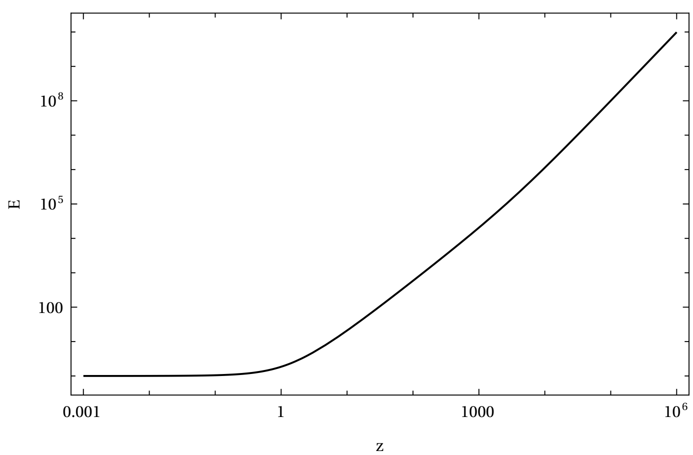
```
In[ ]:= Clear["Global`*"]

In[ ]:= Ωr[z_] := Ωr0 (1 + z)^4 / e[z]^2

In[ ]:= Ωr0 := Ωm0 / (1 + zeq)

In[ ]:= zeq := 2.5 × 10^4 Ωm0 h^2 (Tcmb / (2.7 Kelvin))^-4

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= 24 077.4 h^2 Ωm0

In[ ]:= zeq := 24077.4405856556` h^2 Ωm0

In[ ]:= Q1[z_] := 3 β H[z] × ρde[z]

In[ ]:= Q[z_] := Q1[z]

In[ ]:= ΩI[z_] := Q[z] / (3 MP^2 H[z]^3)

In[ ]:= ρde[z_] := 3 MP^2 H[z]^2 Ωde[z]^2

In[ ]:= Simplify[ΩI[z]]

Out[ ]= 3 β Ωde[z]^2

In[ ]:= ΩI[z_] := 3 β Ωde[z]^2

In[ ]:= Ωm0 = 0.3213;
        c = 0.8294;
        β = 0.0782;
        h = 0.6558;

In[ ]:= Ωr0

Out[ ]= 0.000096542
```

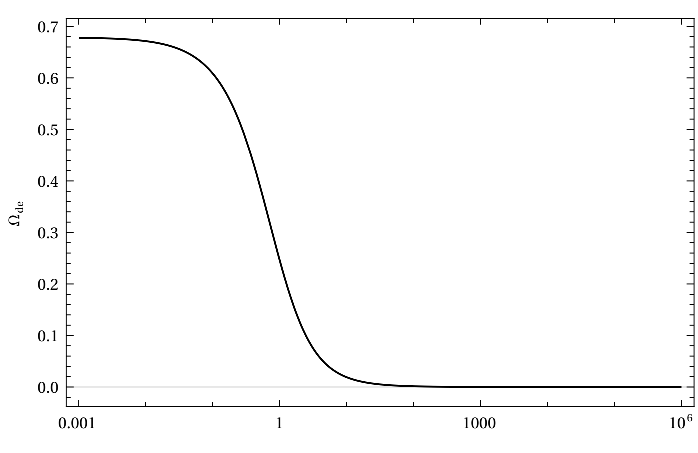
In[ ]:= $\text{sol} = \text{NDSolve}\left[\left\{\frac{1}{e[z]} e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),\right.\right.$
$\Omega de'[z] == -\frac{2\,(1 - \Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1 - \Omega de[z])}\right),$
$\left.\left. e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\right\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^6\}\right]$

Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar],
Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:= LogLogPlot[e[z] /. sol, {z, 10^{-3}, 10^6}, PlotTheme → "Scientific", FrameStyle → Black, PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "E"}]

Out[ ]=


In[ ]:= LogLinearPlot[Ωde[z] /. sol, {z, 10^{-3}, 10^6}, PlotTheme → "Scientific", FrameStyle → Black, PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "Ω_{de}"}]

Out[ ]=


```
In[ ]:= Clear["Global`*"]

In[ ]:= Ωr[z_] := Ωr0 (1 + z)^4 / e[z]^2

In[ ]:= Ωr0 := Ωm0 / (1 + zeq)

In[ ]:= zeq := 2.5 × 10^4 Ωm0 h^2 (Tcmb / (2.7 Kelvin))^-4

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= 24 077.4 h^2 Ωm0

In[ ]:= zeq := 24077.4405856556` h^2 Ωm0

In[ ]:= Q1[z_] := 3 β H[z] × ρde[z]

In[ ]:= Q[z_] := Q1[z]

In[ ]:= ΩI[z_] := Q[z] / (3 MP^2 H[z]^3)

In[ ]:= ρde[z_] := 3 MP^2 H[z]^2 Ωde[z]

In[ ]:= Simplify[ΩI[z]]

Out[ ]= 3 β Ωde[z]

In[ ]:= ΩI[z_] := 3 β Ωde[z]

In[ ]:= Ωm0 = 0.3213;
        c = 0.8294;
        β = 0.0782;
        h = 0.6558;

In[ ]:= Ωr0

Out[ ]= 0.000096542
```

In[ ]:= sol = NDSolve$\left[\left\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-3-\Omega r[z]}{2\,\Omega de[z]}\right),\right.\right.$

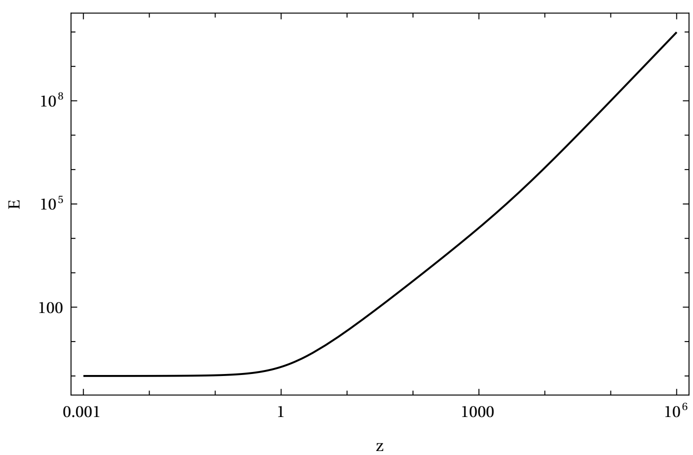
$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-\Omega r[z]}{2\,(1-\Omega de[z])}\right),$

$\left.\left. e[0] == 1,\ \Omega de[0] == 1-\Omega m0-\Omega r0\right\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\right]$

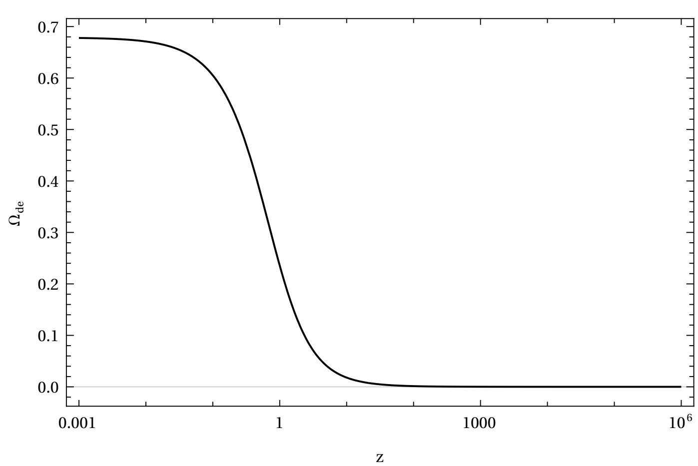
Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:= LogLogPlot[e[z] /. sol, {z, 10^{-3}, 10^6}, PlotTheme → "Scientific", FrameStyle → Black, PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "E"}]

Out[ ]=


In[ ]:= LogLinearPlot[Ωde[z] /. sol, {z, 10^{-3}, 10^6}, PlotTheme → "Scientific", FrameStyle → Black, PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "Ω_{de}"}]

Out[ ]=


In[ ]:= Clear["Global`*"]

In[ ]:= $\Omega r[z\_] := \dfrac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \dfrac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2\left(\dfrac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= zeq := 24077.4405856556` $h^2\,\Omega m0$

In[ ]:= $Q2[z\_] := 3\,\beta\,H[z]\times\rho c[z]$

In[ ]:= Q[z_] := Q2[z]

In[ ]:= $\Omega I[z\_] := \dfrac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[ΩI[z]]

Out[ ]= $\dfrac{3\,(1+z)^3\,\beta\,\Omega c0}{e[z]^2}$

In[ ]:= $\Omega I[z\_] := \dfrac{3\,(1+z)^3\,\beta\,\Omega c0}{e[z]^2}$

In[ ]:= Ωm0 = 0.3225;
Ωb0 = 0.0518;
Ωc0 = Ωm0 - Ωb0;
c = 0.7538;
β = 0.0092;
h = 0.6399;

In[ ]:= Ωr0

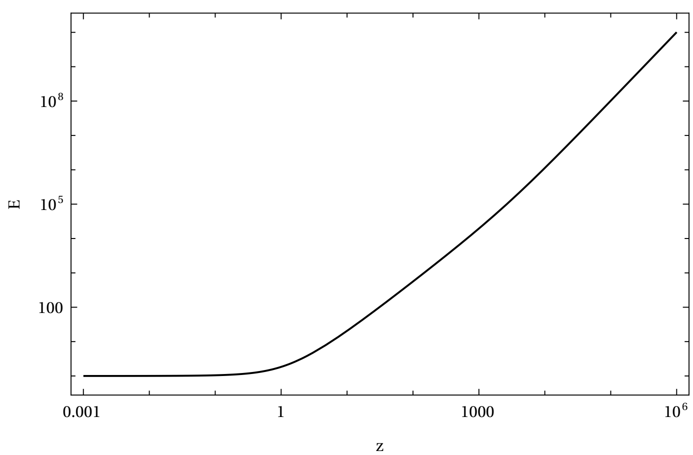
Out[ ]= 0.000101398

In[ ]:=
$$\text{sol} = \text{NDSolve}\Big[\Big\{\frac{1}{\text{e}[z]}\,\text{e}'[z] == -\frac{\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-3-\Omega\text{r}[z]}{2\,\Omega\text{de}[z]}\right),$$
$$\Omega\text{de}'[z] == -\frac{2\,(1-\Omega\text{de}[z])\,\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-\Omega\text{r}[z]}{2\,(1-\Omega\text{de}[z])}\right),$$
$$\text{e}[0] == 1,\ \Omega\text{de}[0] == 1-\Omega\text{m0}-\Omega\text{r0}\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

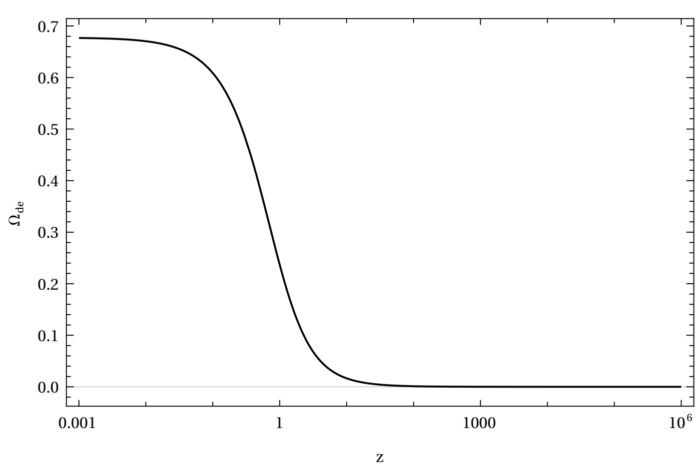
Out[ ]=
$\{\{e \to \text{InterpolatingFunction}[$ Domain : $\{\{0.,\ 1.\times 10^{6}\}\}$ Output : scalar $],$

$\Omega\text{de} \to \text{InterpolatingFunction}[$ Domain : $\{\{0.,\ 1.\times 10^{6}\}\}$ Output : scalar $]\}\}$

In[ ]:=
```
LogLogPlot[e[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "E"}]
```

Out[ ]=


In[ ]:=
```
LogLinearPlot[Ωde[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=


In[ ]:= Clear["Global`*"]

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5 \times 10^4\,\Omega m0\,h^2 \left(\frac{Tcmb}{2.7\ Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\ h^2\ \Omega m0$

In[ ]:= $zeq := 24077.4405856556\text{`}\ h^2\,\Omega m0$

In[ ]:= $Q3[z\_] := 3\,\beta\,H[z]\,(\rho de[z] + \rho c[z])$

In[ ]:= Q[z_] := Q3[z]

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[ΩI[z]]

Out[ ]= $\frac{3\,\beta\left((1+z)^3\,\Omega c0 + e[z]^2\,\Omega de[z]\right)}{e[z]^2}$

In[ ]:= $\Omega I[z\_] := \frac{3\,\beta\left((1+z)^3\,\Omega c0 + e[z]^2\,\Omega de[z]\right)}{e[z]^2}$

In[ ]:=
```
Ωm0 = 0.3235;
Ωb0 = 0.0547;
Ωc0 = Ωm0 - Ωb0;
c = 0.7675;
β = 0.0088;
h = 0.6394;
```

In[ ]:= Ωr0

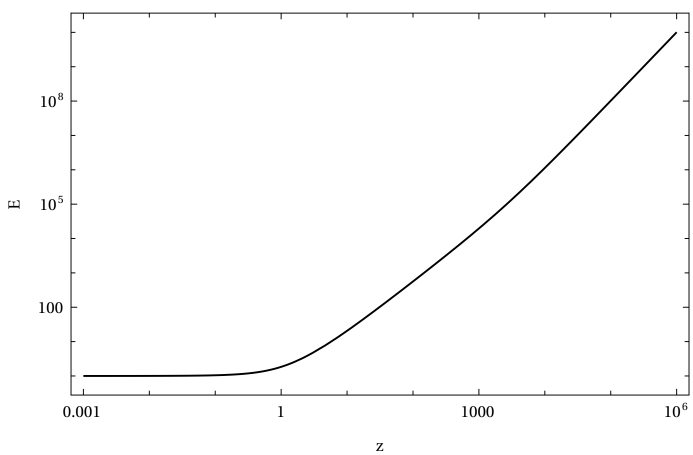
Out[ ]= 0.000101557

In[ ]:=
```
sol = NDSolve[{1/e[z] e'[z] == -Ωde[z]/(1 + z) (1/c Sqrt[Ωde[z]] + 1/2 + (ΩI[z] - 3 - Ωr[z])/(2 Ωde[z])),
    Ωde'[z] == -(2 (1 - Ωde[z]) Ωde[z])/(1 + z) (1/c Sqrt[Ωde[z]] + 1/2 + (ΩI[z] - Ωr[z])/(2 (1 - Ωde[z]))),
    e[0] == 1, Ωde[0] == 1 - Ωm0 - Ωr0}, {e, Ωde}, {z, 0, 10^6}]
```

Out[ ]=
```
{{e → InterpolatingFunction[ Domain : {{0., 1. × 10^6}} Output : scalar ],
  Ωde → InterpolatingFunction[ Domain : {{0., 1. × 10^6}} Output : scalar ]}}
```

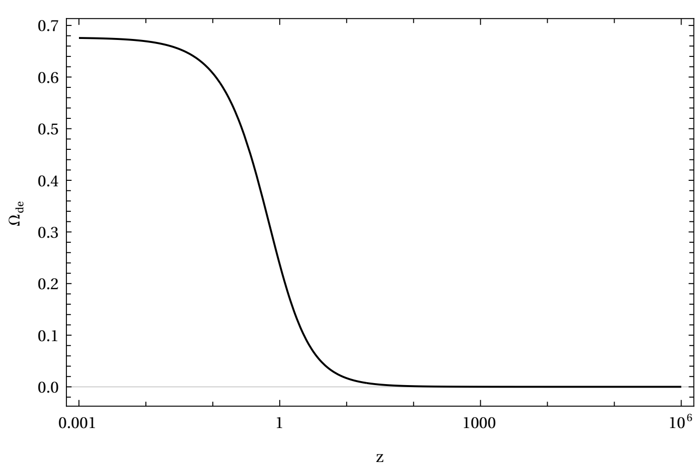
In[ ]:=
```
LogLogPlot[e[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "E"}]
```

Out[ ]=



In[ ]:=
```
LogLinearPlot[Ωde[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=



In[ ]:= Clear["Global`*"]

In[ ]:= $\Omega r[z\_] := \dfrac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \dfrac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2\left(\dfrac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= $zeq := 24077.4405856556\text{`}\;h^2\,\Omega m0$

In[ ]:= $Q4[z\_] := 3\,\beta\,H[z]\,\sqrt{\rho de[z]\times\rho c[z]}$

In[ ]:= Q[z_] := Q4[z]

In[ ]:= $\Omega I[z\_] := \dfrac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[PowerExpand[Simplify[ΩI[z]]]]

Out[ ]= $\dfrac{3\,(1+z)^{3/2}\,\beta\,\sqrt{\Omega c0}\,\sqrt{\Omega de[z]}}{e[z]}$

In[ ]:= $\Omega I[z\_] := \dfrac{3\,(1+z)^{3/2}\,\beta\,\sqrt{\Omega c0}\,\sqrt{\Omega de[z]}}{e[z]}$

In[ ]:=
```
Ωm0 = 0.3233;
Ωb0 = 0.0526;
Ωc0 = Ωm0 - Ωb0;
c = 0.7868;
β = 0.0346;
h = 0.6512;
```

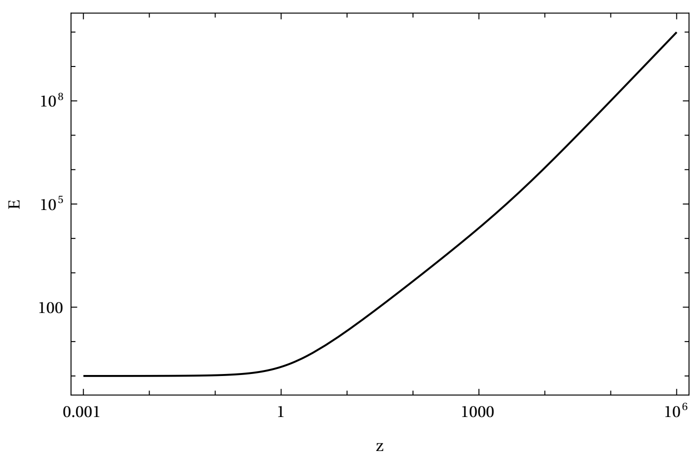
In[ ]:= Ωr0

Out[ ]= 0.0000979105

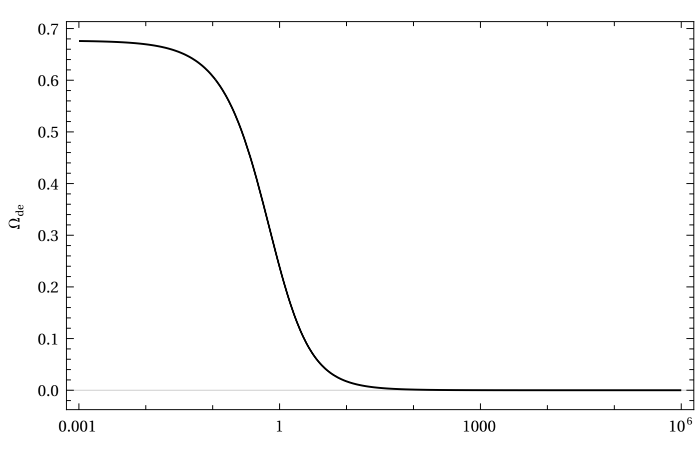
In[ ]:=
```
sol = NDSolve[{1/e[z] e'[z] == -Ωde[z]/(1 + z) (1/c Sqrt[Ωde[z]] + 1/2 + (ΩI[z] - 3 - Ωr[z])/(2 Ωde[z])),
   Ωde'[z] == -(2 (1 - Ωde[z]) Ωde[z])/(1 + z) (1/c Sqrt[Ωde[z]] + 1/2 + (ΩI[z] - Ωr[z])/(2 (1 - Ωde[z]))),
   e[0] == 1, Ωde[0] == 1 - Ωm0 - Ωr0}, {e, Ωde}, {z, 0, 10^6}]
```

