y \delta_x^2 U_{ij}^n + \mathcal{A}_x \delta_y^2 U_{ij}^n + \mathcal{A}_x \mathcal{A}_y f_{ij}^n + (r_4)_{ij}^n, \quad (i,j) \in \omega, \ 1 \leqslant n \leqslant N, \end{aligned} \tag{6.66}$$

and there is a positive constant c_4 such that

$$\left|(r_4)_{ij}^n\right| \leqslant c_4(\tau^2 + h_1^4 + h_2^4 + \Delta\alpha^4), \quad (i,j) \in \omega, \ 1 \leqslant n \leqslant N. \tag{6.67}$$

Noticing the initial-boundary value conditions (6.43)–(6.44), we have

$$\begin{cases} U_{ij}^0 = 0, \quad (i,j) \in \omega, & (6.68) \\ U_{ij}^n = \varphi(x_i, y_j, t_n), \quad (i,j) \in \partial\omega, \ 0 \leqslant n \leqslant N. & (6.69) \end{cases}$$

Omitting the small term $(r_4)_{ij}^n$ in (6.66) and replacing the exact solution U_{ij}^n with its numerical one u_{ij}^n, we can obtain the following difference scheme for solving (6.42)–(6.44):

$$\begin{cases} \mathcal{A}_x \mathcal{A}_y \Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \left[\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} u_{ij}^{n-k} \right] \\ = \mathcal{A}_y \delta_x^2 u_{ij}^n + \mathcal{A}_x \delta_y^2 u_{ij}^n + \mathcal{A}_x \mathcal{A}_y f_{ij}^n, \quad (i,j) \in \omega, \ 1 \leqslant n \leqslant N, & (6.70) \\ u_{ij}^0 = 0, \quad (i,j) \in \omega, & (6.71) \\ u_{ij}^n = \varphi(x_i, y_j, t_n), \quad (i,j) \in \partial\omega, \ 0 \leqslant n \leqslant N. & (6.72) \end{cases}$$

In the subsequent part, the unique solvability, stability and convergence of this difference scheme will be shown.

6.4.2 Solvability of the difference scheme

Theorem 6.4.1. *The difference scheme* (6.70)–(6.72) *is uniquely solvable.*

Proof. Let

$$u^n = \{u_{ij}^n \mid (i,j) \in \bar{\omega}\}.$$

The value of u^0 is uniquely determined by (6.71)–(6.72).

Now suppose the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the system in u^n can be obtained from (6.70) and (6.72). To show its unique solvability, it is sufficient to prove that the corresponding homogeneous one

$$\begin{cases} \nu \mathcal{A}_x \mathcal{A}_y u_{ij}^n = \mathcal{A}_y \delta_x^2 u_{ij}^n + \mathcal{A}_x \delta_y^2 u_{ij}^n, & (i,j) \in \omega, & (6.73) \\ u_{ij}^n = 0, & (i,j) \in \partial\omega & (6.74) \end{cases}$$

has only the trivial solution.

To this end, making the inner product on both hand sides of (6.73) with u^n produces

$$\nu(\mathcal{A}_x \mathcal{A}_y u^n, u^n) = (\mathcal{A}_y \delta_x^2 u^n, u^n) + (\mathcal{A}_x \delta_y^2 u^n, u^n). \tag{6.75}$$

It follows from Lemma 3.10.1 that

$$(\mathcal{A}_x \mathcal{A}_y u^n, u^n) \geqslant \frac{1}{3}\|u^n\|^2. \tag{6.76}$$

By (3.321) and (3.322), we have

$$(\mathcal{A}_y \delta_x^2 u^n, u^n) \leqslant -\frac{2}{3}\|\delta_x u^n\|^2, \quad (\mathcal{A}_x \delta_y^2 u^n, u^n) \leqslant -\frac{2}{3}\|\delta_y u^n\|^2. \tag{6.77}$$

Substituting (6.76) and (6.77) into (6.75) yields

$$\frac{1}{3}\nu\|u^n\|^2 \leqslant -\frac{2}{3}\|\nabla_h u^n\|^2 \leqslant 0,$$

hence, $\|u^n\| = 0$, which implies $u^n = 0$ by noticing (6.74).

By the principle of induction, the theorem is true. The proof ends. □

6.4.3 Stability of the difference scheme

Two useful lemmas are firstly prepared.

Lemma 6.4.1. [48] *For any mesh function $v \in \mathring{\mathcal{V}}_h$, it holds*

$$\frac{2}{3}\|\Delta_h v\|^2 \leqslant (\mathcal{A}_y \delta_x^2 v + \mathcal{A}_x \delta_y^2 v, \Delta_h v) \leqslant \|\Delta_h v\|^2.$$

Lemma 6.4.2. [48] *For any mesh function $v \in \mathring{\mathcal{V}}_h$, it holds*

$$\frac{1}{3}\|\nabla_h v\|^2 \leqslant (\mathcal{A}_x\mathcal{A}_y v, -\Delta_h v) \leqslant \|\nabla_h v\|^2.$$

Next, the stability result is given.

Theorem 6.4.2. *Suppose $\{v_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ is the solution of the difference scheme*

$$\begin{cases} \Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} \mathcal{A}_x\mathcal{A}_y v_{ij}^{n-k} \\ = \mathcal{A}_y\delta_x^2 v_{ij}^n + \mathcal{A}_x\delta_y^2 v_{ij}^n + g_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (6.78) \\ v_{ij}^0 = \varphi_{ij}, \quad (i,j) \in \omega, & (6.79) \\ v_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (6.80) \end{cases}$$

Then it holds

$$\tau \sum_{n=1}^{m} \|\Delta_h v^n\|^2 \leqslant 3\nu\tau\|\nabla_h v^0\|^2 + \frac{9}{4}\tau \sum_{n=1}^{m} \|g^n\|^2, \quad 1 \leqslant m \leqslant N, \tag{6.81}$$

where

$$\|g^n\|^2 = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (g_{ij}^n)^2.$$

Proof. Taking the inner product on both hand sides of (6.78) with $-\Delta_h v^n$, it follows from Lemma 6.4.1 that

$$\begin{aligned} &\Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} (\mathcal{A}_x\mathcal{A}_y v^{n-k}, -\Delta_h v^n) \\ &= (\mathcal{A}_y\delta_x^2 v^n + \mathcal{A}_x\delta_y^2 v^n, -\Delta_h v^n) + (g^n, -\Delta_h v^n) \\ &\leqslant -\frac{2}{3}\|\Delta_h v^n\|^2 + \|g^n\| \cdot \|\Delta_h v^n\| \\ &\leqslant -\frac{2}{3}\|\Delta_h v^n\|^2 + \frac{1}{3}\|\Delta_h v^n\|^2 + \frac{3}{4}\|g^n\|^2 \\ &= -\frac{1}{3}\|\Delta_h v^n\|^2 + \frac{3}{4}\|g^n\|^2, \quad 1 \leqslant n \leqslant N. \end{aligned}$$

Summing up over n from 1 to m and adding the term $\nu(\mathcal{A}_x\mathcal{A}_y v^0, -\Delta_h v^0)$ to both hand sides of the result, we get

$$\Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l} \left[\sum_{n=0}^{m} \sum_{k=0}^{n} w_k^{(\alpha_l)} (\mathcal{A}_x\mathcal{A}_y v^{n-k}, -\Delta_h v^n) \right]$$

$$\leqslant -\frac{1}{3}\sum_{n=1}^{m}\|\Delta_h v^n\|^2 + \nu(\mathcal{A}_x\mathcal{A}_y v^0, -\Delta_h v^0) + \frac{3}{4}\sum_{n=1}^{m}\|g^n\|^2, \quad 1 \leqslant m \leqslant N. \tag{6.82}$$

Since the operators $\mathcal{A}_x$ and $\mathcal{A}_y$ are both positive definite, there exist two positive definite operators $\mathcal{P}_x$ and $\mathcal{P}_y$, such that $\mathcal{A}_x = \mathcal{P}_x^2$, $\mathcal{A}_y = \mathcal{P}_y^2$. Therefore,

$$\begin{aligned}
&(\mathcal{A}_x\mathcal{A}_y v^{n-k}, -\Delta_h v^n)\\
&= (\mathcal{A}_x\mathcal{A}_y v^{n-k}, -\delta_x^2 v^n) + (\mathcal{A}_x\mathcal{A}_y v^{n-k}, -\delta_y^2 v^n)\\
&= (\mathcal{P}_x\mathcal{P}_y v^{n-k}, -\mathcal{P}_x\mathcal{P}_y\delta_x^2 v^n) + (\mathcal{P}_x\mathcal{P}_y v^{n-k}, -\mathcal{P}_x\mathcal{P}_y\delta_y^2 v^n)\\
&= (\mathcal{P}_x\mathcal{P}_y\delta_x v^{n-k}, \mathcal{P}_x\mathcal{P}_y\delta_x v^n) + (\mathcal{P}_x\mathcal{P}_y\delta_y v^{n-k}, \mathcal{P}_x\mathcal{P}_y\delta_y v^n)\\
&= (\delta_x\mathcal{P}_x\mathcal{P}_y v^{n-k}, \delta_x\mathcal{P}_x\mathcal{P}_y v^n) + (\delta_y\mathcal{P}_x\mathcal{P}_y v^{n-k}, \delta_y\mathcal{P}_x\mathcal{P}_y v^n).
\end{aligned}$$

Similar to the proof for (6.61), we have

$$\sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(\mathcal{A}_x\mathcal{A}_y v^{n-k}, -\Delta_h v^n) \geqslant 0. \tag{6.83}$$

By Lemma 6.4.2, it holds

$$(\mathcal{A}_x\mathcal{A}_y v^0, -\Delta_h v^0) \leqslant \|\nabla_h v^0\|^2. \tag{6.84}$$

Substituting (6.83) and (6.84) into (6.82) will yield (6.81). The proof ends. □

6.4.4 Convergence of the difference scheme

Theorem 6.4.3. *Suppose* $\{U_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *and* $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (6.42)–(6.44) *and the difference scheme* (6.70)–(6.72), *respectively. Let*

$$e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N,$$

then it holds

$$\tau\sum_{n=1}^{N}\|e^n\|_\infty \leqslant \frac{3}{2}cT\sqrt{L_1L_2}\,c_4(\tau^2 + h_1^4 + h_2^4 + \Delta\alpha^4), \tag{6.85}$$

where the constant c is defined in Lemma 6.3.1.

Proof. Subtracting (6.70)–(6.72) from (6.66), (6.68)–(6.69), respectively, we get the system of error equations as follows:

$$\begin{cases}
\Delta\alpha\sum_{l=0}^{2J} d_l w(\alpha_l)\left[\tau^{-\alpha_l}\sum_{k=0}^{n} w_k^{(\alpha_l)}\mathcal{A}_x\mathcal{A}_y e_{ij}^{n-k}\right]\\
= \mathcal{A}_y\delta_x^2 e_{ij}^n + \mathcal{A}_x\delta_y^2 e_{ij}^n + (r_4)_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N,\\
e_{ij}^0 = 0, \quad (i,j) \in \omega,\\
e_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N.
\end{cases}$$

Noticing (6.67), the application of Theorem 6.4.2 yields

$$\begin{aligned}\tau\sum_{n=1}^{N}\|\Delta_h e^n\|^2 &\leqslant \frac{9}{4}\tau\sum_{n=1}^{N}\|(r_4)^n\|^2\\ &\leqslant \frac{9}{4}\tau\sum_{n=1}^{N}L_1L_2[c_4(\tau^2+h_1^4+h_2^4+\Delta\alpha^4)]^2\\ &\leqslant \frac{9}{4}TL_1L_2[c_4(\tau^2+h_1^4+h_2^4+\Delta\alpha^4)]^2.\end{aligned} \tag{6.86}$$

By the Cauchy–Schwarz inequality, Lemma 6.3.1 and (6.86), we have

$$\begin{aligned}\left(\tau\sum_{n=1}^{N}\|e^n\|_\infty\right)^2 &\leqslant \left(\tau\sum_{n=1}^{N}1\right)\left(\tau\sum_{n=1}^{N}\|e^n\|_\infty^2\right)\\ &\leqslant Tc^2\tau\sum_{n=1}^{N}\|\Delta_h e^n\|^2\\ &\leqslant \frac{9}{4}c^2T^2L_1L_2[c_4(\tau^2+h_1^4+h_2^4+\Delta\alpha^4)]^2,\end{aligned}$$

which implies (6.85). The proof ends. □

6.5 The second-order ADI method in both space and distributed-order for 2D problem

In this section, an ADI difference scheme for solving the 2D problem of time distributed-order subdiffusion equation (6.42)–(6.44) will be proposed and its unique solvability, stability and convergence will also be shown.

Define the function $\hat{u}(x,y,t)$ like that in Section 2.10. Suppose $\hat{u}(x,y,\cdot)\in\mathscr{C}^{2+1}(\mathcal{R})$ and $u(\cdot,\cdot,t)\in C^{(4,4)}(\bar{\Omega})$.

6.5.1 Derivation of the difference scheme

Adding the small term $\frac{\tau}{\mu}\delta_x^2\delta_y^2\frac{U_{ij}^n-U_{ij}^{n-1}}{\tau}$ to both hand sides of (6.48) arrives at

$$\begin{aligned}&\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)\left[\tau^{-\alpha_l}\sum_{k=0}^{n}w_k^{(\alpha_l)}U_{ij}^{n-k}\right]+\frac{\tau}{\mu}\delta_x^2\delta_y^2\frac{U_{ij}^n-U_{ij}^{n-1}}{\tau}\\ &=\Delta_h U_{ij}^n+f_{ij}^n+(r_5)_{ij}^n,\quad (i,j)\in\omega,\ 1\leqslant n\leqslant N,\end{aligned} \tag{6.87}$$

where

$$(r_5)_{ij}^n=(r_3)_{ij}^n+\frac{\tau}{\mu}\delta_x^2\delta_y^2\frac{U_{ij}^n-U_{ij}^{n-1}}{\tau}.$$

It is easy to see that $\frac{\tau}{\mu} = O(\tau^2|\ln\tau|)$ by Lemma 6.1.4. Hence, there exists a positive constant c_5 such that

$$|(r_5)_{ij}^n| \leqslant c_3(\tau^2 + h_1^2 + h_2^2 + \Delta\alpha^2) + c_5\tau^2|\ln\tau|, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \tag{6.88}$$

Noticing the initial-boundary value conditions (6.43)–(6.44), we have

$$\begin{cases} U_{ij}^0 = 0, \quad (i,j) \in \omega, & (6.89) \\ U_{ij}^n = \varphi(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (6.90) \end{cases}$$

Neglecting the small term $(r_5)_{ij}^n$ in (6.87) and replacing the exact solution U_{ij}^n with its numerical one u_{ij}^n, a difference scheme for (6.42)–(6.44) is obtained as follows:

$$\begin{cases} \Delta\alpha \sum\limits_{l=0}^{2J} c_l w(\alpha_l)\left[\tau^{-\alpha_l} \sum\limits_{k=0}^{n} w_k^{(\alpha_l)} u_{ij}^{n-k}\right] + \dfrac{\tau}{\mu}\delta_x^2\delta_y^2 \dfrac{u_{ij}^n - u_{ij}^{n-1}}{\tau} & \\ = \Delta_h u_{ij}^n + f_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (6.91) \\ u_{ij}^0 = 0, \quad (i,j) \in \omega, & (6.92) \\ u_{ij}^n = \varphi(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (6.93) \end{cases}$$

Rewrite (6.91) in the form of

$$\begin{aligned} &\mu u_{ij}^n - (\delta_x^2 + \delta_y^2)u_{ij}^n + \frac{1}{\mu}\delta_x^2\delta_y^2 u_{ij}^n \\ = &-\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=1}^{n} w_k^{(\alpha_l)} u_{ij}^{n-k} + \frac{1}{\mu}\delta_x^2\delta_y^2 u_{ij}^{n-1} + f_{ij}^n, \end{aligned}$$

or

$$\begin{aligned} &\left(\sqrt{\mu}\,\mathcal{I} - \frac{1}{\sqrt{\mu}}\delta_x^2\right)\left(\sqrt{\mu}\,\mathcal{I} - \frac{1}{\sqrt{\mu}}\delta_y^2\right)u_{ij}^n \\ = &-\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=1}^{n} w_k^{(\alpha_l)} u_{ij}^{n-k} + \frac{1}{\mu}\delta_x^2\delta_y^2 u_{ij}^{n-1} + f_{ij}^n. \end{aligned}$$