Out[ ]=
```
{{e → InterpolatingFunction[ Domain: {{0., 1.×10^6}} Output: scalar ],
  Ωde → InterpolatingFunction[ Domain: {{0., 1.×10^6}} Output: scalar ]}}
```

In[ ]:=
```
LogLogPlot[e[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "E"}]
```

Out[ ]=


In[ ]:=
```
LogLinearPlot[Ωde[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "Ωde"}]
```

Out[ ]=


```
In[ ]:= Clear["Global`*"]
```

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0 \, (1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5 \times 10^4 \, \Omega m0 \, h^2 \left(\frac{Tcmb}{2.7 \, Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4 \, h^2 \, \Omega m0$

In[ ]:= $zeq := 24077.4405856556\text{`} \; h^2 \, \Omega m0$

In[ ]:= $Q5[z\_] := 3 \, \beta \, H[z] \, \frac{\rho de[z] \times \rho c[z]}{\rho de[z] + \rho c[z]}$

In[ ]:= Q[z_] := Q5[z]

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3 \, MP^2 \, H[z]^3}$

In[ ]:= $\rho c[z\_] := 3 \, MP^2 \, H0^2 \, \Omega c0 \, (1+z)^3$

In[ ]:= $\rho de[z\_] := 3 \, MP^2 \, H[z]^2 \, \Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[PowerExpand[Simplify[ΩI[z]]]]

Out[ ]= $\frac{3 \, (1+z)^3 \, \beta \, \Omega c0 \, \Omega de[z]}{(1+z)^3 \, \Omega c0 + e[z]^2 \, \Omega de[z]}$

In[ ]:= $\Omega I[z\_] := \frac{3 \, (1+z)^3 \, \beta \, \Omega c0 \, \Omega de[z]}{(1+z)^3 \, \Omega c0 + e[z]^2 \, \Omega de[z]}$

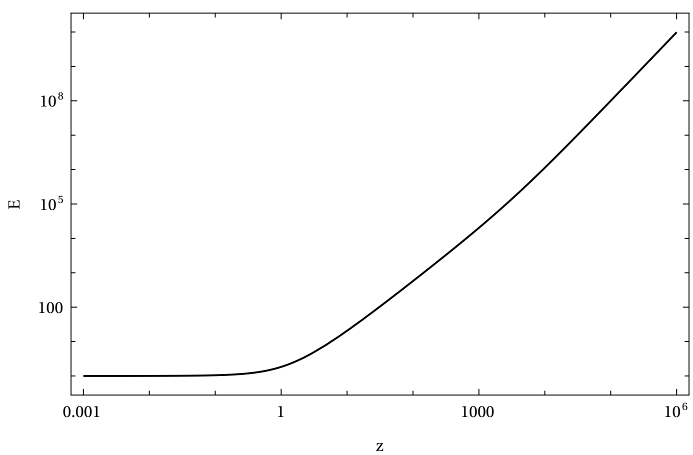
```
In[ ]:= Ωm0 = 0.3224;
        Ωb0 = 0.0521;
        Ωc0 = Ωm0 - Ωb0;
        c = 0.7983;
        β = 0.0958;
        h = 0.6545;
```

```
In[ ]:= Ωr0
Out[ ]= 0.0000969259
```

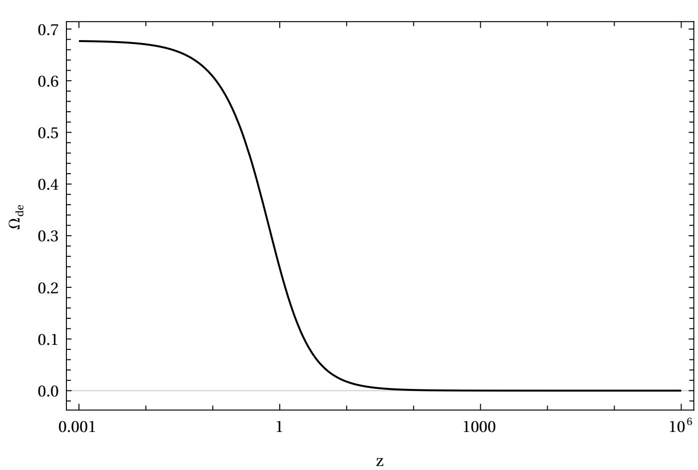
In[ ]:=
$$\text{sol} = \text{NDSolve}\Big[\Big\{\frac{1}{\text{e}[z]}\,\text{e}'[z] == -\frac{\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-3-\Omega\text{r}[z]}{2\,\Omega\text{de}[z]}\right),$$
$$\Omega\text{de}'[z] == -\frac{2\,(1-\Omega\text{de}[z])\,\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-\Omega\text{r}[z]}{2\,(1-\Omega\text{de}[z])}\right),$$
$$\text{e}[0] == 1,\ \Omega\text{de}[0] == 1-\Omega\text{m0}-\Omega\text{r0}\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]=
{{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],
Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:=
```
LogLogPlot[e[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
 PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "E"}]
```

Out[ ]=


In[ ]:=
```
LogLinearPlot[Ωde[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
 PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=


```
(*********************************************************************************)
In[ ]:= Clear["Global`*"]

In[ ]:= Ωr[z_] := Ωr0 (1 + z)^4 / e[z]^2

In[ ]:= Ωr0 := Ωm0 / (1 + zeq)

In[ ]:= zeq := 2.5 × 10^4 Ωm0 h^2 (Tcmb / (2.7 Kelvin))^-4

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= 24 077.4 h^2 Ωm0

In[ ]:= zeq := 24077.4405856556`  h^2 Ωm0

In[ ]:= Q5[z_] := 3 β H[z] (ρde[z] × ρc[z]) / (ρde[z] + ρc[z])

In[ ]:= Q[z_] := Q5[z]

In[ ]:= ΩI[z_] := Q[z] / (3 MP^2 H[z]^3)

In[ ]:= ρc[z_] := 3 MP^2 H0^2 Ωc0 (1 + z)^3

In[ ]:= ρde[z_] := 3 MP^2 H[z]^2 Ωde[z]

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[PowerExpand[Simplify[ΩI[z]]]]

Out[ ]= 3 (1 + z)^3 β Ωc0 Ωde[z] / ((1 + z)^3 Ωc0 + e[z]^2 Ωde[z])

In[ ]:= ΩI[z_] := 3 (1 + z)^3 β Ωc0 Ωde[z] / ((1 + z)^3 Ωc0 + e[z]^2 Ωde[z])

In[ ]:= Ωm0 = 0.3224;
        Ωb0 = 0.0521;
        Ωc0 = Ωm0 - Ωb0;
        c = 0.7983;
        β = 0.0958;
        h = 0.6545;

In[ ]:= Ωr0

Out[ ]= 0.0000969259
```

In[ ]:=

$$\text{sol} = \text{NDSolve}\left[\left\{\frac{1}{\text{e}[z]}\,\text{e}'[z] == -\frac{\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-3-\Omega\text{r}[z]}{2\,\Omega\text{de}[z]}\right),\right.\right.$$
$$\Omega\text{de}'[z] == -\frac{2\,(1-\Omega\text{de}[z])\,\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-\Omega\text{r}[z]}{2\,(1-\Omega\text{de}[z])}\right),$$
$$\left.\left.\text{e}[0] == 1,\ \Omega\text{de}[0] == 1-\Omega\text{m0}-\Omega\text{r0}\right\},\ \{\text{e},\ \Omega\text{de}\},\ \{z,\ 0,\ 10^{6}\}\right]$$

Out[ ]=

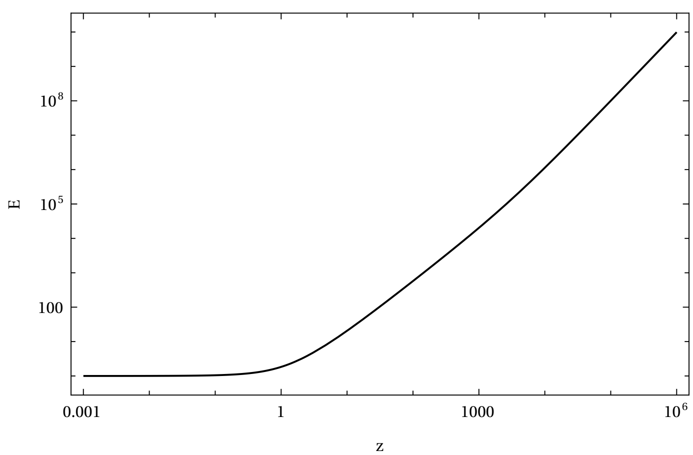
{{e → InterpolatingFunction[Domain: {{0., 1. × 10^6}} Output: scalar],

Ωde → InterpolatingFunction[Domain: {{0., 1. × 10^6}} Output: scalar]}}

In[ ]:=

```
LogLogPlot[e[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
 PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "E"}]
```

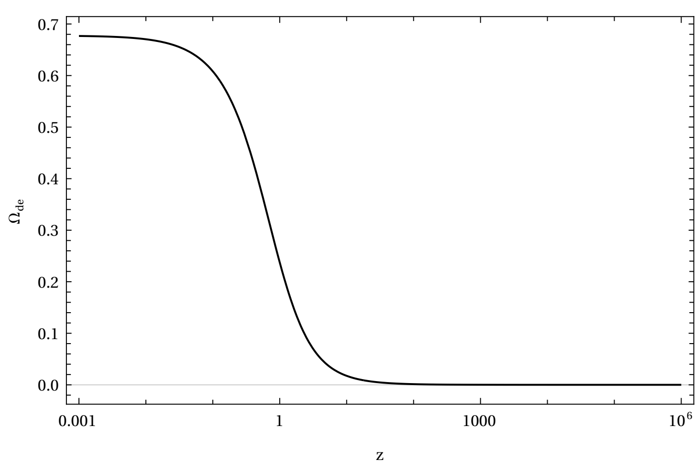
Out[ ]=



In[ ]:=

```
LogLinearPlot[Ωde[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
 PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=



(**)

In[ ]:= Clear["Global`*"]

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2 \left(\frac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= zeq := 24077.4405856556` $h^2\,\Omega m0$

In[ ]:= $Q5[z\_] := 3\,\beta\,H[z]\,\frac{\rho de[z]\times\rho c[z]}{\rho de[z]+\rho c[z]}$

In[ ]:= Q[z_] := Q5[z]

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[PowerExpand[Simplify[ΩI[z]]]]

Out[ ]= $\frac{3\,(1+z)^3\,\beta\,\Omega c0\,\Omega de[z]}{(1+z)^3\,\Omega c0+e[z]^2\,\Omega de[z]}$

In[ ]:= $\Omega I[z\_] := \frac{3\,(1+z)^3\,\beta\,\Omega c0\,\Omega de[z]}{(1+z)^3\,\Omega c0+e[z]^2\,\Omega de[z]}$

In[ ]:=
```
Ωm0 = 0.3224;
Ωb0 = 0.0521;
Ωc0 = Ωm0 - Ωb0;
c = 0.65;
h = 0.6545;
```

In[ ]:= $\beta = 0.02;$

In[ ]:=
$$\text{sol1} = \text{NDSolve}\left[\left\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),\right.\right.$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$\left.\left. e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\right\},\ \{e, \Omega de\},\ \{z, 0, 10^6\}\right]$$

Out[ ]= {{e → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:= $\beta = 0.04;$

In[ ]:=
$$\text{sol2} = \text{NDSolve}\left[\left\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),\right.\right.$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$\left.\left. e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\right\},\ \{e, \Omega de\},\ \{z, 0, 10^6\}\right]$$

Out[ ]= {{e → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:= $\beta = 0.06;$

In[ ]:=
$$\text{sol3} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\,e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-3-\Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-\Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1-\Omega m0-\Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction [Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:= β = 0.08;

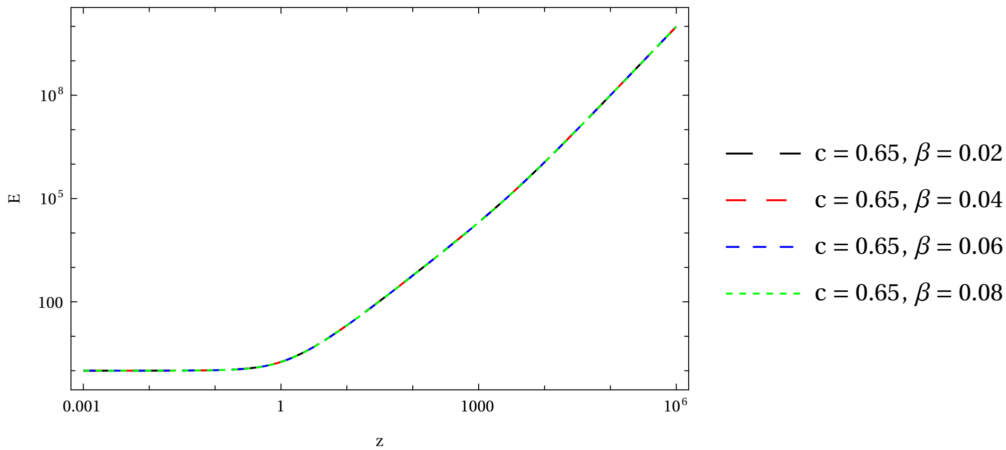
In[ ]:=
$$\text{sol4} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\,e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-3-\Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-\Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1-\Omega m0-\Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction [Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

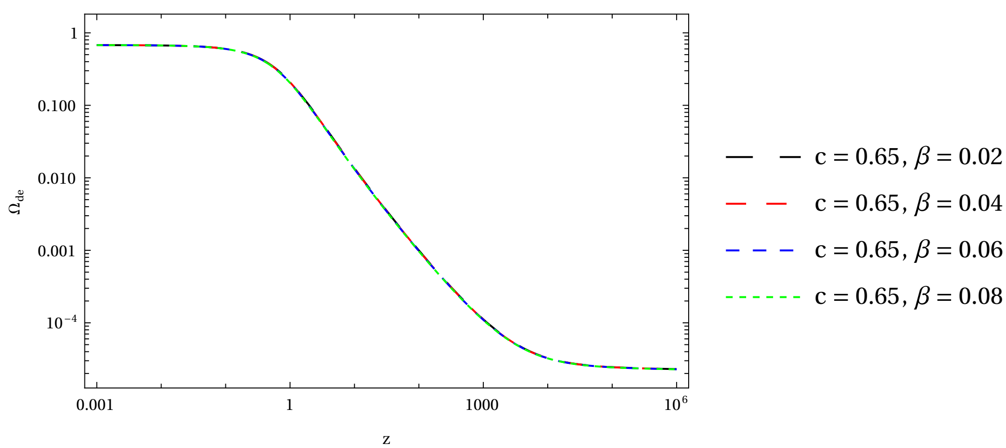
In[ ]:=
```
LogLogPlot[{e[z] /. sol1, e[z] /. sol2, e[z] /. sol3, e[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=



In[ ]:=
```
LogLogPlot[{Ωde[z] /. sol1, Ωde[z] /. sol2, Ωde[z] /. sol3, Ωde[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "Ω_de"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=




```
(******************************************************************************)
```

```
(*************************************************************************************)
```

In[ ]:= Clear["Global`*"]

In[ ]:= $\Omega r[z\_] := \dfrac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \dfrac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2 \left(\dfrac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= $zeq := 24077.4405856556\text{`}\ h^2\,\Omega m0$

In[ ]:= $Q4[z\_] := 3\,\beta\,H[z]\,\sqrt{\rho de[z]\times\rho c[z]}$

In[ ]:= Q[z_] := Q4[z]

In[ ]:= $\Omega I[z\_] := \dfrac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[PowerExpand[Simplify[ΩI[z]]]]

Out[ ]= $\dfrac{3\,(1+z)^{3/2}\,\beta\,\sqrt{\Omega c0}\,\sqrt{\Omega de[z]}}{e[z]}$

In[ ]:= $\Omega I[z\_] := \dfrac{3\,(1+z)^{3/2}\,\beta\,\sqrt{\Omega c0}\,\sqrt{\Omega de[z]}}{e[z]}$

In[ ]:=
```
Ωm0 = 0.3233;
Ωb0 = 0.0526;
Ωc0 = Ωm0 - Ωb0;
c = 0.7868;
β = 0.0346;
h = 0.6512;
```

In[ ]:= Ωr0

Out[ ]= 0.0000979105

In[ ]:=

$$\text{sol} = \text{NDSolve}\Big[\Big\{\frac{1}{\text{e}[z]}\,\text{e}'[z] == -\frac{\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]} + \frac{1}{2} + \frac{\Omega\text{I}[z] - 3 - \Omega\text{r}[z]}{2\,\Omega\text{de}[z]}\right),$$
$$\Omega\text{de}'[z] == -\frac{2\,(1-\Omega\text{de}[z])\,\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]} + \frac{1}{2} + \frac{\Omega\text{I}[z] - \Omega\text{r}[z]}{2\,(1-\Omega\text{de}[z])}\right),$$
$$\text{e}[0] == 1,\ \Omega\text{de}[0] == 1 - \Omega\text{m0} - \Omega\text{r0}\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{z,\ 0,\ 10^6\}\Big]$$

Out[ ]=

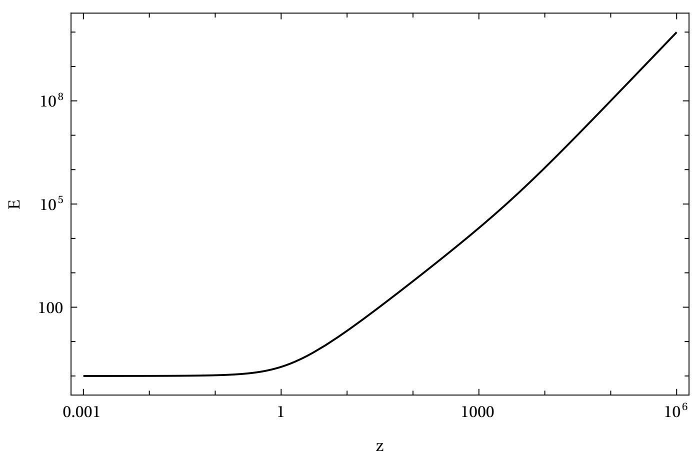
{{e → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:=