Let

$$u_{ij}^* = \left(\sqrt{\mu}\,\mathcal{I} - \frac{1}{\sqrt{\mu}}\delta_y^2\right)u_{ij}^n.$$

Then the difference scheme (6.91)–(6.93) can be decomposed into the following ADI form:

On each time level $t = t_n$ $(1 \leqslant n \leqslant N)$, firstly, for any fixed j from 1 to $M_2 - 1$, solve a series of linear systems in the unknown $\{u_{ij}^* \mid 0 \leqslant i \leqslant M_1\}$ in x direction

$$\begin{cases} \left(\sqrt{\mu}\,\mathcal{I} - \dfrac{1}{\sqrt{\mu}}\delta_x^2\right)u_{ij}^* = -\Delta\alpha \displaystyle\sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=1}^{n} w_k^{(\alpha_l)} u_{ij}^{n-k} \\ \qquad\qquad\qquad + \dfrac{1}{\mu}\delta_x^2\delta_y^2 u_{ij}^{n-1} + f_{ij}^n, \quad 1 \leqslant i \leqslant M_1 - 1, \\ u_{0j}^* = \left(\sqrt{\mu}\,\mathcal{I} - \dfrac{1}{\sqrt{\mu}}\delta_y^2\right)u_{0j}^n, \quad u_{M_1,j}^* = \left(\sqrt{\mu}\,\mathcal{I} - \dfrac{1}{\sqrt{\mu}}\delta_y^2\right)u_{M_1,j}^n \end{cases} \tag{6.94}$$

to get the value of

$$\{u_{ij}^* \mid 1 \leqslant i \leqslant M_1 - 1\};$$

Then, for any fixed i from 1 to M_1-1, compute a series of one-dimensional problems in the unknown $\{u_{ij}^n \mid 0 \leqslant j \leqslant M_2\}$ in y direction

$$\begin{cases} \left(\sqrt{\mu}\,\mathcal{I} - \dfrac{1}{\sqrt{\mu}}\delta_y^2\right)u_{ij}^n = u_{ij}^*, \quad 1 \leqslant j \leqslant M_2 - 1, \\ u_{i0}^n = \varphi(x_i, y_0, t_n), \quad u_{i,M_2}^n = \varphi(x_i, y_{M_2}, t_n) \end{cases} \tag{6.95}$$

to get the desired value of

$$\{u_{ij}^n \mid 1 \leqslant j \leqslant M_2 - 1\}.$$

The linear systems (6.94) and (6.95) are both tridiagonal, which can be easily solved using the Thomas algorithm.

In the subsequent part, we aim to analyze the proposed difference scheme.

6.5.2 Solvability of the difference scheme

Theorem 6.5.1. *The difference scheme* (6.91)–(6.93) *is uniquely solvable.*

Proof. Let

$$u^n = \{u_{ij}^n \mid (i,j) \in \bar{\omega}\}.$$

The value of u^0 is determined by (6.92)–(6.93).

Now suppose the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then we can get the linear system in u^n from (6.91) and (6.93). To show its unique solvability, it suffices to prove that the corresponding homogeneous one

$$\begin{cases} \mu u_{ij}^n + \dfrac{1}{\mu}\delta_x^2\delta_y^2 u_{ij}^n = \Delta_h u_{ij}^n, \quad (i,j) \in \omega, & (6.96) \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega & (6.97) \end{cases}$$

has only the trivial solution.

To this end, making the inner product on both hand sides of (6.96) with u^n leads to

$$\mu(u^n, u^n) + \frac{1}{\mu}(\delta_x^2\delta_y^2 u^n, u^n) = (\Delta_h u^n, u^n) = -\|\nabla_h u^n\|^2.$$

Therefore,

$$\mu\|u^n\|^2 + \frac{1}{\mu}\|\delta_x\delta_y u^n\|^2 = -\|\nabla_h u^n\|^2 \leqslant 0,$$

which implies $\|u^n\| = 0$. Noticing (6.97), it is clear that $u^n = 0$.

By the principle of induction, the theorem is true. The proof ends. □

6.5.3 Stability of the difference scheme

Theorem 6.5.2. *Suppose $\{v_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ is the solution of the difference scheme*

$$\begin{cases} \Delta\alpha \sum\limits_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum\limits_{k=0}^{n} w_k^{(\alpha_l)} v_{ij}^{n-k} + \dfrac{\tau}{\mu}\delta_x^2\delta_y^2 \dfrac{v_{ij}^n - v_{ij}^{n-1}}{\tau} \\ = \Delta_h v_{ij}^n + g_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (6.98) \\ v_{ij}^0 = \varphi_{ij}, \quad (i,j) \in \omega, & (6.99) \\ v_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (6.100) \end{cases}$$

Then it holds

$$\begin{aligned} &\tau \sum_{n=1}^{m} \|\Delta_h v^n\|^2 + \frac{\tau}{\mu}(\|\delta_x^2\delta_y v^m\|^2 + \|\delta_x\delta_y^2 v^m\|^2) \\ &\leqslant 2\mu\tau\|\nabla_h v^0\|^2 + \frac{\tau}{\mu}(\|\delta_x^2\delta_y v^0\|^2 + \|\delta_x\delta_y^2 v^0\|^2) + \tau\sum_{n=1}^{m}\|g^n\|^2, \quad 1 \leqslant m \leqslant N, \end{aligned} \tag{6.101}$$

where

$$\|g^n\|^2 = h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(g_{ij}^n)^2.$$

Proof. Making the inner product on both hand sides of (6.98) with $-\Delta_h v^n$ and using the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned} &\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} (v^{n-k}, -\Delta_h v^n) \\ &+ \frac{\tau}{\mu}\left(\delta_x^2\delta_y^2 \frac{v^n - v^{n-1}}{\tau}, -\Delta_h v^n\right) \end{aligned}$$

$$
\begin{aligned}
&= -(\Delta_h v^n, \Delta_h v^n) + (g^n, -\Delta_h v^n) \\
&\leqslant -\|\Delta_h v^n\|^2 + \frac{1}{2}\|\Delta_h v^n\|^2 + \frac{1}{2}\|g^n\|^2 \\
&= -\frac{1}{2}\|\Delta_h v^n\|^2 + \frac{1}{2}\|g^n\|^2, \quad 1 \leqslant n \leqslant N. \qquad (6.102)
\end{aligned}
$$

For the second term on the left-hand side of the inequality above, by noticing (6.100), we have

$$
\begin{aligned}
&\left(\delta_x^2\delta_y^2\frac{v^n - v^{n-1}}{\tau}, -\Delta_h v^n\right) \\
&= \left(\delta_x^2\delta_y^2\frac{v^n - v^{n-1}}{\tau}, -\delta_x^2 v^n\right) + \left(\delta_x^2\delta_y^2\frac{v^n - v^{n-1}}{\tau}, -\delta_y^2 v^n\right) \\
&= \left(\delta_x^2\delta_y\frac{v^n - v^{n-1}}{\tau}, \delta_x^2\delta_y v^n\right) + \left(\delta_x\delta_y^2\frac{v^n - v^{n-1}}{\tau}, \delta_x\delta_y^2 v^n\right) \\
&\geqslant \frac{1}{2\tau}(\|\delta_x^2\delta_y v^n\|^2 - \|\delta_x^2\delta_y v^{n-1}\|^2) \\
&\quad + \frac{1}{2\tau}(\|\delta_x\delta_y^2 v^n\|^2 - \|\delta_x\delta_y^2 v^{n-1}\|^2). \qquad (6.103)
\end{aligned}
$$

Summing up for n in (6.102) from 1 to m and adding the term $\mu(v^0, -\Delta_h v^0)$ to both hand sides of the obtained inequality, with the aid of (6.103), we get

$$
\begin{aligned}
&\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l}\left[\sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(v^{n-k}, -\Delta_h v^n)\right] \\
&\quad + \frac{1}{2\mu}(\|\delta_x^2\delta_y v^m\|^2 + \|\delta_x\delta_y^2 v^m\|^2 - \|\delta_x^2\delta_y v^0\|^2 - \|\delta_x\delta_y^2 v^0\|^2) \\
&\leqslant -\frac{1}{2}\sum_{n=1}^{m}\|\Delta_h v^n\|^2 + \mu(v^0, -\Delta_h v^0) + \frac{1}{2}\sum_{n=1}^{m}\|g^n\|^2 \\
&= -\frac{1}{2}\sum_{n=1}^{m}\|\Delta_h v^n\|^2 + \mu\|\nabla_h v^0\|^2 + \frac{1}{2}\sum_{n=1}^{m}\|g^n\|^2, \quad 1 \leqslant m \leqslant N. \qquad (6.104)
\end{aligned}
$$

In view of (6.61), it holds

$$\sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(v^{n-k}, -\Delta_h v^n) \geqslant 0,$$

hence, (6.101) is followed from (6.104). The proof ends. □

6.5.4 Convergence of the difference scheme

Theorem 6.5.3. *Suppose* $\{U_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *and* $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (6.42)–(6.44) *and the difference scheme* (6.91)–(6.93), *respectively. Let*

$$e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N,$$

then it holds

$$\tau \sum_{n=1}^{N} \|e^n\|_\infty \leqslant cT\sqrt{L_1L_2}\,(c_3 + c_5)(\tau^2|\ln\tau| + h_1^2 + h_2^2 + \Delta\alpha^2), \tag{6.105}$$

where the constant c is defined in Lemma 6.3.1.

Proof. The subtraction of (6.91)–(6.93) from (6.87), (6.89)–(6.90), respectively, will produce the system of error equations as follows:

$$\begin{cases} \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\left[\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} e_{ij}^{n-k}\right] + \frac{\tau}{\mu}\delta_x^2\delta_y^2 \frac{e_{ij}^n - e_{ij}^{n-1}}{\tau} \\ = \Delta_h e_{ij}^n + (r_5)_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ e_{ij}^0 = 0, \quad (i,j) \in \omega, \\ e_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (6.88), the application of Theorem 6.5.2 immediately yields

$$\begin{aligned} \tau \sum_{n=1}^{N} \|\Delta_h e^n\|^2 &\leqslant \tau \sum_{n=1}^{N} \|(r_5)^n\|^2 \\ &\leqslant \tau \sum_{n=1}^{N} L_1L_2[c_3(\tau^2 + h_1^2 + h_2^2 + \Delta\alpha^2) + c_5\tau^2|\ln\tau|]^2 \\ &\leqslant TL_1L_2(c_3 + c_5)^2(\tau^2|\ln\tau| + h_1^2 + h_2^2 + \Delta\alpha^2)^2. \end{aligned} \tag{6.106}$$

By the Cauchy–Schwarz inequality, Lemma 6.3.1 and (6.106), we have

$$\begin{aligned} \left(\tau \sum_{n=1}^{N} \|e^n\|_\infty\right)^2 &\leqslant T\left(\tau \sum_{n=1}^{N} \|e^n\|_\infty^2\right) \\ &\leqslant Tc^2\tau \sum_{n=1}^{N} \|\Delta_h e^n\|^2 \\ &\leqslant c^2T^2L_1L_2(c_3 + c_5)^2(\tau^2|\ln\tau| + h_1^2 + h_2^2 + \Delta\alpha^2)^2, \end{aligned}$$

which implies (6.105). The proof ends. □

6.6 The fourth-order ADI method in both space and distributed-order for 2D problem

This section is devoted to the derivation of another high order ADI difference scheme for solving the problem (6.42)–(6.44) and the corresponding analysis on the resultant scheme.

Define the function $\hat{u}(x,y,t)$ in the same way as that in Section 2.10. Suppose $\hat{u}(x,y,\cdot) \in \mathscr{C}^{2+1}(\mathcal{R})$ and $u(\cdot,\cdot,t) \in C^{(6,6)}(\bar{\Omega})$.

6.6.1 Derivation of the difference scheme

Adding the small term $\frac{\tau}{\nu}\delta_x^2\delta_y^2\frac{U_{ij}^n-U_{ij}^{n-1}}{\tau}$ to both hand sides of (6.66) gives

$$\mathcal{A}_x\mathcal{A}_y\Delta\alpha\sum_{l=0}^{2J} d_l w(\alpha_l)\left[\tau^{-\alpha_l}\sum_{k=0}^{n} w_k^{(\alpha_l)}U_{ij}^{n-k}\right]+\frac{\tau}{\nu}\delta_x^2\delta_y^2\frac{U_{ij}^n-U_{ij}^{n-1}}{\tau}$$
$$=\mathcal{A}_y\delta_x^2U_{ij}^n+\mathcal{A}_x\delta_y^2U_{ij}^n+\mathcal{A}_x\mathcal{A}_yf_{ij}^n+(r_6)_{ij}^n,\quad (i,j)\in\omega,\ 1\leqslant n\leqslant N, \tag{6.107}$$

where

$$(r_6)_{ij}^n=(r_4)_{ij}^n+\frac{\tau}{\nu}\delta_x^2\delta_y^2\frac{U_{ij}^n-U_{ij}^{n-1}}{\tau},$$

by noticing Lemma 6.2.2, there is a positive constant c_6 such that

$$\left|(r_6)_{ij}^n\right|\leqslant c_4(\tau^2+h_1^4+h_2^4+\Delta\alpha^4)+c_6\tau^2|\ln\tau|,\quad (i,j)\in\omega,\ 1\leqslant n\leqslant N. \tag{6.108}$$

Noticing the initial-boundary value conditions (6.43)–(6.44), we have

$$\begin{cases} U_{ij}^0=0,\quad (i,j)\in\omega, & (6.109)\\ U_{ij}^n=\varphi(x_i,y_j,t_n),\quad (i,j)\in\partial\omega,\ 0\leqslant n\leqslant N. & (6.110)\end{cases}$$

Omitting the small term $(r_6)_{ij}^n$ in (6.107) and replacing the exact solution U_{ij}^n with its numerical one u_{ij}^n, we can obtain the difference scheme for solving (6.42)–(6.44) as follows:

$$\begin{cases} \mathcal{A}_x\mathcal{A}_y\Delta\alpha\sum\limits_{l=0}^{2J} d_l w(\alpha_l)\left[\tau^{-\alpha_l}\sum\limits_{k=0}^{n} w_k^{(\alpha_l)}u_{ij}^{n-k}\right]+\frac{\tau}{\nu}\delta_x^2\delta_y^2\frac{u_{ij}^n-u_{ij}^{n-1}}{\tau} & \\ =\mathcal{A}_y\delta_x^2u_{ij}^n+\mathcal{A}_x\delta_y^2u_{ij}^n+\mathcal{A}_x\mathcal{A}_yf_{ij}^n,\quad (i,j)\in\omega,\ 1\leqslant n\leqslant N, & (6.111)\\ u_{ij}^0=0,\quad (i,j)\in\omega, & (6.112)\\ u_{ij}^n=\varphi(x_i,y_j,t_n),\quad (i,j)\in\partial\omega,\ 0\leqslant n\leqslant N. & (6.113)\end{cases}$$

Equation (6.111) can be reformulated as

$$\nu\mathcal{A}_x\mathcal{A}_yu_{ij}^n-\left(\mathcal{A}_y\delta_x^2u_{ij}^n+\mathcal{A}_x\delta_y^2u_{ij}^n\right)+\frac{1}{\nu}\delta_x^2\delta_y^2u_{ij}^n$$
$$=-\Delta\alpha\sum_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l}\sum_{k=1}^{n} w_k^{(\alpha_l)}\mathcal{A}_x\mathcal{A}_yu_{ij}^{n-k}+\frac{1}{\nu}\delta_x^2\delta_y^2u_{ij}^{n-1}+\mathcal{A}_x\mathcal{A}_yf_{ij}^n,$$

or

$$\left(\sqrt{\nu}\mathcal{A}_x-\frac{1}{\sqrt{\nu}}\delta_x^2\right)\left(\sqrt{\nu}\mathcal{A}_y-\frac{1}{\sqrt{\nu}}\delta_y^2\right)u_{ij}^n$$

$$= -\Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=1}^{n} w_k^{(\alpha_l)} \mathcal{A}_x\mathcal{A}_y u_{ij}^{n-k} + \frac{1}{\nu}\delta_x^2\delta_y^2 u_{ij}^{n-1} + \mathcal{A}_x\mathcal{A}_y f_{ij}^n.$$