LogLogPlot[e[z] /. sol, {z, 10^{-3}, 10^6}, PlotTheme → "Scientific", FrameStyle → Black, PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "E"}]

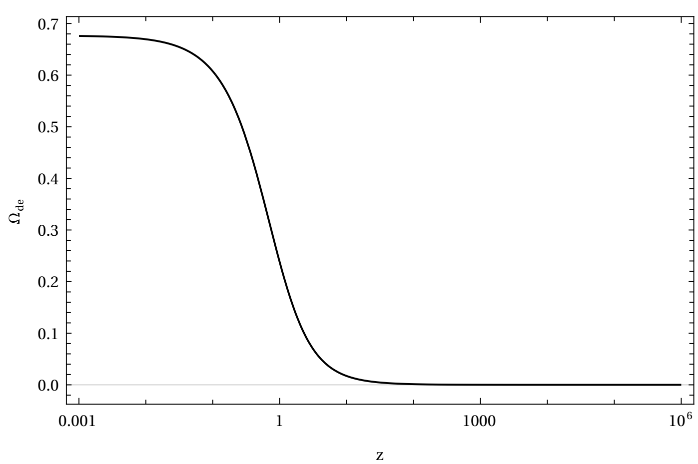
Out[ ]=



In[ ]:=

LogLinearPlot[Ωde[z] /. sol, {z, 10^{-3}, 10^6}, PlotTheme → "Scientific", FrameStyle → Black, PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "Ω_{de}"}]

Out[ ]=



(***)

In[ ]:= `Clear["Global`*"]`

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2\left(\frac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

In[ ]:= `Tcmb := 2.7255 Kelvin`

In[ ]:= `Simplify[zeq]`

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= $zeq := 24077.4405856556`\ h^2\,\Omega m0$

In[ ]:= $Q4[z\_] := 3\,\beta\,H[z]\,\sqrt{\rho de[z]\times\rho c[z]}$

In[ ]:= `Q[z_] := Q4[z]`

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= `H[z_] := H0 e[z]`

In[ ]:= `Simplify[PowerExpand[Simplify[ΩI[z]]]]`

Out[ ]= $\frac{3\,(1+z)^{3/2}\,\beta\,\sqrt{\Omega c0}\,\sqrt{\Omega de[z]}}{e[z]}$

In[ ]:= $\Omega I[z\_] := \frac{3\,(1+z)^{3/2}\,\beta\,\sqrt{\Omega c0}\,\sqrt{\Omega de[z]}}{e[z]}$

In[ ]:=
```
Ωm0 = 0.3233;
Ωb0 = 0.0526;
Ωc0 = Ωm0 - Ωb0;
c = 0.65;
h = 0.6512;
```

In[ ]:= $\beta = 0.02;$

In[ ]:=
$$\text{sol1} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\,e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^6\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain: {{0., 1. × 10^6}} Output: scalar],

Ωde → InterpolatingFunction[Domain: {{0., 1. × 10^6}} Output: scalar]}}

In[ ]:= $\beta = 0.04;$

In[ ]:=
$$\text{sol2} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\,e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^6\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain: {{0., 1. × 10^6}} Output: scalar],

Ωde → InterpolatingFunction[Domain: {{0., 1. × 10^6}} Output: scalar]}}

In[ ]:= $\beta = 0.06;$

In[ ]:=
$$\text{sol3} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:= $\beta = 0.08;$

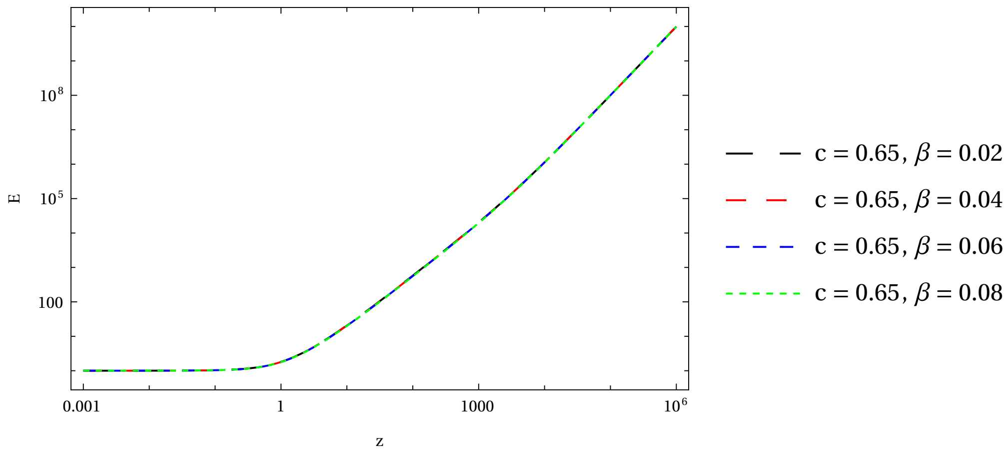
In[ ]:=
$$\text{sol4} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

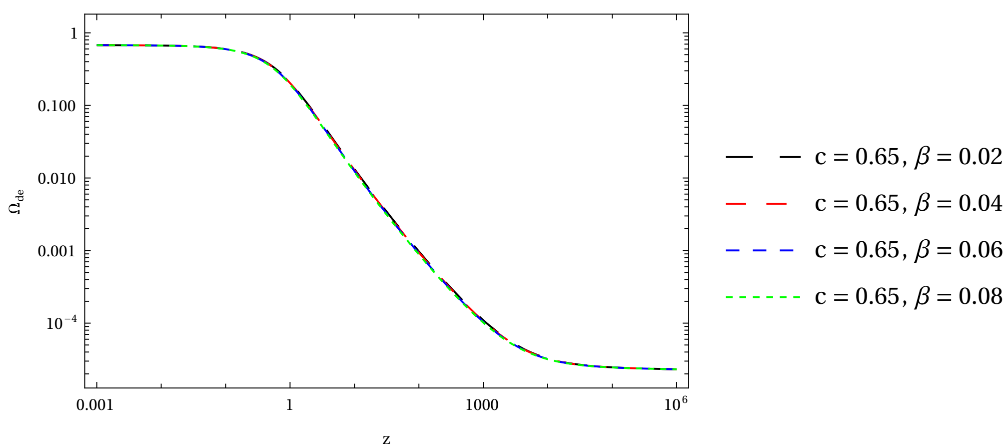
In[ ]:=
```
LogLogPlot[{e[z] /. sol1, e[z] /. sol2, e[z] /. sol3, e[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=



In[ ]:=
```
LogLogPlot[{Ωde[z] /. sol1, Ωde[z] /. sol2, Ωde[z] /. sol3, Ωde[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "Ω_de"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=




```
(*******************************************************************************)
```

```
(*********************************************************************************)
In[ ]:= Clear["Global`*"]
```

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2\left(\frac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= zeq := 24077.4405856556` $h^2\,\Omega m0$

In[ ]:= $Q3[z\_] := 3\,\beta\,H[z]\,(\rho de[z] + \rho c[z])$

In[ ]:= Q[z_] := Q3[z]

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[ΩI[z]]

Out[ ]= $\frac{3\,\beta\left((1+z)^3\,\Omega c0 + e[z]^2\,\Omega de[z]\right)}{e[z]^2}$

In[ ]:= $\Omega I[z\_] := \frac{3\,\beta\left((1+z)^3\,\Omega c0 + e[z]^2\,\Omega de[z]\right)}{e[z]^2}$

```
In[ ]:= Ωm0 = 0.3235;
        Ωb0 = 0.0547;
        Ωc0 = Ωm0 - Ωb0;
        c = 0.7675;
        β = 0.0088;
        h = 0.6394;

In[ ]:= Ωr0

Out[ ]= 0.000101557
```

In[ ]:=

$$\texttt{sol} = \texttt{NDSolve}\Big[\Big\{\frac{1}{\texttt{e[z]}}\,\texttt{e'[z]} == -\frac{\Omega\texttt{de[z]}}{1+\texttt{z}}\left(\frac{1}{\texttt{c}}\sqrt{\Omega\texttt{de[z]}}+\frac{1}{2}+\frac{\Omega\texttt{I[z]}-3-\Omega\texttt{r[z]}}{2\,\Omega\texttt{de[z]}}\right),$$

$$\Omega\texttt{de'[z]} == -\frac{2\,(1-\Omega\texttt{de[z]})\,\Omega\texttt{de[z]}}{1+\texttt{z}}\left(\frac{1}{\texttt{c}}\sqrt{\Omega\texttt{de[z]}}+\frac{1}{2}+\frac{\Omega\texttt{I[z]}-\Omega\texttt{r[z]}}{2\,(1-\Omega\texttt{de[z]})}\right),$$

$$\texttt{e[0]} == 1,\ \Omega\texttt{de[0]} == 1-\Omega\texttt{m0}-\Omega\texttt{r0}\Big\},\ \{\texttt{e},\ \Omega\texttt{de}\},\ \{\texttt{z},\ 0,\ 10^{6}\}\Big]$$

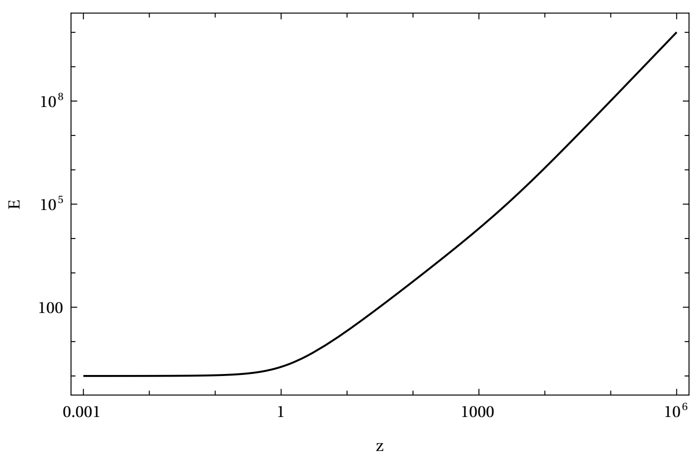
Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:=

```
LogLogPlot[e[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "E"}]
```

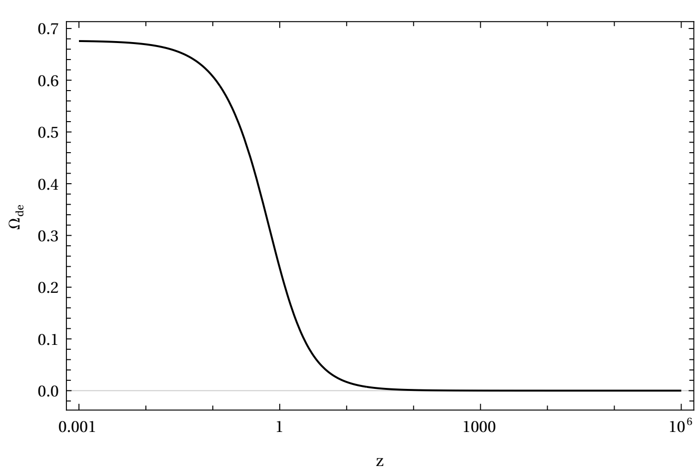
Out[ ]=



In[ ]:=

```
LogLinearPlot[Ωde[z] /. sol, {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "Ωde"}]
```

Out[ ]=



(***)

In[ ]:= Clear["Global`*"]

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2 \left(\frac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= zeq := 24077.4405856556` h^2 Ωm0

In[ ]:= Q3[z_] := 3 β H[z] (ρde[z] + ρc[z])

In[ ]:= Q[z_] := Q3[z]

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[ΩI[z]]

Out[ ]= $\frac{3\,\beta\left((1+z)^3\,\Omega c0 + e[z]^2\,\Omega de[z]\right)}{e[z]^2}$

In[ ]:= $\Omega I[z\_] := \frac{3\,\beta\left((1+z)^3\,\Omega c0 + e[z]^2\,\Omega de[z]\right)}{e[z]^2}$

In[ ]:=
```
Ωm0 = 0.3235;
Ωb0 = 0.0547;
Ωc0 = Ωm0 - Ωb0;
c = 0.65;
h = 0.6394;
```

In[ ]:= $\beta = 0.02;$

In[ ]:= $$\text{sol1} = \text{NDSolve}\left[\left\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),\right.\right.$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$\left.\left. e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\right\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\right]$$

Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:= $\beta = 0.04;$

In[ ]:= $$\text{sol2} = \text{NDSolve}\left[\left\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),\right.\right.$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$\left.\left. e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\right\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\right]$$

Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:= $\beta = 0.06;$

In[ ]:=
$$\texttt{sol3} = \texttt{NDSolve}\Big[\Big\{\frac{1}{e[z]}\,e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-3-\Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-\Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1-\Omega m0-\Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain: {{0., 1. × 10^6}} Output: scalar],

Ωde → InterpolatingFunction[Domain: {{0., 1. × 10^6}} Output: scalar]}}

$\beta = 0.08;$

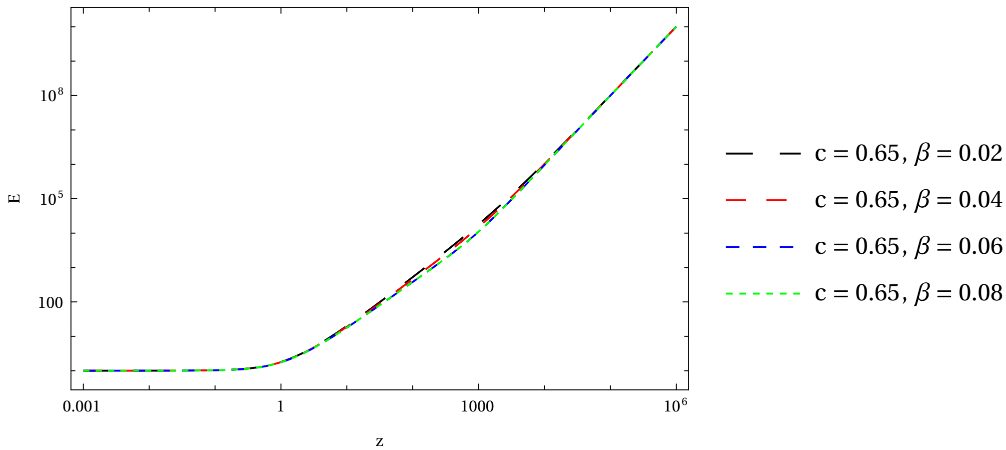
In[ ]:=
$$\texttt{sol4} = \texttt{NDSolve}\Big[\Big\{\frac{1}{e[z]}\,e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-3-\Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-\Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1-\Omega m0-\Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain: {{0., 1. × 10^6}} Output: scalar],

Ωde → InterpolatingFunction[Domain: {{0., 1. × 10^6}} Output: scalar]}}

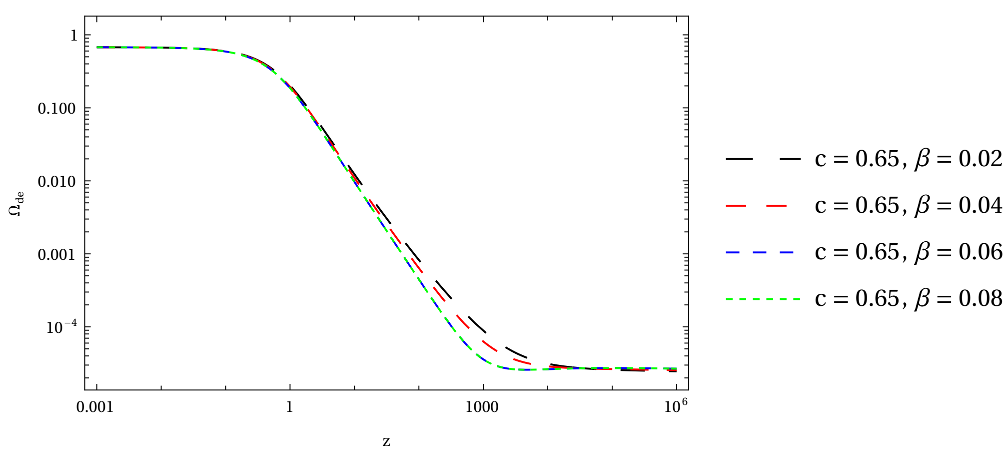
In[ ]:=
```
LogLogPlot[{e[z] /. sol1, e[z] /. sol2, e[z] /. sol3, e[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=



In[ ]:=
```
LogLogPlot[{Ωde[z] /. sol1, Ωde[z] /. sol2, Ωde[z] /. sol3, Ωde[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "Ω_de"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=




```
(**********************************************************************************)
```

```
(*******************************************************************************)
```

In[ ]:= `Clear["Global`*"]`

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5 \times 10^4\,\Omega m0\,h^2 \left(\frac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

In[ ]:= `Tcmb := 2.7255 Kelvin`

In[ ]:= `Simplify[zeq]`

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= $zeq := 24077.4405856556`\ h^2\,\Omega m0$

In[ ]:= $Q2[z\_] := 3\,\beta\,H[z] \times \rho c[z]$

In[ ]:= `Q[z_] := Q2[z]`

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= `H[z_] := H0 e[z]`

In[ ]:= `Simplify[ΩI[z]]`

Out[ ]= $\frac{3\,(1+z)^3\,\beta\,\Omega c0}{e[z]^2}$

In[ ]:= $\Omega I[z\_] := \frac{3\,(1+z)^3\,\beta\,\Omega c0}{e[z]^2}$

In[ ]:=
```
Ωm0 = 0.3225;
Ωb0 = 0.0518;
Ωc0 = Ωm0 - Ωb0;
c = 0.7538;
β = 0.0092;
h = 0.6399;
```

In[ ]:= `Ωr0`

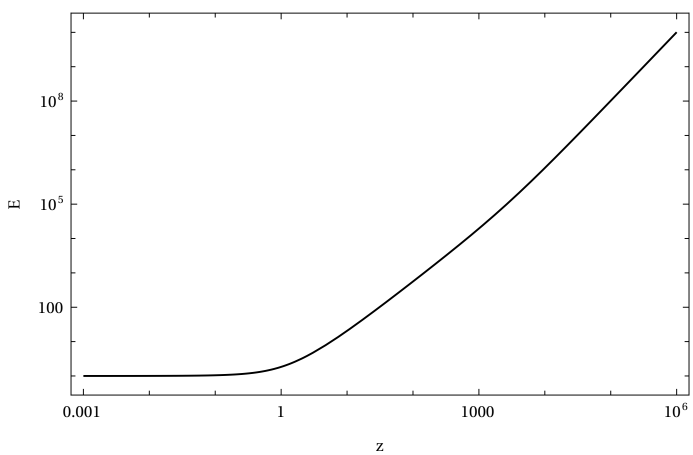
Out[ ]= 0.000101398

In[ ]:=
$$\text{sol} = \text{NDSolve}\Big[\Big\{\frac{1}{\text{e}[z]}\,\text{e}'[z] == -\frac{\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-3-\Omega\text{r}[z]}{2\,\Omega\text{de}[z]}\right),$$
$$\Omega\text{de}'[z] == -\frac{2\,(1-\Omega\text{de}[z])\,\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-\Omega\text{r}[z]}{2\,(1-\Omega\text{de}[z])}\right),$$
$$\text{e}[0] == 1,\ \Omega\text{de}[0] == 1-\Omega\text{m0}-\Omega\text{r0}\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