Let

$$u_{ij}^* = \left(\sqrt{\nu}\mathcal{A}_y - \frac{1}{\sqrt{\nu}}\delta_y^2\right)u_{ij}^n.$$

Then the difference scheme (6.111)–(6.113) can be written as the following ADI form:

On each time level $t = t_n$ $(1 \leqslant n \leqslant N)$, at first, for any fixed j from 1 to $M_2 - 1$, solve a series of one-dimensional problems in the unknown $\{u_{ij}^* \mid 0 \leqslant i \leqslant M_1\}$ in x direction

$$\begin{cases} \left(\sqrt{\nu}\mathcal{A}_x - \frac{1}{\sqrt{\nu}}\delta_x^2\right)u_{ij}^* = -\Delta\alpha \sum\limits_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l} \sum\limits_{k=1}^{n} w_k^{(\alpha_l)} \mathcal{A}_x\mathcal{A}_y u_{ij}^{n-k} \\ \qquad\qquad + \frac{1}{\nu}\delta_x^2\delta_y^2 u_{ij}^{n-1} + \mathcal{A}_x\mathcal{A}_y f_{ij}^n, \quad 1 \leqslant i \leqslant M_1 - 1, \\ u_{0j}^* = \left(\sqrt{\nu}\mathcal{A}_y - \frac{1}{\sqrt{\nu}}\delta_y^2\right)u_{0j}^n, \quad u_{M_1,j}^* = \left(\sqrt{\nu}\mathcal{A}_y - \frac{1}{\sqrt{\nu}}\delta_y^2\right)u_{M_1,j}^n \end{cases} \tag{6.114}$$

to get the value of

$$\{u_{ij}^* \mid 1 \leqslant i \leqslant M_1 - 1\}$$

on an intermediate time level.

Then, for any fixed i from 1 to $M_1 - 1$, perform some calculations on a series of one-dimensional problems in the unknown $\{u_{ij}^n \mid 0 \leqslant j \leqslant M_2\}$ in y direction

$$\begin{cases} \left(\sqrt{\nu}\mathcal{A}_y - \frac{1}{\sqrt{\nu}}\delta_y^2\right)u_{ij}^n = u_{ij}^*, \quad 1 \leqslant j \leqslant M_2 - 1, \\ u_{i0}^n = \varphi(x_i, y_0, t_n), \quad u_{i,M_2}^n = \varphi(x_i, y_{M_2}, t_n) \end{cases} \tag{6.115}$$

to get the desired value of

$$\{u_{ij}^n \mid 1 \leqslant j \leqslant M_2 - 1\}.$$

The linear systems (6.114) and (6.115) are both tridiagonal, which can be easily solved using the Thomas algorithm.

6.6.2 Solvability of the difference scheme

Theorem 6.6.1. *The difference scheme* (6.111)–(6.113) *is uniquely solvable.*

Proof. Let

$$u^n = \{u_{ij}^n \mid (i,j) \in \bar{\omega}\}.$$

The value of u^0 is determined by (6.112)–(6.113).

Now assume that the values of $u^0, u^1, \ldots, u^{n-1}$ have been uniquely determined, then the system in u^n can be obtained from (6.111) and (6.113). To show its unique solvability, it is sufficient to prove that the corresponding homogeneous one

$$\begin{cases} \nu \mathcal{A}_x \mathcal{A}_y u_{ij}^n + \dfrac{1}{\nu}\delta_x^2 \delta_y^2 u_{ij}^n = \mathcal{A}_y \delta_x^2 u_{ij}^n + \mathcal{A}_x \delta_y^2 u_{ij}^n, \quad (i,j) \in \omega, & (6.116) \\ u_{ij}^n = 0, \quad (i,j) \in \partial\omega & (6.117) \end{cases}$$

has only the trivial solution.

To this end, making the inner product on both hand sides of (6.116) with u^n arrives at

$$\nu(\mathcal{A}_x \mathcal{A}_y u^n, u^n) + \frac{1}{\nu}(\delta_x^2 \delta_y^2 u^n, u^n) = (\mathcal{A}_y \delta_x^2 u^n, u^n) + (\mathcal{A}_x \delta_y^2 u^n, u^n).$$

Noticing (6.76), (6.77) and $(\delta_x^2 \delta_y^2 u^n, u^n) = \|\delta_x \delta_y u^n\|^2$, we have

$$\frac{1}{3}\nu\|u^n\|^2 + \frac{1}{\nu}\|\delta_x \delta_y u^n\|^2 \leqslant -\frac{2}{3}\|\nabla_h u^n\|^2 \leqslant 0,$$

thus, $\|u^n\| = 0$. Then $u^n = 0$ is followed by noticing (6.117).

By the principle of induction, the theorem is true. The proof ends. □

6.6.3 Stability of the difference scheme

Theorem 6.6.2. *Suppose* $\{v_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *is the solution of the difference scheme*

$$\begin{cases} \Delta\alpha \displaystyle\sum_{l=0}^{2J} d_l w(\alpha_l) \tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)} \mathcal{A}_x \mathcal{A}_y v_{ij}^{n-k} + \frac{\tau}{\nu}\delta_x^2 \delta_y^2 \frac{v_{ij}^n - v_{ij}^{n-1}}{\tau} & \\ = \mathcal{A}_y \delta_x^2 v_{ij}^n + \mathcal{A}_x \delta_y^2 v_{ij}^n + g_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, & (6.118) \\ v_{ij}^0 = \varphi_{ij}, \quad (i,j) \in \omega, & (6.119) \\ v_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. & (6.120) \end{cases}$$

Then it holds

$$\begin{aligned} &\tau \sum_{n=1}^{m} \|\Delta_h v^n\|^2 + \frac{3\tau}{2\nu}\left(\|\delta_x^2 \delta_y v^m\|^2 + \|\delta_x \delta_y^2 v^m\|^2\right) \\ &\leqslant 3\nu\tau\|\nabla_h v^0\|^2 + \frac{3\tau}{2\nu}\left(\|\delta_x^2 \delta_y v^0\|^2 + \|\delta_x \delta_y^2 v^0\|^2\right) \\ &\quad + \frac{9}{4}\tau \sum_{n=1}^{m} \|g^n\|^2, \quad 1 \leqslant m \leqslant N, \end{aligned} \tag{6.121}$$

where

$$\|g^n\|^2 = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (g_{ij}^n)^2.$$

Proof. Taking the inner product on both hand sides of (6.118) with $-\Delta_h v^n$, it follows from Lemma 6.4.1 that

$$\begin{aligned}
&\Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=0}^{n} w_k^{(\alpha_l)}(\mathcal{A}_x\mathcal{A}_y v^{n-k}, -\Delta_h v^n) \\
&\quad + \frac{\tau}{\nu}\left(\delta_x^2\delta_y^2 \frac{v^n - v^{n-1}}{\tau}, -\Delta_h v^n\right) \\
&= (\mathcal{A}_y\delta_x^2 v^n + \mathcal{A}_x\delta_y^2 v^n, -\Delta_h v^n) + (g^n, -\Delta_h v^n) \\
&\leqslant -\frac{2}{3}\|\Delta_h v^n\|^2 + \frac{1}{3}\|\Delta_h v^n\|^2 + \frac{3}{4}\|g^n\|^2 \\
&= -\frac{1}{3}\|\Delta_h v^n\|^2 + \frac{3}{4}\|g^n\|^2, \quad 1 \leqslant n \leqslant N.
\end{aligned} \tag{6.122}$$

By (6.103), we have

$$\begin{aligned}
&\left(\delta_x^2\delta_y^2 \frac{v^n - v^{n-1}}{\tau}, -\Delta_h v^n\right) \\
&\geqslant \frac{1}{2\tau}[(\|\delta_x^2\delta_y v^n\|^2 + \|\delta_x\delta_y^2 v^n\|^2) - (\|\delta_x^2\delta_y v^{n-1}\|^2 + \|\delta_x\delta_y^2 v^{n-1}\|^2)].
\end{aligned} \tag{6.123}$$