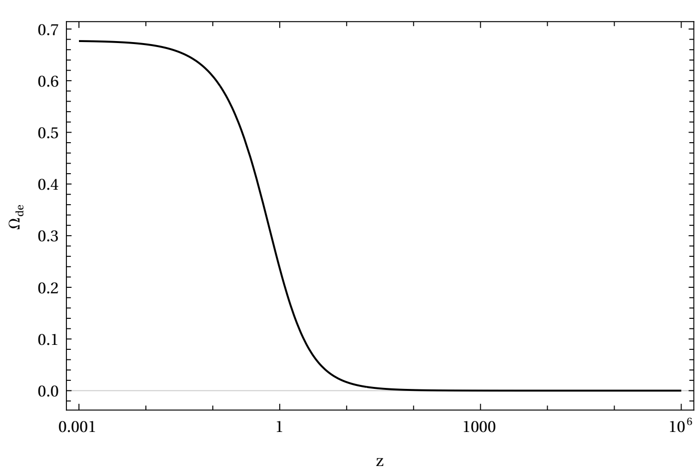
Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:= LogLogPlot[e[z] /. sol, {z, 10^{-3}, 10^6}, PlotTheme → "Scientific", FrameStyle → Black, PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "E"}]

Out[ ]=


In[ ]:= LogLinearPlot[Ωde[z] /. sol, {z, 10^{-3}, 10^6}, PlotTheme → "Scientific", FrameStyle → Black, PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "Ω_{de}"}]

Out[ ]=


(**)

```
In[ ]:= Clear["Global`*"]

In[ ]:= Ωr[z_] := Ωr0 (1 + z)^4 / e[z]^2

In[ ]:= Ωr0 := Ωm0 / (1 + zeq)

In[ ]:= zeq := 2.5 × 10^4 Ωm0 h^2 (Tcmb / (2.7 Kelvin))^-4

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= 24 077.4 h^2 Ωm0

In[ ]:= zeq := 24077.4405856556`  h^2 Ωm0

In[ ]:= Q2[z_] := 3 β H[z] × ρc[z]

In[ ]:= Q[z_] := Q2[z]

In[ ]:= ΩI[z_] := Q[z] / (3 MP^2 H[z]^3)

In[ ]:= ρc[z_] := 3 MP^2 H0^2 Ωc0 (1 + z)^3

In[ ]:= ρde[z_] := 3 MP^2 H[z]^2 Ωde[z]

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[ΩI[z]]

Out[ ]= 3 (1 + z)^3 β Ωc0 / e[z]^2

In[ ]:= ΩI[z_] := 3 (1 + z)^3 β Ωc0 / e[z]^2



In[ ]:= Ωm0 = 0.3225;
        Ωb0 = 0.0518;
        Ωc0 = Ωm0 - Ωb0;
        h = 0.6399;
        c = 0.65;
```

In[ ]:= $\beta = 0.02;$

In[ ]:=
$$\text{sol1} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\,e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-3-\Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-\Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1-\Omega m0-\Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction [Domain : {{0., $1. \times 10^{6}$}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., $1. \times 10^{6}$}} Output : scalar]}}

In[ ]:= $\beta = 0.04;$

In[ ]:=
$$\text{sol2} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\,e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-3-\Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-\Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1-\Omega m0-\Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction [Domain : {{0., $1. \times 10^{6}$}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., $1. \times 10^{6}$}} Output : scalar]}}

In[ ]:= $\beta = 0.06;$

In[ ]:=
$$\texttt{sol3} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]=
$$\{\{e \to \text{InterpolatingFunction}[\ \text{Domain}: \{\{0.,\ 1.\times 10^{6}\}\}\ \text{Output}: \text{scalar}\ ],$$
$$\Omega de \to \text{InterpolatingFunction}[\ \text{Domain}: \{\{0.,\ 1.\times 10^{6}\}\}\ \text{Output}: \text{scalar}\ ]\}\}$$

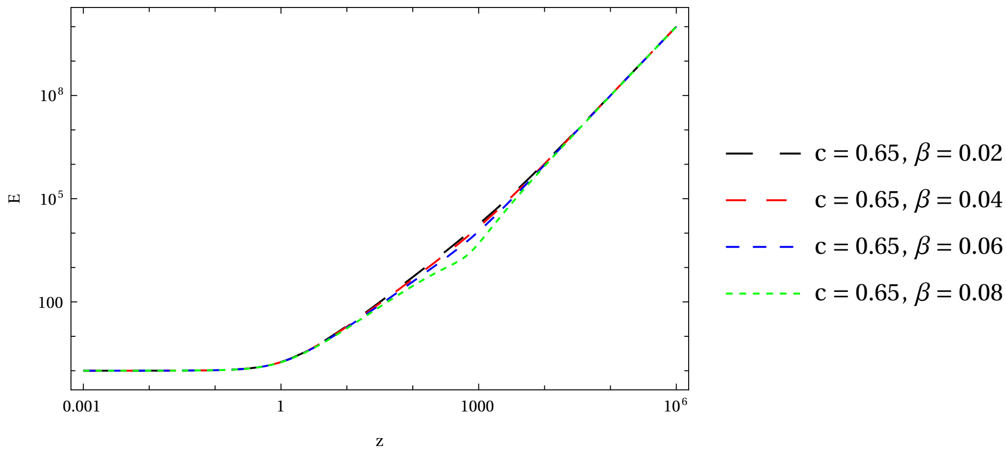
In[ ]:= $\beta = 0.08;$

In[ ]:=
$$\texttt{sol4} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]=
$$\{\{e \to \text{InterpolatingFunction}[\ \text{Domain}: \{\{0.,\ 1.\times 10^{6}\}\}\ \text{Output}: \text{scalar}\ ],$$
$$\Omega de \to \text{InterpolatingFunction}[\ \text{Domain}: \{\{0.,\ 1.\times 10^{6}\}\}\ \text{Output}: \text{scalar}\ ]\}\}$$

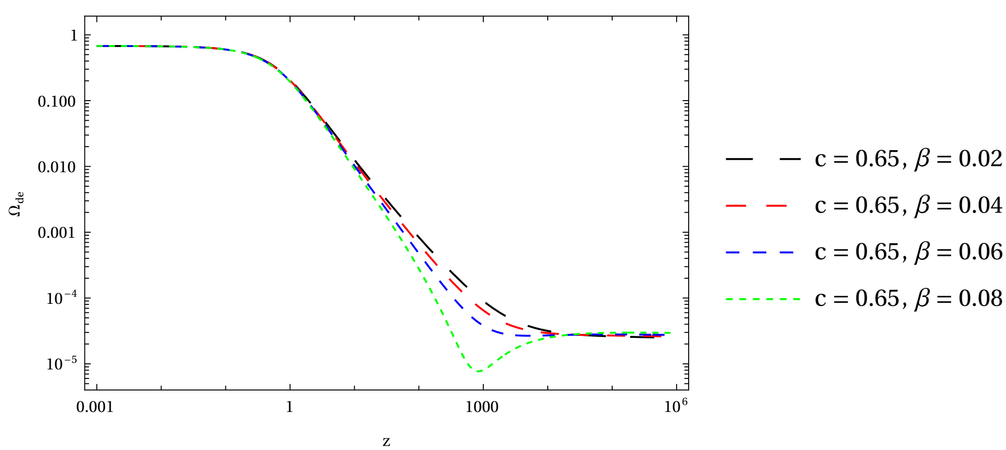
In[ ]:=
```
LogLogPlot[{e[z] /. sol1, e[z] /. sol2, e[z] /. sol3, e[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=



In[ ]:=
```
LogLogPlot[{Ωde[z] /. sol1, Ωde[z] /. sol2, Ωde[z] /. sol3, Ωde[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "Ω_de"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=



(**)

```
(*********************************************************************************)
In[ ]:= Clear["Global`*"]
```

In[ ]:= $\Omega r[z\_] := \dfrac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \dfrac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5 \times 10^4\,\Omega m0\,h^2 \left(\dfrac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

```
In[ ]:= Tcmb := 2.7255 Kelvin
In[ ]:= Simplify[zeq]
```

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= $zeq := 24077.4405856556`\ h^2\,\Omega m0$

In[ ]:= $Q1[z\_] := 3\,\beta\,H[z] \times \rho de[z]$

```
In[ ]:= Q[z_] := Q1[z]
```

In[ ]:= $\Omega I[z\_] := \dfrac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= Simplify[ΩI[z]]

Out[ ]= $3\,\beta\,\Omega de[z]$

In[ ]:= $\Omega I[z\_] := 3\,\beta\,\Omega de[z]$

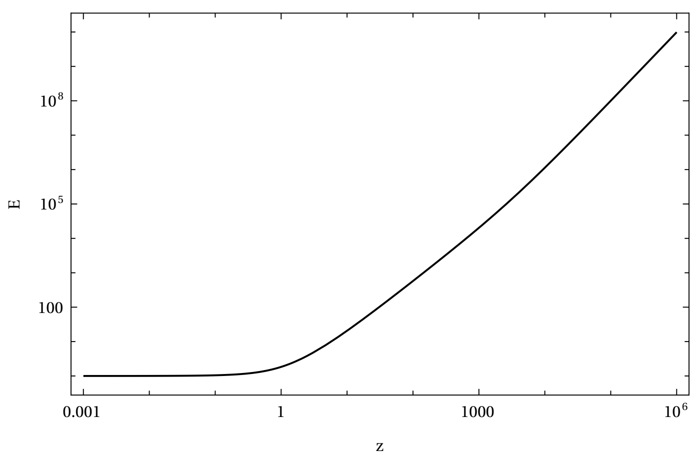
```
In[ ]:= Ωm0 = 0.3213;
        c = 0.8294;
        β = 0.0782;
        h = 0.6558;
In[ ]:= Ωr0
Out[ ]= 0.000096542
```

In[ ]:=
$$\text{sol} = \text{NDSolve}\Big[\Big\{\frac{1}{\text{e}[z]}\,\text{e}'[z] == -\frac{\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-3-\Omega\text{r}[z]}{2\,\Omega\text{de}[z]}\right),$$
$$\Omega\text{de}'[z] == -\frac{2\,(1-\Omega\text{de}[z])\,\Omega\text{de}[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-\Omega\text{r}[z]}{2\,(1-\Omega\text{de}[z])}\right),$$
$$\text{e}[0] == 1,\ \Omega\text{de}[0] == 1-\Omega\text{m0}-\Omega\text{r0}\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

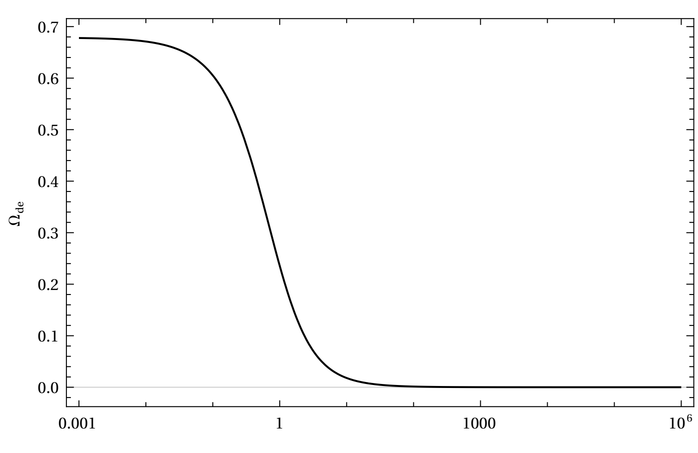
Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:= LogLogPlot[e[z] /. sol, {z, 10^{-3}, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "E"}]

Out[ ]=


In[ ]:= LogLinearPlot[Ωde[z] /. sol, {z, 10^{-3}, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
PlotStyle → {Directive[Black, Thickness[Medium]]}, FrameLabel → {"z", "Ω_{de}"}]

Out[ ]=


(***)

In[ ]:= Clear["Global`*"]

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2\left(\frac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= $zeq := 24077.4405856556\text{`}\;h^2\,\Omega m0$

In[ ]:= $Q1[z\_] := 3\,\beta\,H[z]\times\rho de[z]$

In[ ]:= Q[z_] := Q1[z]

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= Simplify[ΩI[z]]

Out[ ]= $3\,\beta\,\Omega de[z]$

In[ ]:= $\Omega I[z\_] := 3\,\beta\,\Omega de[z]$

In[ ]:=

In[ ]:= Ωm0 = 0.3213;
h = 0.6558;
c = 0.65;

In[ ]:=

In[ ]:= β = 0.02;

In[ ]:=

$$\text{sol1} = \text{NDSolve}\Big[\Big\{\frac{1}{\text{e[z]}}\,\text{e}'\text{[z]} == -\frac{\Omega\text{de[z]}}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de[z]}}+\frac{1}{2}+\frac{\Omega\text{I[z]}-3-\Omega\text{r[z]}}{2\,\Omega\text{de[z]}}\right),$$
$$\Omega\text{de}'\text{[z]} == -\frac{2\,(1-\Omega\text{de[z]})\,\Omega\text{de[z]}}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de[z]}}+\frac{1}{2}+\frac{\Omega\text{I[z]}-\Omega\text{r[z]}}{2\,(1-\Omega\text{de[z]})}\right),$$
$$\text{e[0]} == 1,\ \Omega\text{de[0]} == 1-\Omega\text{m0}-\Omega\text{r0}\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{\text{z},\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:= β = 0.04;

In[ ]:=

$$\text{sol2} = \text{NDSolve}\Big[\Big\{\frac{1}{\text{e[z]}}\,\text{e}'\text{[z]} == -\frac{\Omega\text{de[z]}}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de[z]}}+\frac{1}{2}+\frac{\Omega\text{I[z]}-3-\Omega\text{r[z]}}{2\,\Omega\text{de[z]}}\right),$$
$$\Omega\text{de}'\text{[z]} == -\frac{2\,(1-\Omega\text{de[z]})\,\Omega\text{de[z]}}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de[z]}}+\frac{1}{2}+\frac{\Omega\text{I[z]}-\Omega\text{r[z]}}{2\,(1-\Omega\text{de[z]})}\right),$$
$$\text{e[0]} == 1,\ \Omega\text{de[0]} == 1-\Omega\text{m0}-\Omega\text{r0}\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{\text{z},\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:= β = 0.06;

In[ ]:=

$$\text{sol3} = \text{NDSolve}\left[\left\{\frac{1}{\text{e}[z]}\,\text{e}'[z] == -\frac{\Omega\text{de}[z]}{1+z}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-3-\Omega\text{r}[z]}{2\,\Omega\text{de}[z]}\right),\right.\right.$$

$$\Omega\text{de}'[z] == -\frac{2\,(1-\Omega\text{de}[z])\,\Omega\text{de}[z]}{1+z}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-\Omega\text{r}[z]}{2\,(1-\Omega\text{de}[z])}\right),$$

$$\left.\left.\text{e}[0] == 1,\ \Omega\text{de}[0] == 1-\Omega\text{m0}-\Omega\text{r0}\right\},\ \{\text{e},\ \Omega\text{de}\},\ \{z,\ 0,\ 10^6\}\right]$$

Out[ ]= {{e → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:= $\beta = 0.08$;

In[ ]:=

$$\text{sol4} = \text{NDSolve}\left[\left\{\frac{1}{\text{e}[z]}\,\text{e}'[z] == -\frac{\Omega\text{de}[z]}{1+z}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-3-\Omega\text{r}[z]}{2\,\Omega\text{de}[z]}\right),\right.\right.$$

$$\Omega\text{de}'[z] == -\frac{2\,(1-\Omega\text{de}[z])\,\Omega\text{de}[z]}{1+z}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de}[z]}+\frac{1}{2}+\frac{\Omega\text{I}[z]-\Omega\text{r}[z]}{2\,(1-\Omega\text{de}[z])}\right),$$

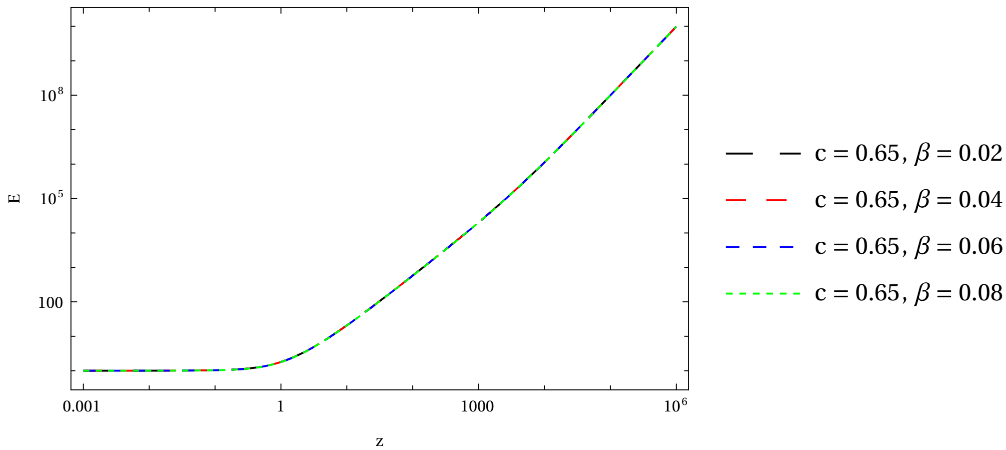
$$\left.\left.\text{e}[0] == 1,\ \Omega\text{de}[0] == 1-\Omega\text{m0}-\Omega\text{r0}\right\},\ \{\text{e},\ \Omega\text{de}\},\ \{z,\ 0,\ 10^6\}\right]$$

Out[ ]= {{e → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar]}}

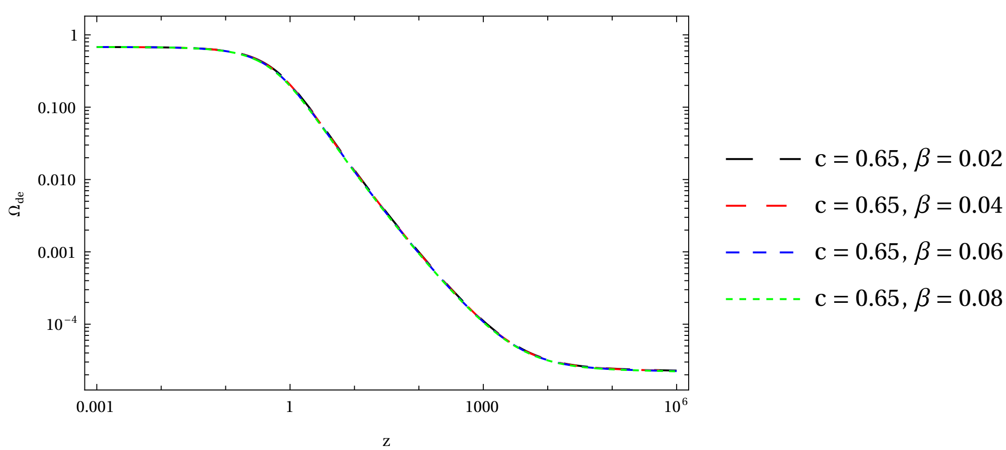
In[◦]:=
```
LogLogPlot[{e[z] /. sol1, e[z] /. sol2, e[z] /. sol3, e[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[◦]=



In[◦]:=
```
LogLogPlot[{Ωde[z] /. sol1, Ωde[z] /. sol2, Ωde[z] /. sol3, Ωde[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "Ω_de"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[◦]=



(***)

```
In[ ]:= (*******************************************************************************)

In[ ]:= Clear["Global`*"]

In[ ]:= (*******************************************************************************)

In[ ]:= (* Q1 *)
```

In[ ]:= $\Omega r[z\_] := \dfrac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \dfrac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2 \left(\dfrac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

In[ ]:= $Tcmb := 2.7255\,Kelvin$

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= $zeq := 24077.4405856556\text{`}\ h^2\,\Omega m0$

In[ ]:= $Q1[z\_] := 3\,\beta\,H[z]\times\rho de[z]$

In[ ]:= $Q[z\_] := Q1[z]$

In[ ]:= $\Omega I[z\_] := \dfrac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= Simplify[ΩI[z]]

Out[ ]= $3\,\beta\,\Omega de[z]$

In[ ]:= $\Omega I[z\_] := 3\,\beta\,\Omega de[z]$

In[ ]:=

```
In[ ]:= Ωm0 = 0.3213;
h = 0.6558;
c = 0.65;

In[ ]:=

In[ ]:= β = 0.02;
```

In[▫]:=

$$\text{sol1} = \text{NDSolve}\Big[\Big\{\frac{1}{\text{e}[\text{z}]}\,\text{e}'[\text{z}] == -\frac{\Omega\text{de}[\text{z}]}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de}[\text{z}]}+\frac{1}{2}+\frac{\Omega\text{I}[\text{z}]-3-\Omega\text{r}[\text{z}]}{2\,\Omega\text{de}[\text{z}]}\right),$$
$$\Omega\text{de}'[\text{z}] == -\frac{2\,(1-\Omega\text{de}[\text{z}])\,\Omega\text{de}[\text{z}]}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de}[\text{z}]}+\frac{1}{2}+\frac{\Omega\text{I}[\text{z}]-\Omega\text{r}[\text{z}]}{2\,(1-\Omega\text{de}[\text{z}])}\right),$$
$$\text{e}[0] == 1,\ \Omega\text{de}[0] == 1-\Omega\text{m}0-\Omega\text{r}0\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{\text{z},\ 0,\ 10^{6}\}\Big]$$

Out[▫]=

{{e → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[▫]:=

In[▫]:= $\beta = 0.04;$

In[▫]:=

$$\text{sol2} = \text{NDSolve}\Big[\Big\{\frac{1}{\text{e}[\text{z}]}\,\text{e}'[\text{z}] == -\frac{\Omega\text{de}[\text{z}]}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de}[\text{z}]}+\frac{1}{2}+\frac{\Omega\text{I}[\text{z}]-3-\Omega\text{r}[\text{z}]}{2\,\Omega\text{de}[\text{z}]}\right),$$
$$\Omega\text{de}'[\text{z}] == -\frac{2\,(1-\Omega\text{de}[\text{z}])\,\Omega\text{de}[\text{z}]}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de}[\text{z}]}+\frac{1}{2}+\frac{\Omega\text{I}[\text{z}]-\Omega\text{r}[\text{z}]}{2\,(1-\Omega\text{de}[\text{z}])}\right),$$
$$\text{e}[0] == 1,\ \Omega\text{de}[0] == 1-\Omega\text{m}0-\Omega\text{r}0\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{\text{z},\ 0,\ 10^{6}\}\Big]$$

Out[▫]=

{{e → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[▫]:=

In[▫]:= $\beta = 0.06;$

In[ ]:=
$$\text{sol3} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:=

In[ ]:= β = 0.08;

In[ ]:=
$$\text{sol4} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

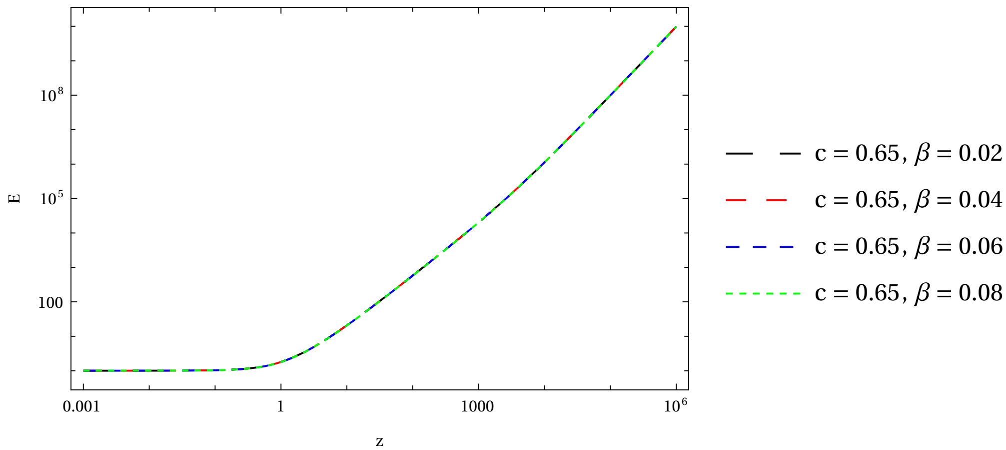
Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:=

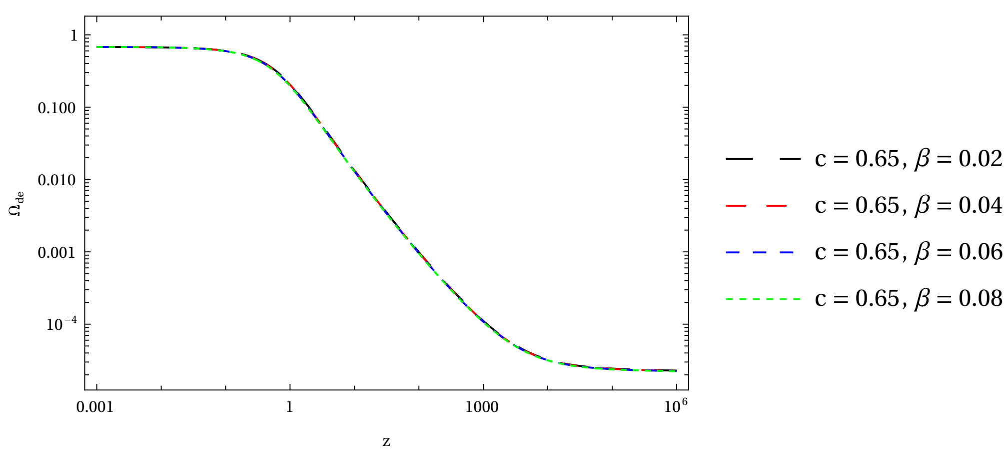
In[ ]:=
```
LogLogPlot[{e[z] /. sol1, e[z] /. sol2, e[z] /. sol3, e[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=



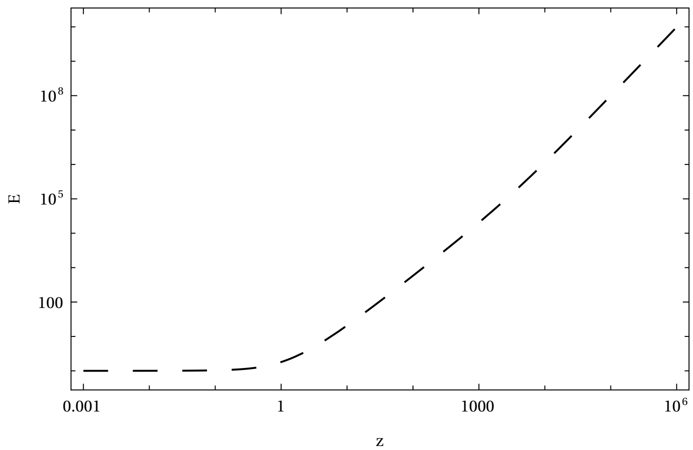
In[ ]:=
```
LogLogPlot[{Ωde[z] /. sol1, Ωde[z] /. sol2, Ωde[z] /. sol3, Ωde[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "Ω_de"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=

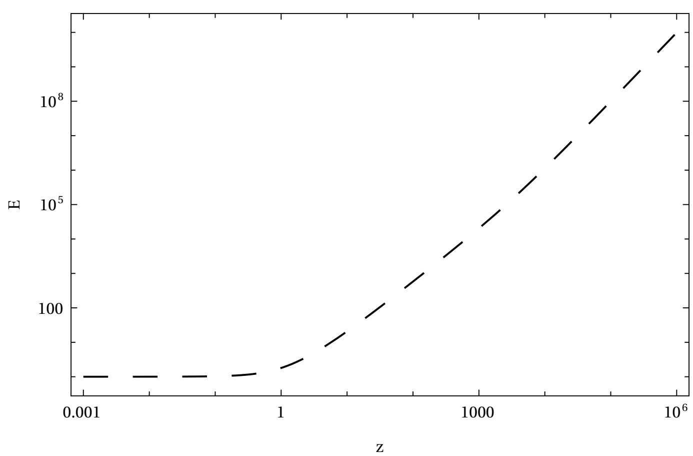


In[ ]:=

In[ ]:=
```
p11 = LogLogPlot[{e[z] /. sol1}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.04], Black, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

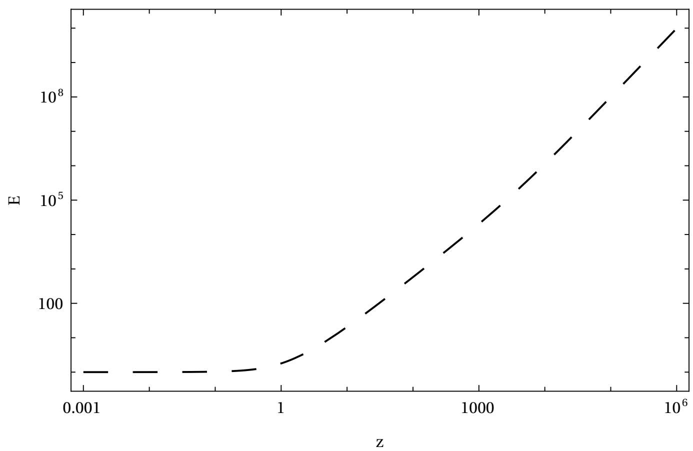
Out[ ]=


In[ ]:=
```
p12 = LogLogPlot[{e[z] /. sol2}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.04], Black, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=


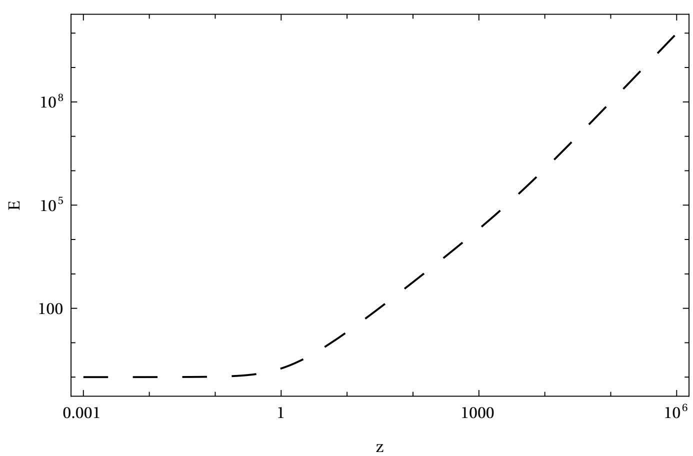
In[ ]:=
```
p13 = LogLogPlot[{e[z] /. sol3}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.04], Black, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=



In[ ]:=
```
p14 = LogLogPlot[{e[z] /. sol4}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.04], Black, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

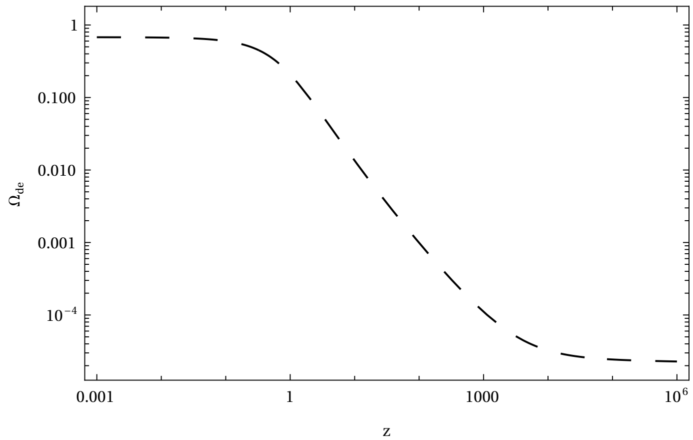
Out[ ]=



In[ ]:=

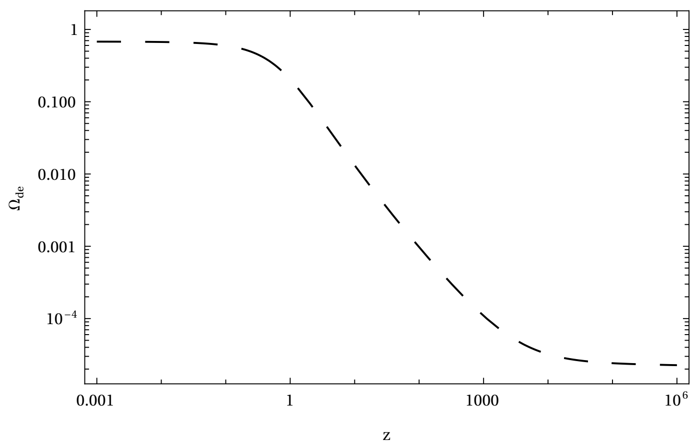
```
In[ ]:= p15 = LogLogPlot[{Ωde[z] /. sol1}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
    FrameStyle → Black, PlotStyle → {Directive[Dashing[0.04], Black, Thickness[Medium]]},
    FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=

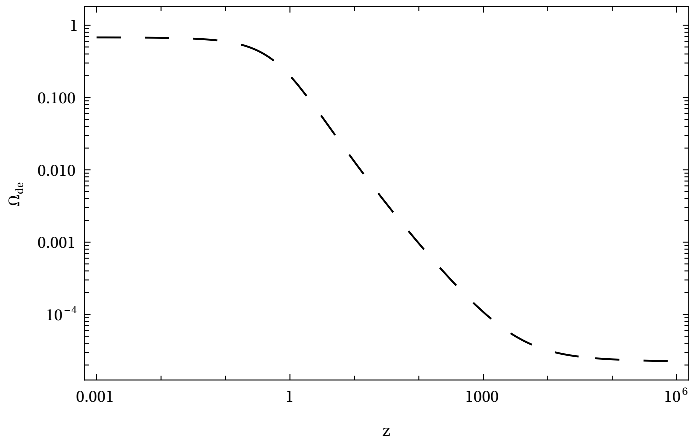



```
In[ ]:= p16 = LogLogPlot[{Ωde[z] /. sol2}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
    FrameStyle → Black, PlotStyle → {Directive[Dashing[0.04], Black, Thickness[Medium]]},
    FrameLabel → {"z", "Ω_de"}]
```

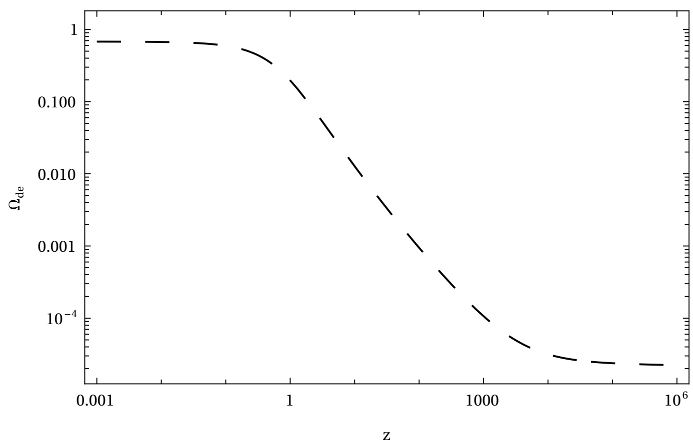
Out[ ]=



In[ ]:= p17 = LogLogPlot[{Ωde[z] /. sol3}, {z, 10^{-3}, 10^{6}}, PlotTheme → "Scientific ",
FrameStyle → Black , PlotStyle → {Directive [Dashing[0.04], Black , Thickness [Medium]]},
FrameLabel → {"z", "Ω_{de}"}]

Out[ ]=


In[ ]:= p18 = LogLogPlot[{Ωde[z] /. sol4}, {z, 10^{-3}, 10^{6}}, PlotTheme → "Scientific ",
FrameStyle → Black , PlotStyle → {Directive [Dashing[0.04], Black , Thickness [Medium]]},
FrameLabel → {"z", "Ω_{de}"}]

Out[ ]=


In[ ]:= (**)

In[ ]:= (* Q2 *)

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5 \times 10^4\, \Omega m0\, h^2 \left(\frac{Tcmb}{2.7\ Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

```
In[ ]:= Simplify[zeq]
Out[ ]= 3327.08
In[ ]:= zeq := 24077.4405856556` h^2 Ωm0
In[ ]:= Q2[z_] := 3 β H[z] × ρc[z]
In[ ]:= Q[z_] := Q2[z]
```

$$\Omega I[z\_] := \frac{Q[z]}{3\, MP^2\, H[z]^3}$$

$$\rho c[z\_] := 3\, MP^2\, H0^2\, \Omega c0\, (1+z)^3$$

$$\rho de[z\_] := 3\, MP^2\, H[z]^2\, \Omega de[z]$$

```
In[ ]:= H[z_] := H0 e[z]
In[ ]:= Simplify[ΩI[z]]
```

$$\text{Out[ ]=}\ \frac{0.24\,(1+z)^3\,\Omega c0}{e[z]^2}$$

$$\Omega I[z\_] := \frac{3\,(1+z)^3\,\beta\,\Omega c0}{e[z]^2}$$

```
In[ ]:=

In[ ]:= Ωm0 = 0.3225;
        Ωb0 = 0.0518;
        Ωc0 = Ωm0 - Ωb0;
        h = 0.6399;
        c = 0.65;