Summing up for n in (6.122) from 1 to m and adding the term $\nu(\mathcal{A}_x\mathcal{A}_y v^0, -\Delta_h v^0)$ to both hand sides of the obtained inequality, it follows by noticing (6.123) that

$$\begin{aligned}
&\Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l}\left[\sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(\mathcal{A}_x\mathcal{A}_y v^{n-k}, -\Delta_h v^n)\right] \\
&\quad + \frac{1}{2\nu}(\|\delta_x^2\delta_y v^m\|^2 + \|\delta_x\delta_y^2 v^m\|^2 - \|\delta_x^2\delta_y v^0\|^2 - \|\delta_x\delta_y^2 v^0\|^2) \\
&\leqslant -\frac{1}{3}\sum_{n=1}^{m}\|\Delta_h v^n\|^2 + \nu(\mathcal{A}_x\mathcal{A}_y v^0, -\Delta_h v^0) + \frac{3}{4}\sum_{n=1}^{m}\|g^n\|^2, \quad 1 \leqslant m \leqslant N.
\end{aligned} \tag{6.124}$$

It is clear from (6.83) and (6.84), respectively, that

$$\sum_{n=0}^{m}\sum_{k=0}^{n} w_k^{(\alpha_l)}(\mathcal{A}_x\mathcal{A}_y v^{n-k}, -\Delta_h v^n) \geqslant 0 \tag{6.125}$$

and

$$(\mathcal{A}_x\mathcal{A}_y v^0, -\Delta_h v^0) \leqslant \|\nabla_h v^0\|^2. \tag{6.126}$$

Substituting (6.125) and (6.126) into (6.124) produces (6.121). The proof ends. □

6.6.4 Convergence of the difference scheme

Theorem 6.6.3. *Suppose* $\{U_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *and* $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ *are solutions of the problem* (6.42)–(6.44) *and the difference scheme* (6.111)–(6.113), *respec-*

tively. Let

$$e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j) \in \bar{\omega},\ 0 \leqslant n \leqslant N,$$

then it holds

$$\tau \sum_{n=1}^{N} \|e^n\|_\infty \leqslant \frac{3}{2} cT \sqrt{L_1 L_2}\,(c_4 + c_6)(\tau^2 |\ln \tau| + h_1^4 + h_2^4 + \Delta\alpha^4), \tag{6.127}$$

where the constant c is defined in Lemma 6.3.1.

Proof. The subtraction of (6.111)–(6.113) from (6.107), (6.109)–(6.110), respectively, gives the system of error equations as follows:

$$\begin{cases} \Delta\alpha \sum\limits_{l=0}^{2J} d_l w(\alpha_l) \left[\tau^{-\alpha_l} \sum\limits_{k=0}^{n} w_k^{(\alpha_l)} \mathcal{A}_x \mathcal{A}_y e_{ij}^{n-k} \right] + \dfrac{\tau}{\nu} \delta_x^2 \delta_y^2 \dfrac{e_{ij}^n - e_{ij}^{n-1}}{\tau} \\ = \mathcal{A}_y \delta_x^2 e_{ij}^n + \mathcal{A}_x \delta_y^2 e_{ij}^n + (r_6)_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ e_{ij}^0 = 0, \quad (i,j) \in \omega, \\ e_{ij}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

Noticing (6.108), the application of Theorem 6.6.2 yields

$$\begin{aligned} \tau \sum_{n=1}^{N} \|\Delta_h e^n\|^2 &\leqslant \frac{9}{4} \tau \sum_{n=1}^{N} \|(r_6)^n\|^2 \\ &\leqslant \frac{9}{4} \tau \sum_{n=1}^{N} L_1 L_2 [c_4(\tau^2 + h_1^4 + h_2^4 + \Delta\alpha^4) + c_6 \tau^2 |\ln \tau|]^2 \\ &\leqslant \frac{9}{4} T L_1 L_2 (c_4 + c_6)^2 (\tau^2 |\ln \tau| + h_1^4 + h_2^4 + \Delta\alpha^4)^2. \end{aligned} \tag{6.128}$$

By the Cauchy–Schwarz inequality, Lemma 6.3.1 and (6.128), we have

$$\begin{aligned} \left(\tau \sum_{n=1}^{N} \|e^n\|_\infty \right)^2 &\leqslant T \left(\tau \sum_{n=1}^{N} \|e^n\|_\infty^2 \right) \\ &\leqslant T c^2 \tau \sum_{n=1}^{N} \|\Delta_h e^n\|^2 \\ &\leqslant \frac{9}{4} c^2 T^2 L_1 L_2 (c_4 + c_6)^2 (\tau^2 |\ln \tau| + h_1^4 + h_2^4 + \Delta\alpha^4)^2, \end{aligned}$$

which implies (6.127). The proof ends. □

6.7 Supplementary remarks and discussions

1. In this chapter, the difference methods for solving 1D and 2D time distributed-order subdiffusion equations have been introduced[24, 29]. For the approximation of the distributed integral, the composite trapezoid formula or the composite Simpson formula

was used; For the discretization of time Caputo derivatives, the second-order WSGL formula was applied and several higher order difference schemes were derived. For 2D problem, two ADI difference schemes were also discussed. For each scheme, the unique solvability, stability and convergence were proved. Indeed, one can also directly apply the G-L formula to approximate the time Caputo derivatives and the first-order difference scheme in time can be obtained, then the techniques of Richardson extrapolation can be used to improve the accuracy in time[23, 25].

2. For the approximation of distributed integral, the composite mid-point formula can also be used. For details, see [103]. Also, the Gauss quadrature can be applied. For the approximation of time Caputo derivatives, the L1 formula can also be used, please refer to [60, 103].

3. In this chapter, we only discussed the numerical solutions of time distributed-order subdiffusion equations. Now the numerical solutions of time distributed-order wave equations are briefly introduced. Consider the following problem:

$$\begin{cases} \mathcal{D}_t^w u(x,t) = u_{xx}(x,t) + f(x,t), \quad 0 < x < L,\ 0 < t \leqslant T, & (6.129)\\ u(x,0) = 0, \quad u_t(x,0) = 0, \quad 0 < x < L, & (6.130)\\ u(0,t) = \varphi_1(t), \quad u(L,t) = \varphi_2(t), \quad 0 \leqslant t \leqslant T, & (6.131) \end{cases}$$

where $\varphi_1(0) = \varphi_2(0) = 0$, $\varphi_1'(0) = \varphi_2'(0) = 0$ and

$$\mathcal{D}_t^w u(x,t) = \int_1^2 w(\gamma)\, {}_0^C D_t^\gamma u(x,t) \mathrm{d}\gamma,$$

$${}_0^C D_t^\gamma u(x,t) = \begin{cases} \frac{1}{\Gamma(2-\gamma)} \int_0^t (t-\xi)^{1-\gamma} u_{\xi\xi}(x,\xi)\mathrm{d}\xi, & 1 \leqslant \gamma < 2,\\ u_{tt}(x,t), & \gamma = 2, \end{cases}$$

$w(\gamma) \geqslant 0$, $\int_1^2 w(\gamma)\mathrm{d}\gamma = c_0 > 0$, the functions f, φ_1 and φ_2 are all given.

Denote $\Delta\gamma = \frac{1}{2J}$, $\gamma_l = 1 + l\Delta\gamma \ (0 \leqslant l \leqslant 2J)$.