In[ ]:=

In[ ]:= β = 0.02;
```

In[◦]:=
$$\text{sol1} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[◦]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[◦]:=

In[◦]:= $\beta = 0.04$;

In[◦]:=
$$\text{sol2} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[◦]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[◦]:=

In[◦]:= $\beta = 0.06$;

In[ ]:=
$$\text{sol3} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e, \Omega de\},\ \{z, 0, 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain: $\{\{0., 1.\times 10^{6}\}\}$ Output: scalar],

Ωde → InterpolatingFunction[Domain: $\{\{0., 1.\times 10^{6}\}\}$ Output: scalar]}}

In[ ]:=

In[ ]:= β = 0.08;

In[ ]:=
$$\text{sol4} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e, \Omega de\},\ \{z, 0, 10^{6}\}\Big]$$

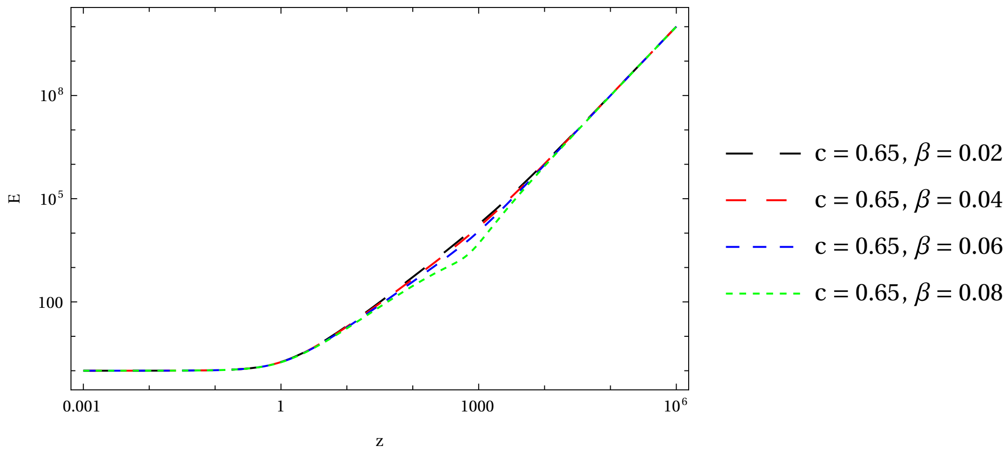
Out[ ]= {{e → InterpolatingFunction[Domain: $\{\{0., 1.\times 10^{6}\}\}$ Output: scalar],

Ωde → InterpolatingFunction[Domain: $\{\{0., 1.\times 10^{6}\}\}$ Output: scalar]}}

In[ ]:=

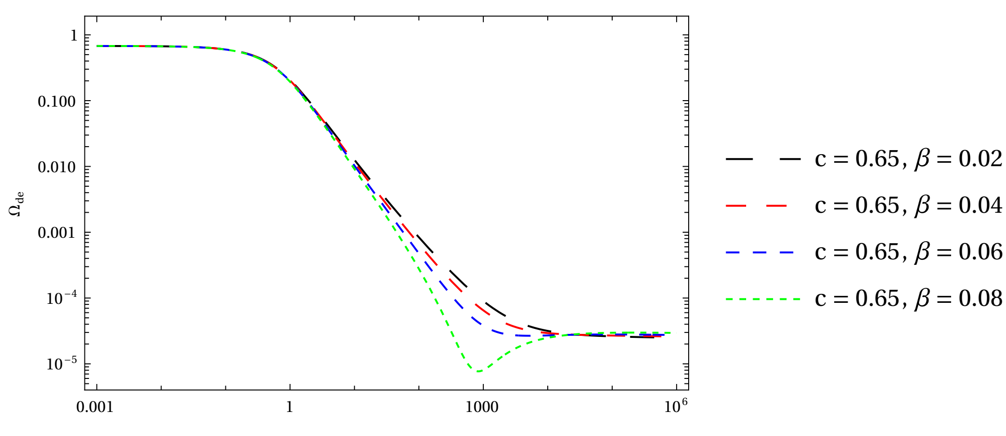
In[∘]:=
```
LogLogPlot[{e[z] /. sol1, e[z] /. sol2, e[z] /. sol3, e[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[∘]=



In[∘]:=
```
LogLogPlot[{Ωde[z] /. sol1, Ωde[z] /. sol2, Ωde[z] /. sol3, Ωde[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "Ω_de"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

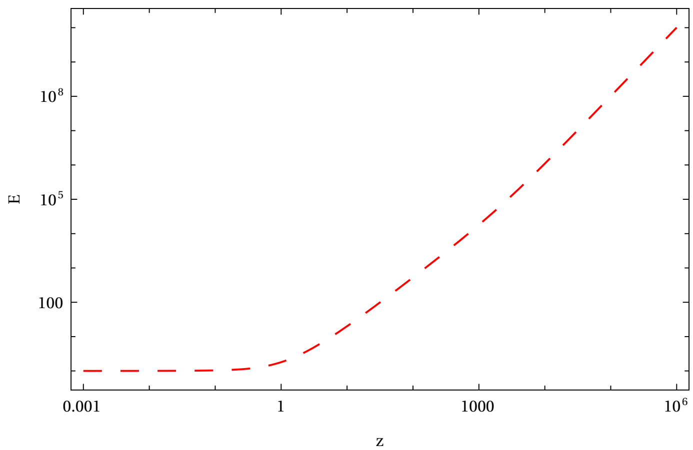
Out[∘]=



In[ ]:=

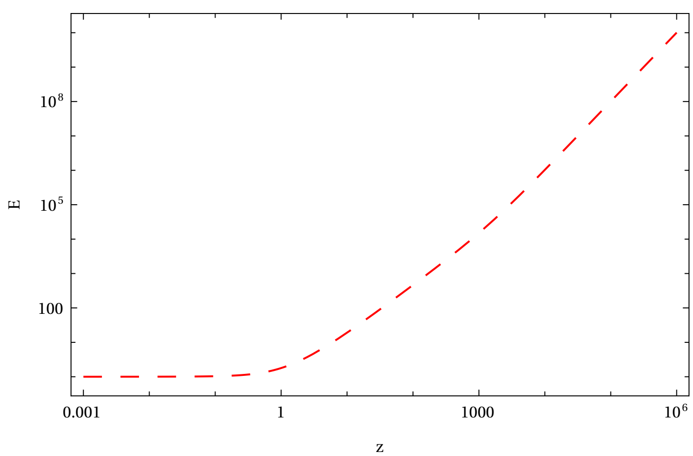
In[ ]:=
```
p21 = LogLogPlot[{e[z] /. sol1}, {z, 10^-3, 10^6},
  PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Directive[Dashing[0.03], Red, Thickness[Medium]]}, FrameLabel → {"z", "E"}]
```

Out[ ]=



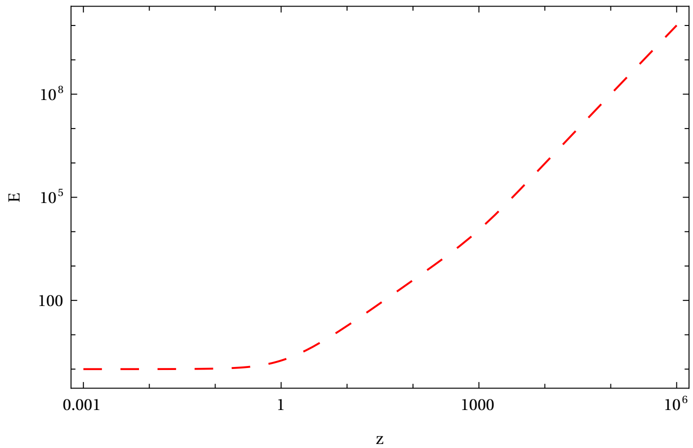
In[ ]:=
```
p22 = LogLogPlot[{e[z] /. sol2}, {z, 10^-3, 10^6},
  PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Directive[Dashing[0.03], Red, Thickness[Medium]]}, FrameLabel → {"z", "E"}]
```

Out[ ]=



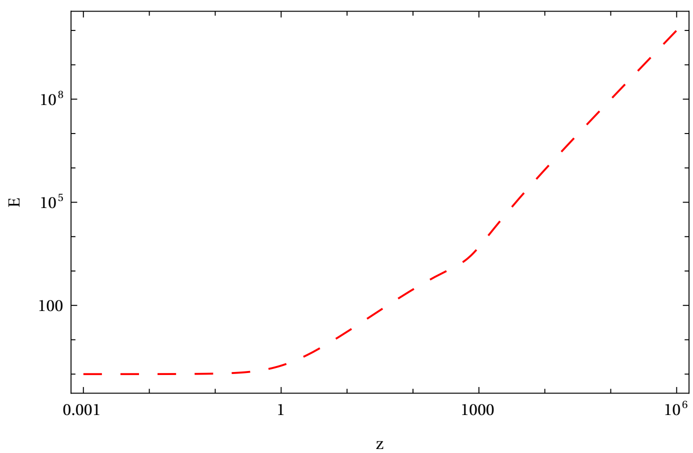
```
In[ ]:= p23 = LogLogPlot[{e[z] /. sol3}, {z, 10^-3, 10^6},
   PlotTheme → "Scientific", FrameStyle → Black,
   PlotStyle → {Directive[Dashing[0.03], Red, Thickness[Medium]]}, FrameLabel → {"z", "E"}]
```

Out[ ]=




```
In[ ]:= p24 = LogLogPlot[{e[z] /. sol4}, {z, 10^-3, 10^6},
   PlotTheme → "Scientific", FrameStyle → Black,
   PlotStyle → {Directive[Dashing[0.03], Red, Thickness[Medium]]}, FrameLabel → {"z", "E"}]
```

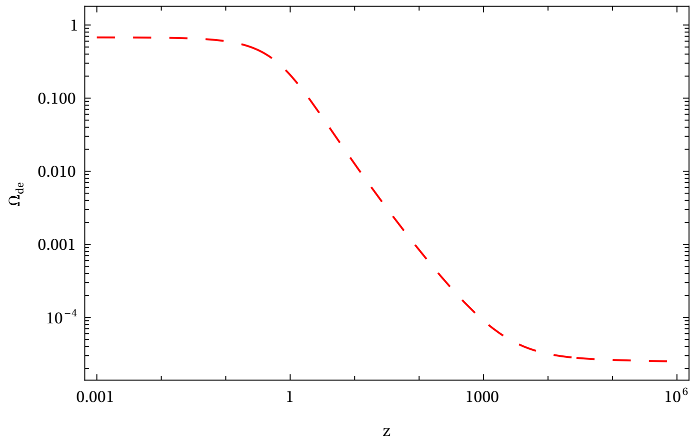
Out[ ]=



In[ ]:=

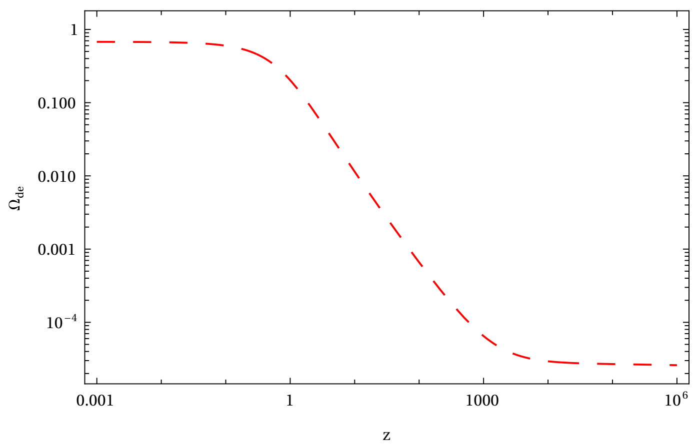
```
In[ ]:= p25 = LogLogPlot[{Ωde[z] /. sol1}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
   FrameStyle → Black, PlotStyle → {Directive[Dashing[0.03], Red, Thickness[Medium]]},
   FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=



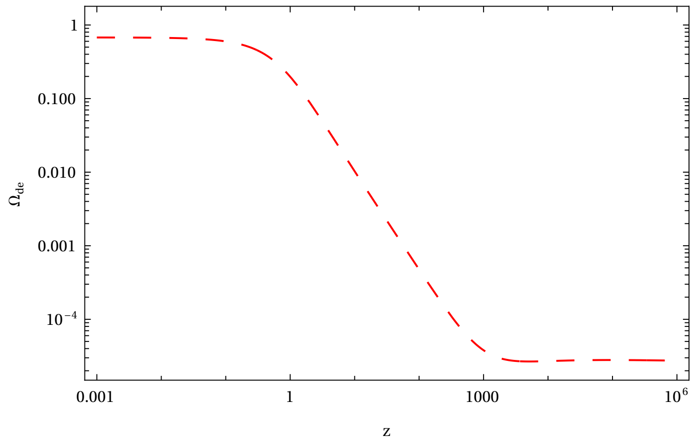

```
In[ ]:= p26 = LogLogPlot[{Ωde[z] /. sol2}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
   FrameStyle → Black, PlotStyle → {Directive[Dashing[0.03], Red, Thickness[Medium]]},
   FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=



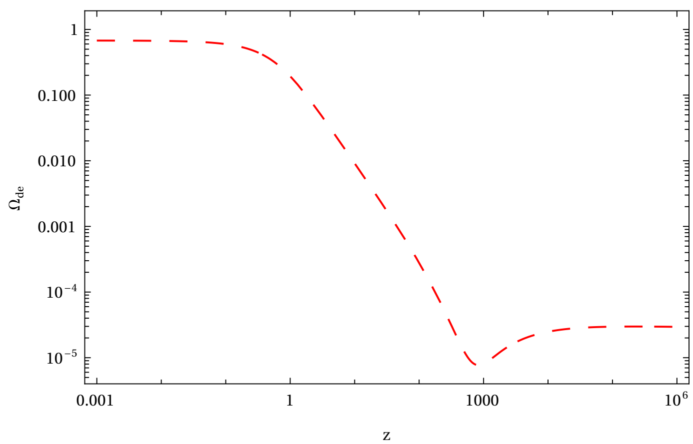
In[ ]:=
```
p27 = LogLogPlot[{Ωde[z] /. sol3}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.03], Red, Thickness[Medium]]},
  FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=



In[ ]:=
```
p28 = LogLogPlot[{Ωde[z] /. sol4}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.03], Red, Thickness[Medium]]},
  FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=



In[ ]:=
```
(*********************************************************************************)
```

In[ ]:=
```
(* Q3 *)
```

In[ ]:= $\Omega r[z\_] := \dfrac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \dfrac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2\left(\dfrac{Tcmb}{2.7\ Kelvin}\right)^{-4}$

In[ ]:=
```
Tcmb := 2.7255 Kelvin
```

In[ ]:=
```
Simplify[zeq]
```
Out[ ]= 3179.54

In[ ]:=
```
zeq := 24077.4405856556` h^2 Ωm0
```

In[ ]:=
```
Q3[z_] := 3 β H[z] (ρde[z] + ρc[z])
```

In[ ]:=
```
Q[z_] := Q3[z]
```

In[ ]:= $\Omega I[z\_] := \dfrac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:=
```
H[z_] := H0 e[z]
```

In[ ]:=
```
Simplify[ΩI[z]]
```
Out[ ]= $\dfrac{0.064968\,(1.+1.\,z)^3 + 0.24\,e[z]^2\,\Omega de[z]}{e[z]^2}$

In[ ]:= $\Omega I[z\_] := \dfrac{3\,\beta\,\left((1+z)^3\,\Omega c0 + e[z]^2\,\Omega de[z]\right)}{e[z]^2}$

In[ ]:=

In[ ]:=
```
Ωm0 = 0.3235;
Ωb0 = 0.0547;
Ωc0 = Ωm0 - Ωb0;
c = 0.65;
h = 0.6394;
```

In[ ]:=

In[ ]:=
```
β = 0.02;
```

In[ ]:= $$\text{sol1} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^6\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:=

In[ ]:= $\beta = 0.04;$

In[ ]:= $$\text{sol2} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^6\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:=

In[ ]:= $\beta = 0.06;$

In[ ]:=

$$\text{sol3} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

Out[ ]= {{e → InterpolatingFunction [Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:=

In[ ]:= β = 0.08;

In[ ]:=

$$\text{sol4} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\Big]$$

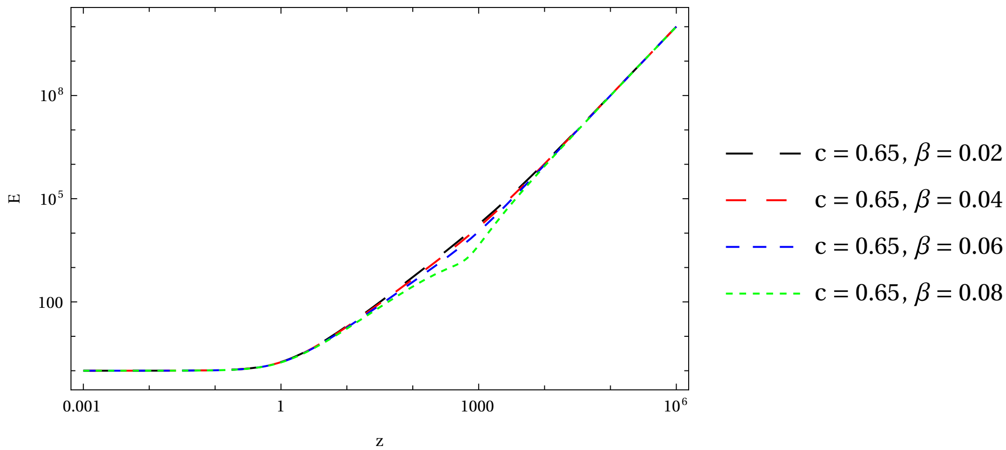
Out[ ]= {{e → InterpolatingFunction [Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction [Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:=

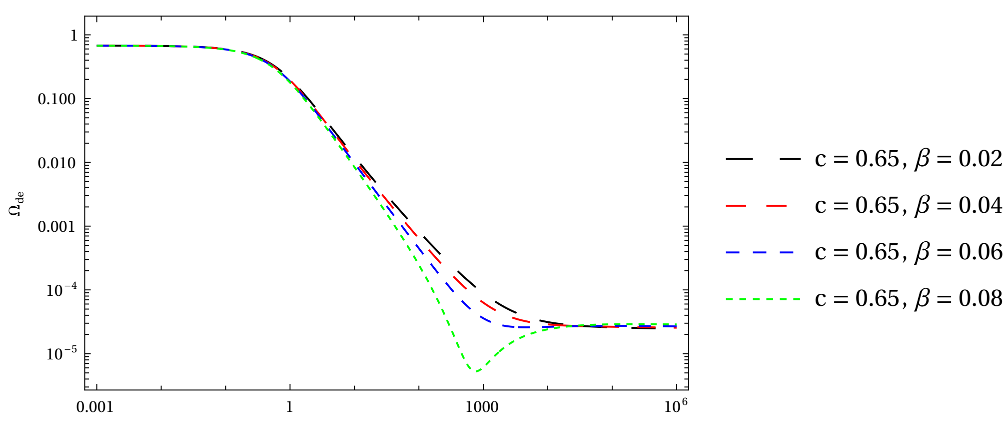
In[ ]:=
```
LogLogPlot[{e[z] /. sol1, e[z] /. sol2, e[z] /. sol3, e[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=



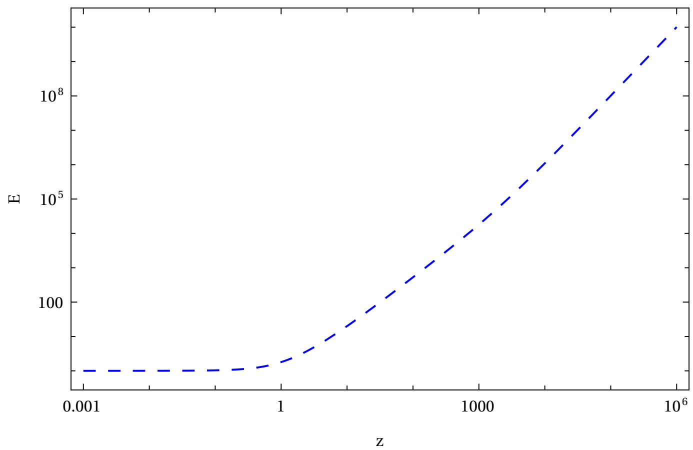
In[ ]:=
```
LogLogPlot[{Ωde[z] /. sol1, Ωde[z] /. sol2, Ωde[z] /. sol3, Ωde[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "Ωde"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=

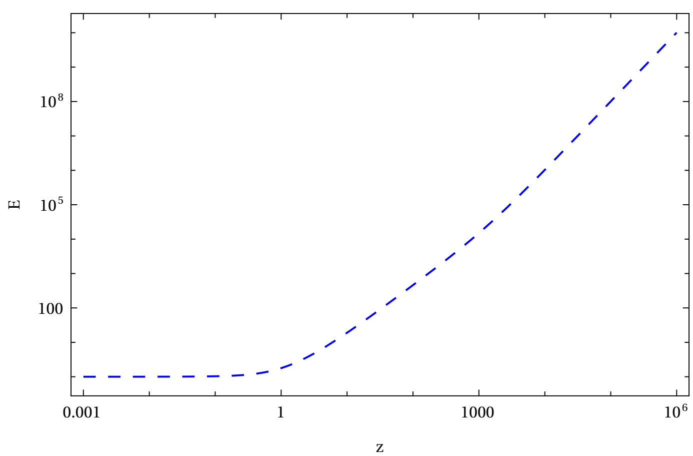


In[ ]:=

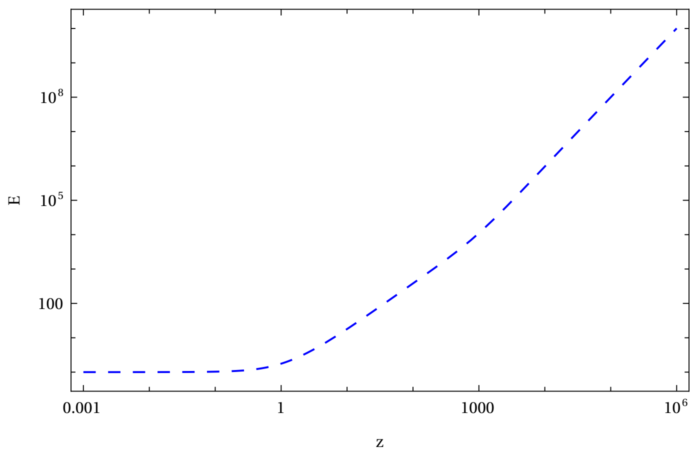
In[ ]:=
```
p31 = LogLogPlot[{e[z] /. sol1}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=


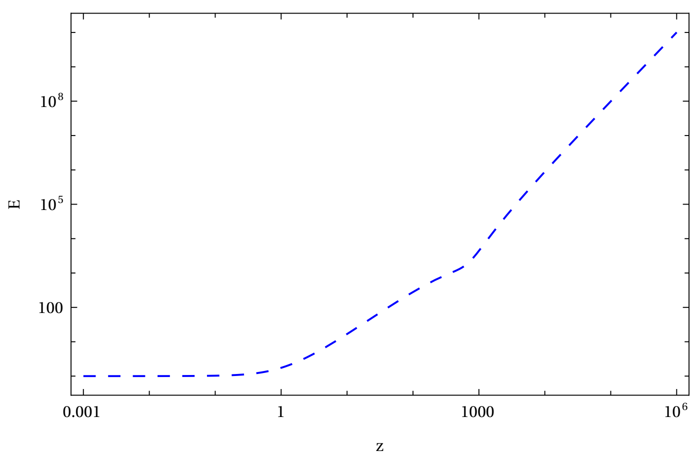
In[ ]:=
```
p32 = LogLogPlot[{e[z] /. sol2}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=


In[ ]:=
```
p33 = LogLogPlot[{e[z] /. sol3}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

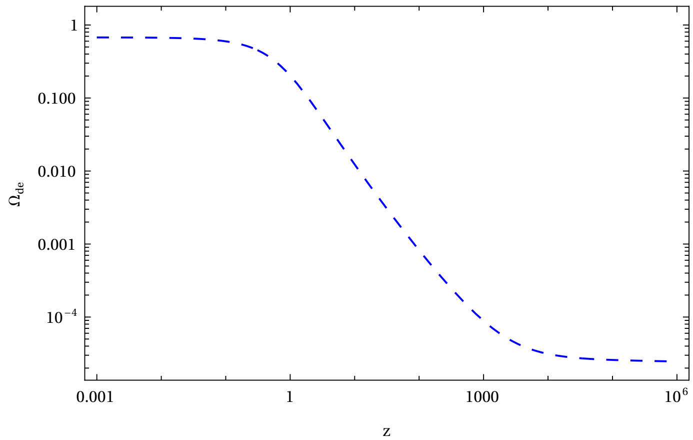
Out[ ]=


In[ ]:=
```
p34 = LogLogPlot[{e[z] /. sol4}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

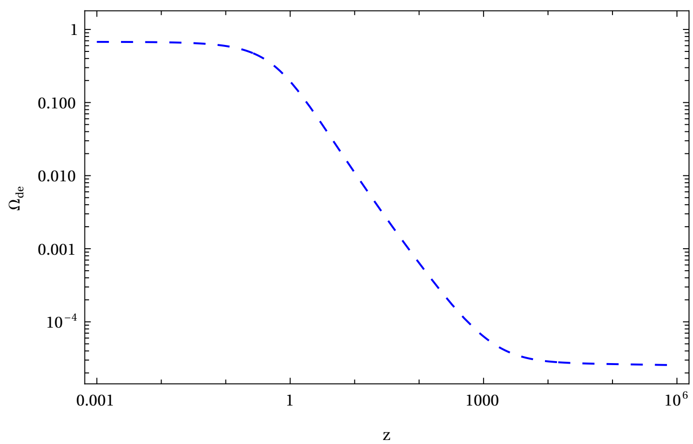
Out[ ]=


In[ ]:=

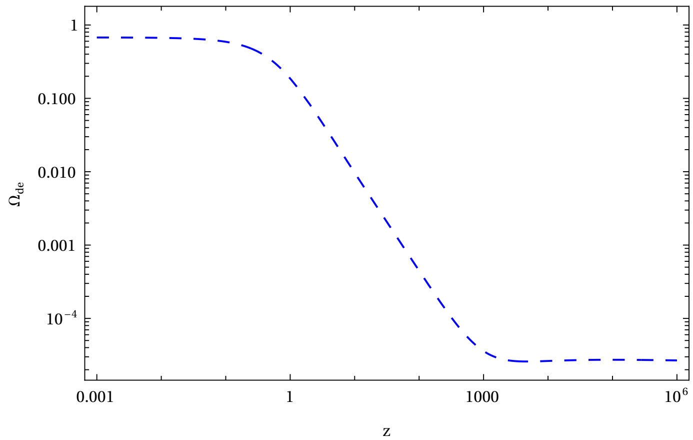
In[ ]:= p35 = LogLogPlot[{Ωde[z] /. sol1}, {z, 10^{-3}, 10^{6}}, PlotTheme → "Scientific",
FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
FrameLabel → {"z", "Ω_{de}"}]

Out[ ]=


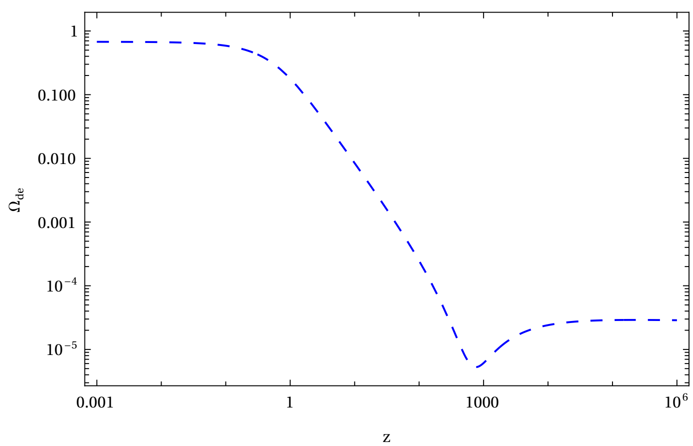
In[ ]:= p36 = LogLogPlot[{Ωde[z] /. sol2}, {z, 10^{-3}, 10^{6}}, PlotTheme → "Scientific",
FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
FrameLabel → {"z", "Ω_{de}"}]

Out[ ]=


In[ ]:=
```
p37 = LogLogPlot[{Ωde[z] /. sol3}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
  FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=


In[ ]:=
```
p38 = LogLogPlot[{Ωde[z] /. sol4}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
  FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=


In[ ]:= (***)

In[ ]:= (* Q4 *)

In[ ]:= $\Omega r[z\_] := \dfrac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \dfrac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2\left(\dfrac{Tcmb}{2.7\ Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:=
```
Simplify[zeq]
```
Out[ ]= 3184.42

In[ ]:= $zeq := 24077.4405856556`\ h^2\ \Omega m0$

In[ ]:= $Q4[z\_] := 3\,\beta\, H[z]\ \sqrt{\rho de[z] \times \rho c[z]}$

In[ ]:= $Q[z\_] := Q4[z]$

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3\ MP^2\ H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\ MP^2\ H0^2\ \Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\ MP^2\ H[z]^2\ \Omega de[z]$

In[ ]:= $H[z\_] := H0\ e[z]$

In[ ]:=
```
Simplify[PowerExpand[Simplify[ΩI[z]]]]
```
Out[ ]= $\frac{0.12443\ (1+z)^{3/2}\ \sqrt{\Omega de[z]}}{e[z]}$

In[ ]:= $\Omega I[z\_] := \frac{3\,(1+z)^{3/2}\ \beta\ \sqrt{\Omega c0}\ \sqrt{\Omega de[z]}}{e[z]}$

In[ ]:=

In[ ]:=
```
Ωm0 = 0.3233;
Ωb0 = 0.0526;
Ωc0 = Ωm0 - Ωb0;
c = 0.65;
h = 0.6512;
```

In[ ]:=

In[ ]:=
```
β = 0.02;
```

In[ ]:=

$$sol1 = NDSolve\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\,\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$

$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\,\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$

$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^6\}\Big]$$

Out[ ]=

```
{{e → InterpolatingFunction[ Domain : {{0., 1. × 10^6}} Output : scalar ],
  Ωde → InterpolatingFunction[ Domain : {{0., 1. × 10^6}} Output : scalar ]}}
```

In[ ]:=

In[ ]:=

```
β = 0.04;
```

In[ ]:=

$$sol2 = NDSolve\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\,\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$

$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\,\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$

$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^6\}\Big]$$

Out[ ]=

```
{{e → InterpolatingFunction[ Domain : {{0., 1. × 10^6}} Output : scalar ],
  Ωde → InterpolatingFunction[ Domain : {{0., 1. × 10^6}} Output : scalar ]}}
```

In[ ]:=

In[ ]:=

```
β = 0.06;
```

In[ ]:= sol3 = NDSolve

$$\left[\left\{\frac{1}{e[z]}\,e'[z]==-\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\,\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-3-\Omega r[z]}{2\,\Omega de[z]}\right),\right.\right.$$

$$\Omega de'[z]==-\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\,\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-\Omega r[z]}{2\,(1-\Omega de[z])}\right),$$

$$\left.\left.e[0]==1,\ \Omega de[0]==1-\Omega m0-\Omega r0\right\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\right]$$

Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:=

In[ ]:= β = 0.08;

In[ ]:= sol4 = NDSolve

$$\left[\left\{\frac{1}{e[z]}\,e'[z]==-\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\,\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-3-\Omega r[z]}{2\,\Omega de[z]}\right),\right.\right.$$

$$\Omega de'[z]==-\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\,\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-\Omega r[z]}{2\,(1-\Omega de[z])}\right),$$

$$\left.\left.e[0]==1,\ \Omega de[0]==1-\Omega m0-\Omega r0\right\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^{6}\}\right]$$

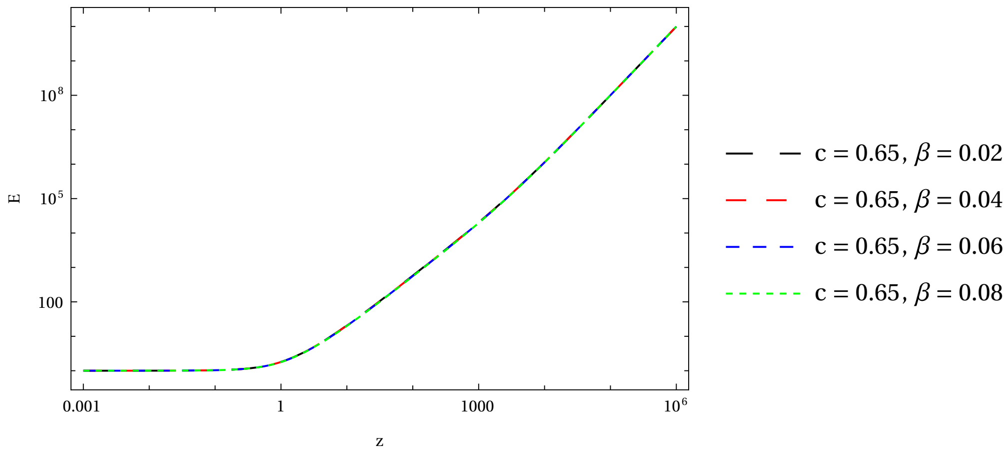
Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^{6}}} Output : scalar]}}

In[ ]:=

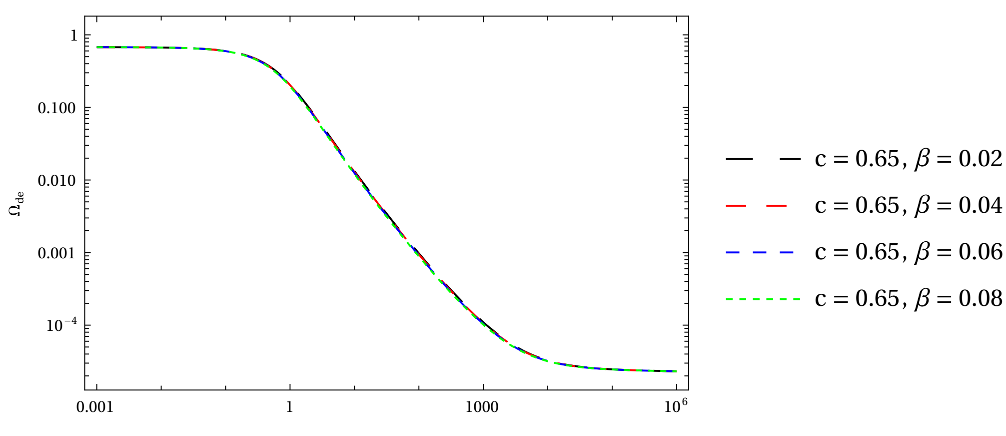
In[ ]:=
```
LogLogPlot[{e[z] /. sol1, e[z] /. sol2, e[z] /. sol3, e[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=



In[ ]:=
```
LogLogPlot[{Ωde[z] /. sol1, Ωde[z] /. sol2, Ωde[z] /. sol3, Ωde[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "Ω_de"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

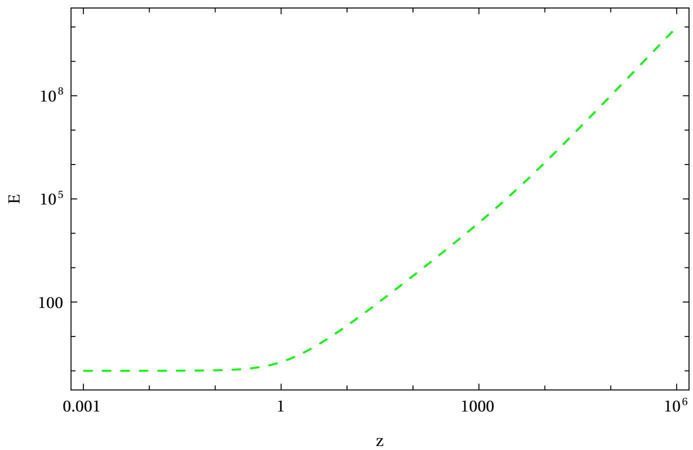
Out[ ]=



In[ ]:=

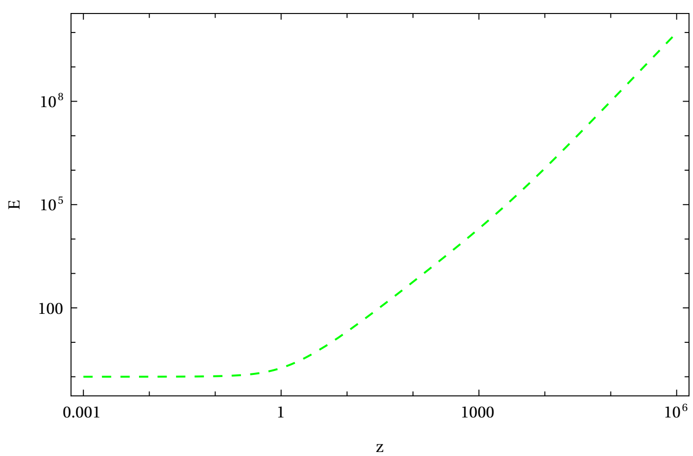
In[ ]:=
```
p41 = LogLogPlot[{e[z] /. sol1}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.015], Green, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=



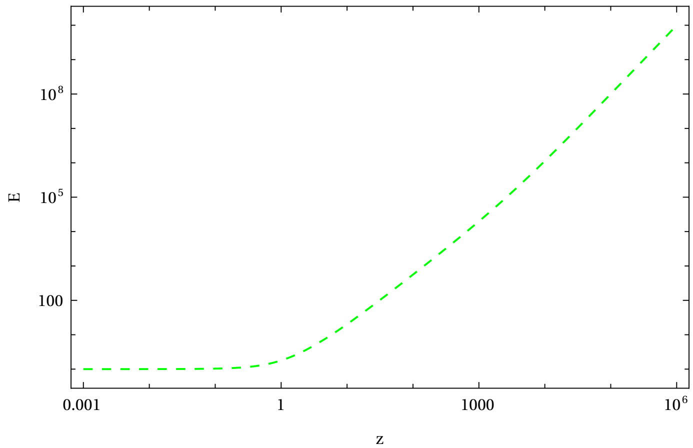
In[ ]:=
```
p42 = LogLogPlot[{e[z] /. sol2}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.015], Green, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=



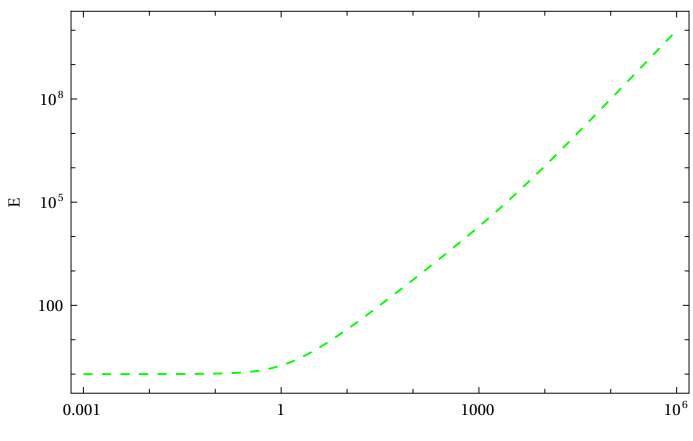
In[ ]:=
```
p43 = LogLogPlot[{e[z] /. sol3}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.015], Green, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=