Considering equation (6.129) at the point (x_i, t_n), we have

$$\mathcal{D}_t^w u(x_i,t_n) = u_{xx}(x_i,t_n) + f(x_i,t_n), \quad 1 \leqslant i \leqslant M-1,\ 0 \leqslant n \leqslant N.$$

Taking an average on two adjacent time levels gives

$$\frac{1}{2}[\mathcal{D}_t^w u(x_i,t_n) + \mathcal{D}_t^w u(x_i,t_{n-1})] = \frac{1}{2}[u_{xx}(x_i,t_n) + u_{xx}(x_i,t_{n-1})] + \frac{1}{2}[f(x_i,t_n) + f(x_i,t_{n-1})], \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N. \tag{6.132}$$

Let

$$s(\gamma, x_i, t_n) = w(\gamma)\, {}_0^C D_t^\gamma u(x_i, t_n).$$

Suppose function $s(\cdot, x_i, t_n) \in C^2[1,2]$. Applying the composite trapezoid formula to discretize the distributed integral, it follows from Lemma 6.1.1 that

$$\mathcal{D}_t^w u(x_i, t_n) = \Delta\gamma \sum_{l=0}^{2J} c_l w(\gamma_l)\ {}_0^C D_t^{\gamma_l} u(x_i, t_n) + O(\Delta\gamma^2).$$

Substituting the result above into (6.132) produces

$$\begin{aligned}
&\Delta\gamma \sum_{l=0}^{2J} c_l w(\gamma_l) \cdot \frac{1}{2}\big[{}_0^C D_t^{\gamma_l} u(x_i, t_n) + {}_0^C D_t^{\gamma_l} u(x_i, t_{n-1})\big] \\
&= \frac{1}{2}\big[u_{xx}(x_i, t_n) + u_{xx}(x_i, t_{n-1})\big] + \frac{1}{2}\big[f(x_i, t_n) + f(x_i, t_{n-1})\big] \\
&\quad + O(\Delta\gamma^2), \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N. \qquad (6.133)
\end{aligned}$$

Define the function $\hat{u}(x,t)$ like that in Exercise 3.1 in Chapter 3. Suppose $\hat{u}(x,\cdot) \in \mathscr{C}^{2+2}(\mathcal{R})$ and $u(\cdot, t) \in C^4[0, L]$.

Let $v(x,t) = u_t(x,t)$, then

$${}_0^C D_t^{\gamma_l} u(x_i, t_n) = {}_0^C D_t^{\gamma_l - 1} v(x_i, t_n) = \tau^{-(\gamma_l - 1)} \sum_{k=0}^{n} w_k^{(\gamma_l - 1)} v(x_i, t_{n-k}) + O(\tau^2).$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\big[{}_0^C D_t^{\gamma_l} u(x_i, t_n) + {}_0^C D_t^{\gamma_l} u(x_i, t_{n-1})\big] \\
&= \frac{1}{2}\left[\tau^{-(\gamma_l-1)} \sum_{k=0}^{n} w_k^{(\gamma_l-1)} v(x_i, t_{n-k}) + \tau^{-(\gamma_l-1)} \sum_{k=0}^{n-1} w_k^{(\gamma_l-1)} v(x_i, t_{n-1-k})\right] \\
&\quad + O(\tau^2) \\
&= \tau^{-(\gamma_l-1)} \sum_{k=0}^{n-1} w_k^{(\gamma_l-1)} \cdot \frac{1}{2}\big[v(x_i, t_{n-k}) + v(x_i, t_{n-1-k})\big] + O(\tau^2) \\
&= \tau^{-(\gamma_l-1)} \sum_{k=0}^{n-1} w_k^{(\gamma_l-1)} \cdot \frac{u(x_i, t_{n-k}) - u(x_i, t_{n-1-k})}{\tau} + O(\tau^2). \qquad (6.134)
\end{aligned}$$

Substituting (6.134) into (6.133) and denoting

$$f_i^{n-\frac{1}{2}} = \frac{1}{2}\big[f(x_i, t_n) + f(x_i, t_{n-1})\big],$$

we have

$$\begin{aligned}
&\Delta\gamma \sum_{l=0}^{2J} c_l w(\gamma_l) \tau^{-(\gamma_l-1)} \sum_{k=0}^{n-1} w_k^{(\gamma_l-1)} \delta_t U_i^{n-k-\frac{1}{2}} \\
&= \delta_x^2 U_i^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}} + (r_7)_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N,
\end{aligned}$$

and there exists a positive constant c_7 such that

$$\left|(r_7)_i^{n-\frac{1}{2}}\right| \leqslant c_7(\tau^2 + h^2 + \Delta\gamma^2), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N.$$

Noticing the initial-boundary value conditions (6.130)–(6.131), we can get a difference scheme for (6.129)–(6.131) as follows:

$$\begin{cases} \Delta\gamma \sum\limits_{l=0}^{2J} c_l w(\gamma_l)\tau^{-(\gamma_l-1)} \sum\limits_{k=0}^{n-1} w_k^{(\gamma_l-1)} \delta_t u_i^{n-k-\frac{1}{2}} \\ = \delta_x^2 u_i^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (6.135) \\ u_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, & (6.136) \\ u_0^n = \varphi_1(t_n), \quad u_M^n = \varphi_2(t_n), \quad 0 \leqslant n \leqslant N. & (6.137) \end{cases}$$

It can be proved that the difference scheme (6.135)–(6.137) is uniquely solvable, unconditionally stable and convergent. More details can be found in [26]. Regarding the ADI difference scheme for solving 2D time distributed-order wave equations, interested readers can refer to [27].

4. Ye et al.[104] and Hu et al.[37] studied the difference methods for the one-dimensional and multidimensional distributed-order fractional wave equations based on the L1 approximation, respectively.

5. There are some works on the difference methods for the space distributed-order equations. The interested readers may refer to [94].

Exercises 6

6.1 For the problem (6.1)–(6.3), construct the following difference scheme:

$$\begin{cases} \Delta\alpha \sum\limits_{l=0}^{2J} c_l w(\alpha_l)\tau^{-\alpha_l} \sum\limits_{k=0}^{n} g_k^{(\alpha_l)} u_i^{n-k} = \delta_x^2 u_i^n + f_i^n, \\ \qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = \varphi_1(t_n), \quad u_M^n = \varphi_2(t_n), \quad 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\hat{u}(x,t)$ like that in Section 2.1 and suppose $\hat{u}(x,\cdot) \in \mathscr{C}^{1+1}(\mathcal{R})$. For this difference scheme, try to

(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the function f;
(4) show the convergence.

6.2 For the problem (6.1)–(6.3), construct the following difference scheme:

$$\begin{cases} \Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\tau^{-\alpha_l} \sum_{k=0}^{n} g_k^{(\alpha_l)} \mathcal{A} u_i^{n-k} = \delta_x^2 u_i^n + \mathcal{A} f_i^n, \\ \qquad\qquad\qquad\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_i^0 = 0, \quad 1 \leqslant i \leqslant M-1, \\ u_0^n = \varphi_1(t_n), \quad u_M^n = \varphi_2(t_n), \quad 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\hat{u}(x,t)$ like that in Section 2.1 and suppose $\hat{u}(x,\cdot) \in \mathscr{C}^{1+1}(\mathcal{R})$. For this difference scheme, try to

(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the function f;
(4) show the convergence.

6.3 For the problem (6.42)–(6.44), construct the following difference scheme:

$$\begin{cases} \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\left[\tau^{-\alpha_l} \sum_{k=0}^{n} g_k^{(\alpha_l)} u_{ij}^{n-k}\right] = \Delta_h u_{ij}^n + f_{ij}^n, \\ \qquad\qquad\qquad\qquad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ u_{ij}^0 = 0, \quad (i,j) \in \omega, \\ u_{ij}^n = \varphi(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\hat{u}(x,y,t)$ like that in Section 2.10 and suppose $\hat{u}(x,y,\cdot) \in \mathscr{C}^{1+1}(\mathcal{R})$.

For this difference scheme, try to

(1) analyze the truncation error;
(2) show the unique solvability;
(3) show the stability with respect to the function f;
(4) show the convergence.

6.4 For the problem (6.42)–(6.44), construct the following difference scheme:

$$\begin{cases} \Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l)\left[\tau^{-\alpha_l} \sum_{k=0}^{n} g_k^{(\alpha_l)} \mathcal{A}_x \mathcal{A}_y u_{ij}^{n-k}\right] \\ = \mathcal{A}_y \delta_x^2 u_{ij}^n + \mathcal{A}_x \delta_y^2 u_{ij}^n + \mathcal{A}_x \mathcal{A}_y f_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N, \\ u_{ij}^0 = 0, \quad (i,j) \in \omega, \\ u_{ij}^n = \varphi(x_i, y_j, t_n), \quad (i,j) \in \partial\omega,\ 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\hat{u}(x,y,t)$ like that in Section 2.10 and suppose $\hat{u}(x,y,\cdot) \in \mathscr{C}^{1+1}(\mathcal{R})$.

For this difference scheme, try to

(1) analyze the truncation error;

(2) show the unique solvability;
(3) show the stability with respect to the function f;
(4) show the convergence.

6.5 Denote

$$\bar{\mu} = \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \tau^{-\alpha_l} g_0^{(\alpha_l)}.$$

Try to show that

$$\bar{\mu}\tau = O\left(\frac{1}{|\ln \tau|}\right).$$

For the problem (6.42)–(6.44), construct the following difference scheme:

$$\begin{cases} \Delta\alpha \sum\limits_{l=0}^{2J} c_l w(\alpha_l) \left[\tau^{-\alpha_l} \sum\limits_{k=0}^{n} g_k^{(\alpha_l)} u_{ij}^{n-k}\right] + \dfrac{\tau}{\bar{\mu}} \delta_x^2 \delta_y^2 \dfrac{u_{ij}^n - u_{ij}^{n-1}}{\tau} = \Delta_h u_{ij}^n + f_{ij}^n, \\ \qquad (i,j) \in \omega, \ 1 \leqslant n \leqslant N, \\ u_{ij}^0 = 0, \quad (i,j) \in \omega, \\ u_{ij}^n = \varphi(x_i, y_j, t_n), \quad (i,j) \in \partial\omega, \ 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\hat{u}(x,y,t)$ like that in Section 2.10 and suppose $\hat{u}(x,y,\cdot) \in \mathscr{C}^{1+1}(\mathcal{R})$.

For this difference scheme, try to
(1) analyze the truncation error;
(2) write the ADI form;
(3) show the unique solvability;
(4) show the stability with respect to the function f;
(5) show the convergence.

6.6 Denote

$$\bar{\nu} = \Delta\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \tau^{-\alpha_l} g_0^{(\alpha_l)}.$$

Try to show that

$$\bar{\nu}\tau = O\left(\frac{1}{|\ln \tau|}\right).$$

For the problem (6.42)–(6.44), construct the following difference scheme:

$$\begin{cases} \Delta\alpha \sum\limits_{l=0}^{2J} d_l w(\alpha_l) \left[\tau^{-\alpha_l} \sum\limits_{k=0}^{n} g_k^{(\alpha_l)} \mathcal{A}_x \mathcal{A}_y u_{ij}^{n-k}\right] + \dfrac{\tau}{\bar{\nu}} \delta_x^2 \delta_y^2 \dfrac{u_{ij}^n - u_{ij}^{n-1}}{\tau} \\ = \mathcal{A}_y \delta_x^2 u_{ij}^n + \mathcal{A}_x \delta_y^2 u_{ij}^n + \mathcal{A}_x \mathcal{A}_y f_{ij}^n, \quad (i,j) \in \omega, \ 1 \leqslant n \leqslant N, \\ u_{ij}^0 = 0, \quad (i,j) \in \omega, \\ u_{ij}^n = \varphi(x_i, y_j, t_n), \quad (i,j) \in \partial\omega, \ 0 \leqslant n \leqslant N. \end{cases}$$

Define the function $\hat{u}(x,y,t)$ like that in Section 2.10 and suppose $\hat{u}(x,y,\cdot) \in \mathscr{C}^{1+1}(\mathcal{R})$.

For this difference scheme, try to

(1) analyze the truncation error;

(2) write the ADI form;

(3) show the unique solvability;

(4) show the stability with respect to the function f;

(5) show the convergence.

A The Matlab code of sum-of-exponentials approximations for the kernel $t^{-\alpha}$ in the Caputo fractional derivative

```
function [xs,ws,nexp] = sumofexpappr2new(alpha,reps,dt,Tfinal )

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Copyright: all rights reserved by Shidong Jiang, Jiwei Zhang,
% Qian Zhang and Zhimin Zhang.
% Citation: please cite the following papers:
% [1] S. Jiang, J. Zhang, Q. Zhang and Z. Zhang. Fast evaluation
% of the Caputo fractional derivative and its applications to
% fractional diffusion equations. Commun. Comput. Phys., 21(2017),
% 650--678.
% [2] G. Beylkin and L. Monzn. On approximation of functions by
%  exponential sums. Appl. Comput. Harmon. Anal. 19(2005), 17--48.
% [3] G. Beylkin and L. Monzn. Approximation by exponential sums
% revisited. Appl. Comput. Harmon. Anal., 28(2):131--149, 2010.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% For given positive parameters: alpha, reps, dt and T, return
% sum-of-exponentials approximation for 1/t^alpha for the inverval
% dt<t<T  under relative error bounded by reps, i.\,e.,
%% |1/t^alpha - \sum_{l=1}^nexp ws(l)*exp(-xs(l))| <= reps,
% for all t in [dt,T]
% The following parameters will be calculated with
% xs: SOE approximation nodes
% ws: SOE approximation weights
% nexp: the number of SOE approximation weights or nodes
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
delta = dt/Tfinal;
h = 2*pi/(log(3) + alpha*log(1/cos(1)) + log(1/reps));
tlower = 1/alpha*log(reps*gamma(1+alpha));
if alpha>=1,
    tupper = log(1/delta) + log(log(1/reps)) + log(alpha) + 1/2;
else
    tupper = log(1/delta)+log(log(1/reps));
end
M = floor(tlower/h);
N = ceil(tupper/h);
```

https://doi.org/10.1515/9783110616064-007

```
n1 = M:-1;
xs1 = -exp(h*n1);
ws1 = h/gamma(alpha)*exp(alpha*h*n1);
% use prony's method to reduce the number of SOE
% approximation nodes
[ws1new,xs1new] = prony(xs1,ws1);
n2= 0:N;
xs2 = -exp(h*n2);
ws2 = h/gamma(alpha)*exp(alpha*h*n2);
xs = [-real(xs1new); -real(xs2.')];
ws = [real(ws1new); real(ws2.')];
xs = xs/Tfinal;
ws = ws/Tfinal^alpha;
nexp = length(ws);
return;
end

function [wsnew, xsnew] = prony(xs,ws)
M = length(xs);
errbnd = 1d-12;
h=zeros(2*M,1);
for j=1:2*M
    h(j)=xs.^(j-1)*ws';
end
C=h(1:M);
R=h(M:2*M-1);
H=hankel(C,R);
b=-h;
q = myls_qr(H, b, errbnd);
r = length(q);
A=zeros(2*M,r);
Coef = [1; flipud(q)];
xsnew=roots(Coef);
for j=1:2*M
    A(j,:)= xsnew.^(j-1);
end
wsnew = myls_svd(A,h,errbnd);
ind = find(real(xsnew)>=0);
p = length(ind);
assert(sum(abs(wsnew(ind))<1d-15) == p)
ind = find(real(xsnew)<0);
xsnew = xsnew(ind);
```

```
wsnew = wsnew(ind);
end

function x = myls_qr(A,b,eps)
% solve the rank deficient least squares problem by QR
% x is the LS solution, res is the residue
[m,n] = size(A);
[Q,R] = qr(A,0);
if nargin < 3
    eps = 1e-13;
end
s = diag(R);
r = sum(abs(s)>eps);
Q = Q(:, 1:r);
R = R(1:r,1:r);
b1 = b(r+1:m+r);
x = R\(Q.'*b1);
end

function [x,res] = myls_svd(A,b,eps)
% solve the rank deficient least squares problem by SVD
% x is the LS solution, res is the residue
[m,n] = size(A);
[U,S,V] = svd(A,0);
if nargin < 3
    eps = 1e-12;
end
s = diag(S);
r = sum(s>eps);
x = zeros(n,1);
for i=1:r
    x = x + (U(:,i)'*b)/s(i)*V(:,i);
end
if (nargout>1)
    res = norm(A*x-b)/norm(b);
end
end
```

This code is provided by the authors of [41]. We remark that the parameters in Lemma 1.7.1 of this book correspond to the parameters in the above code as α = alpha, $T = T$, ϵ = reps, $\hat{\tau}$ = dt, $N_{\exp}^{(\alpha)}$ = nexp, , $\omega_l^{(\alpha)}$ = ws(l), $s_l^{(\alpha)}$ = xs(l).

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