In[ ]:=
```
p44 = LogLogPlot[{e[z] /. sol4}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.015], Green, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

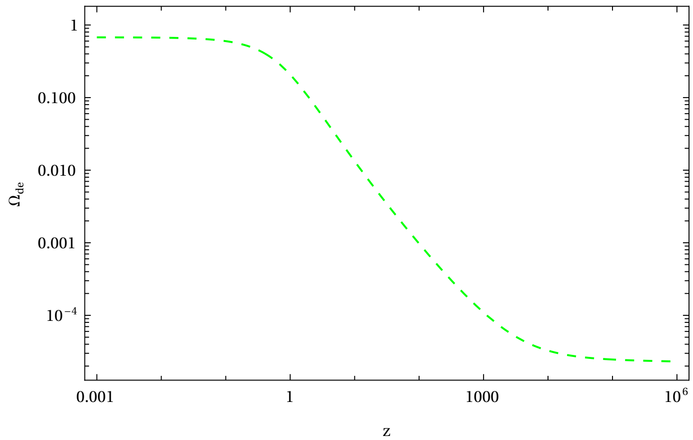
Out[ ]=



In[ ]:=

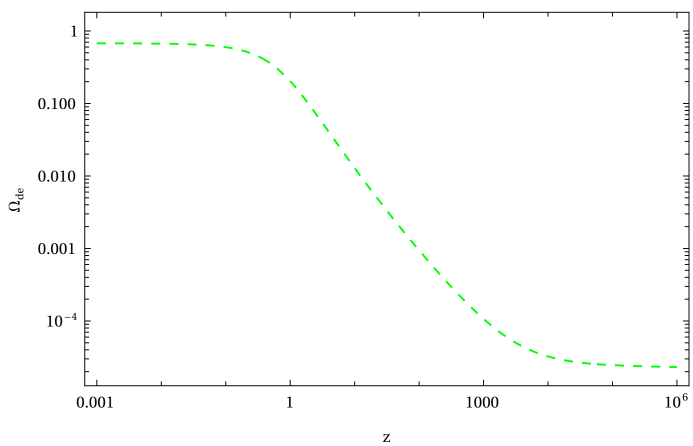
In[ ]:=
```
p45 = LogLogPlot[{Ωde[z] /. sol1}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.015], Green, Thickness[Medium]]},
  FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=

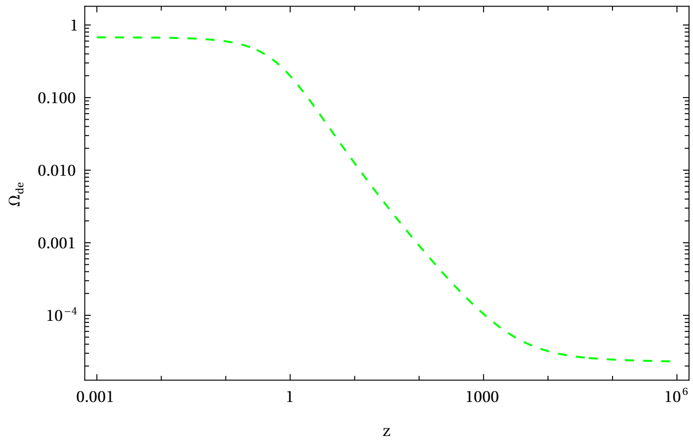


In[ ]:=
```
p46 = LogLogPlot[{Ωde[z] /. sol2}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.015], Green, Thickness[Medium]]},
  FrameLabel → {"z", "Ω_de"}]
```

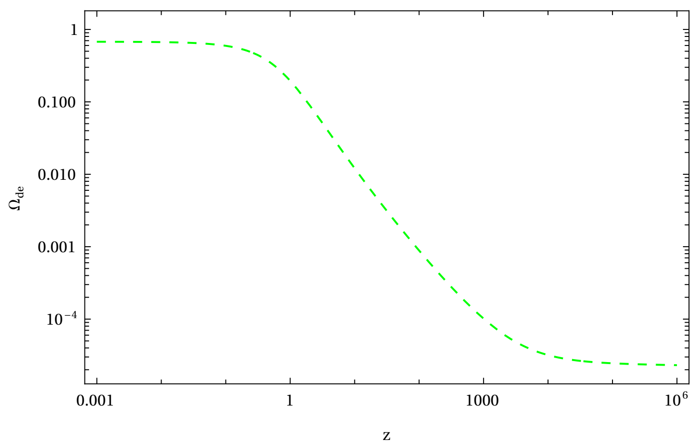
Out[ ]=



In[ ]:=
```
p47 = LogLogPlot[{Ωde[z] /. sol3}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.015], Green, Thickness[Medium]]},
  FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=


In[ ]:=
```
p48 = LogLogPlot[{Ωde[z] /. sol4}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.015], Green, Thickness[Medium]]},
  FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=


In[ ]:=
```
(*****************************************************************************)
```

In[ ]:=
```
(* Q5 *)
```

In[ ]:= $\Omega r[z\_] := \dfrac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \dfrac{\Omega m0}{(1 + zeq)}$

In[ ]:= $zeq := 2.5 \times 10^4\ \Omega m0\ h^2 \left(\dfrac{Tcmb}{2.7\ Kelvin}\right)^{-4}$

In[ ]:=
```
Tcmb := 2.7255 Kelvin
```

In[ ]:= Simplify[zeq]

Out[ ]= 3300.99

In[ ]:= zeq := 24077.4405856556` h^2 Ωm0

In[ ]:= $Q5[z\_] := 3\,\beta\,H[z]\,\frac{\rho de[z] \times \rho c[z]}{\rho de[z] + \rho c[z]}$

In[ ]:= Q[z_] := Q5[z]

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[PowerExpand[Simplify[ΩI[z]]]]

Out[ ]= $\frac{0.24\,(1.+z)^3\,\Omega de[z]}{1.\,(1.+1.\,z)^3 + 3.69413\,e[z]^2\,\Omega de[z]}$

In[ ]:= $\Omega I[z\_] := \frac{3\,(1+z)^3\,\beta\,\Omega c0\,\Omega de[z]}{(1+z)^3\,\Omega c0 + e[z]^2\,\Omega de[z]}$

In[ ]:=

In[ ]:= Ωm0 = 0.3224;
Ωb0 = 0.0521;
Ωc0 = Ωm0 - Ωb0;
c = 0.65;
h = 0.6545;

In[ ]:=

In[ ]:= β = 0.02;

In[ ]:=

$$\text{sol1} = \text{NDSolve}\left[\left\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),\right.\right.$$

$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$

$$\left.\left. e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\right\},\ \{e, \Omega de\},\ \{z, 0, 10^6\}\right]$$

Out[ ]=

{{e → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:=

In[ ]:= $\beta = 0.04;$

In[ ]:=

$$\text{sol2} = \text{NDSolve}\left[\left\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),\right.\right.$$

$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$

$$\left.\left. e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\right\},\ \{e, \Omega de\},\ \{z, 0, 10^6\}\right]$$

Out[ ]=

{{e → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:=

In[ ]:= $\beta = 0.06;$

In[◦]:= sol3 = NDSolve$\Big[\Big\{\frac{1}{\text{e[z]}}\,\text{e}'\text{[z]} == -\frac{\Omega\text{de[z]}}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de[z]}}+\frac{1}{2}+\frac{\Omega\text{I[z]}-3-\Omega\text{r[z]}}{2\,\Omega\text{de[z]}}\right),$

$\Omega\text{de}'\text{[z]} == -\frac{2\,(1-\Omega\text{de[z]})\,\Omega\text{de[z]}}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de[z]}}+\frac{1}{2}+\frac{\Omega\text{I[z]}-\Omega\text{r[z]}}{2\,(1-\Omega\text{de[z]})}\right),$

$\text{e[0]} == 1,\ \Omega\text{de[0]} == 1-\Omega\text{m0}-\Omega\text{r0}\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{\text{z},\ 0,\ 10^{6}\}\Big]$

Out[◦]= {{e → InterpolatingFunction [Domain : $\{\{0.,\ 1.\times 10^{6}\}\}$ Output : scalar],

Ωde → InterpolatingFunction [Domain : $\{\{0.,\ 1.\times 10^{6}\}\}$ Output : scalar]}}

In[◦]:=

In[◦]:= β = 0.08;

In[◦]:= sol4 = NDSolve$\Big[\Big\{\frac{1}{\text{e[z]}}\,\text{e}'\text{[z]} == -\frac{\Omega\text{de[z]}}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de[z]}}+\frac{1}{2}+\frac{\Omega\text{I[z]}-3-\Omega\text{r[z]}}{2\,\Omega\text{de[z]}}\right),$

$\Omega\text{de}'\text{[z]} == -\frac{2\,(1-\Omega\text{de[z]})\,\Omega\text{de[z]}}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de[z]}}+\frac{1}{2}+\frac{\Omega\text{I[z]}-\Omega\text{r[z]}}{2\,(1-\Omega\text{de[z]})}\right),$

$\text{e[0]} == 1,\ \Omega\text{de[0]} == 1-\Omega\text{m0}-\Omega\text{r0}\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{\text{z},\ 0,\ 10^{6}\}\Big]$

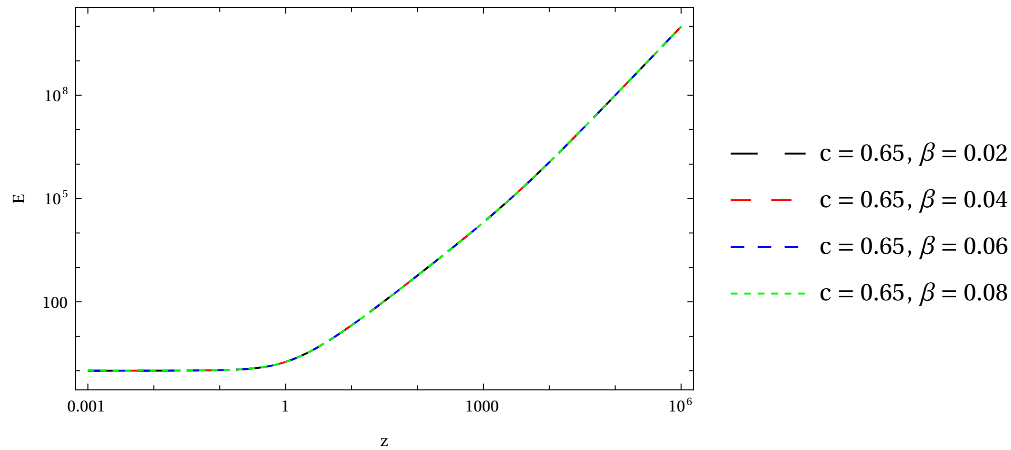
Out[◦]= {{e → InterpolatingFunction [Domain : $\{\{0.,\ 1.\times 10^{6}\}\}$ Output : scalar],

Ωde → InterpolatingFunction [Domain : $\{\{0.,\ 1.\times 10^{6}\}\}$ Output : scalar]}}

In[◦]:=

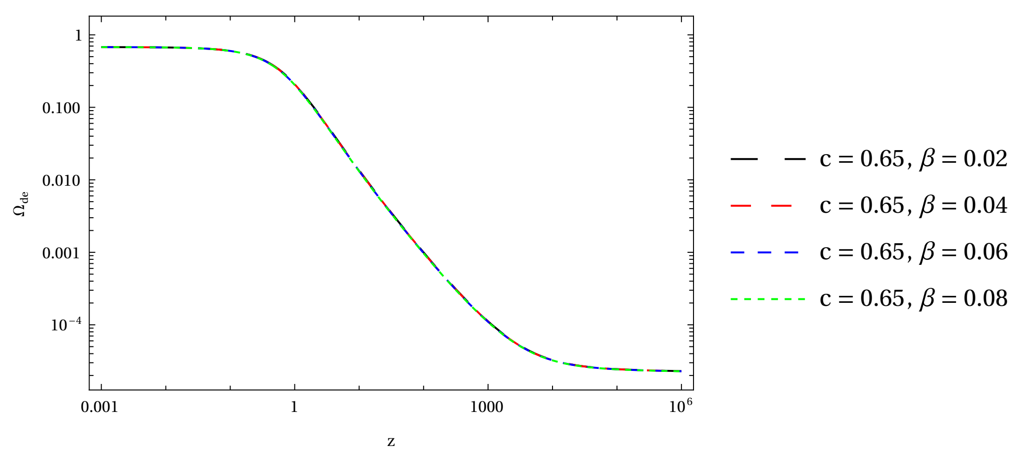
In[ ]:=
```
LogLogPlot[{e[z] /. sol1, e[z] /. sol2, e[z] /. sol3, e[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

Out[ ]=



In[ ]:=
```
LogLogPlot[{Ωde[z] /. sol1, Ωde[z] /. sol2, Ωde[z] /. sol3, Ωde[z] /. sol4},
  {z, 10^-3, 10^6}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.01], Green, Thickness[Medium]]}},
  FrameLabel → {"z", "Ω_de"}, PlotLegends → {"c = 0.65, β = 0.02",
    "c = 0.65, β = 0.04", "c = 0.65, β = 0.06", "c = 0.65, β = 0.08"}]
```

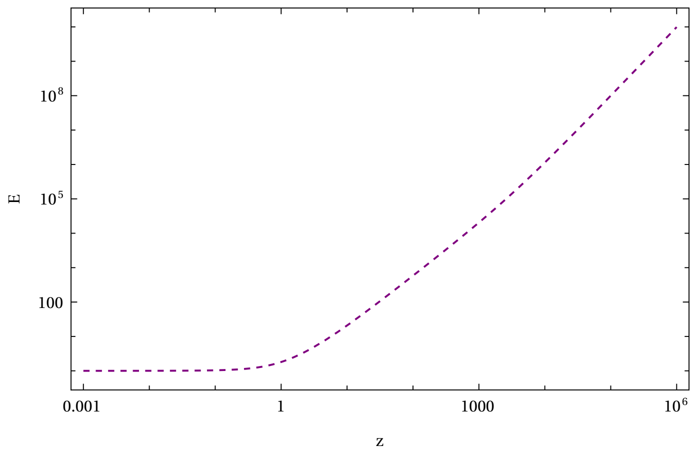
Out[ ]=



In[ ]:=

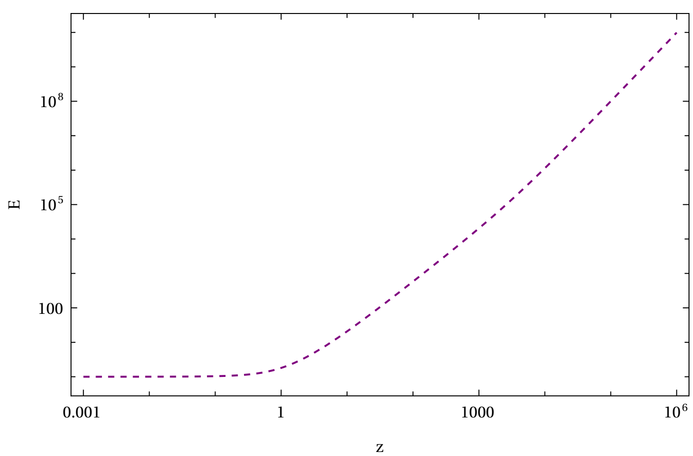
In[ ]:=
```
p51 = LogLogPlot[{e[z] /. sol1}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.01], Purple, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=



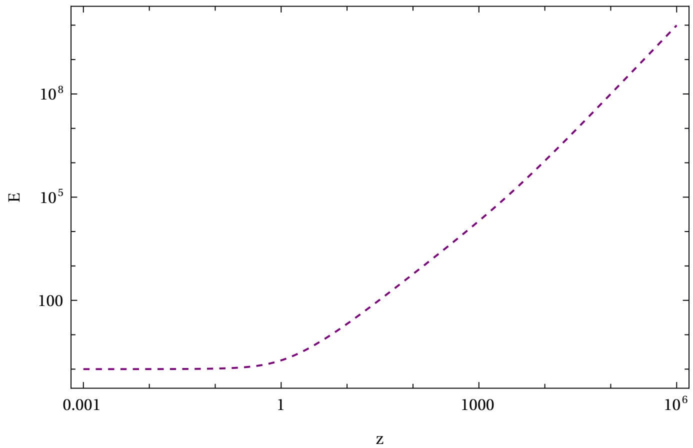
In[ ]:=
```
p52 = LogLogPlot[{e[z] /. sol2}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.01], Purple, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=



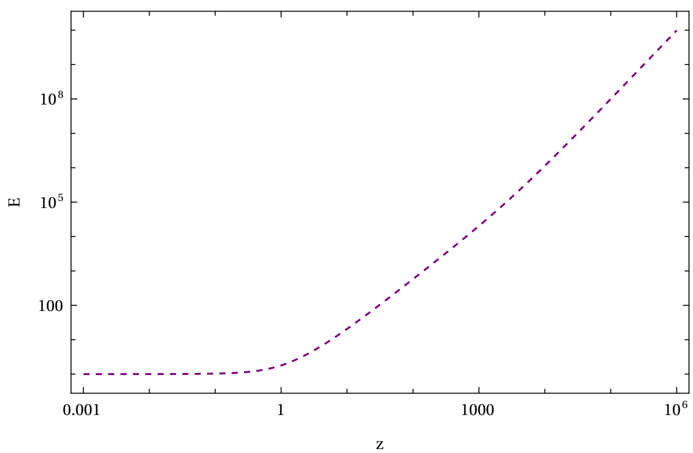
In[ ]:=
```
p53 = LogLogPlot[{e[z] /. sol3}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.01], Purple, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=



In[ ]:=
```
p54 = LogLogPlot[{e[z] /. sol4}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.01], Purple, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

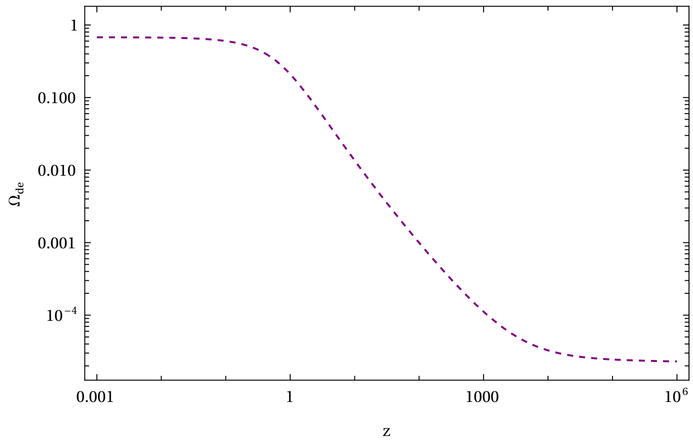
Out[ ]=



In[ ]:=

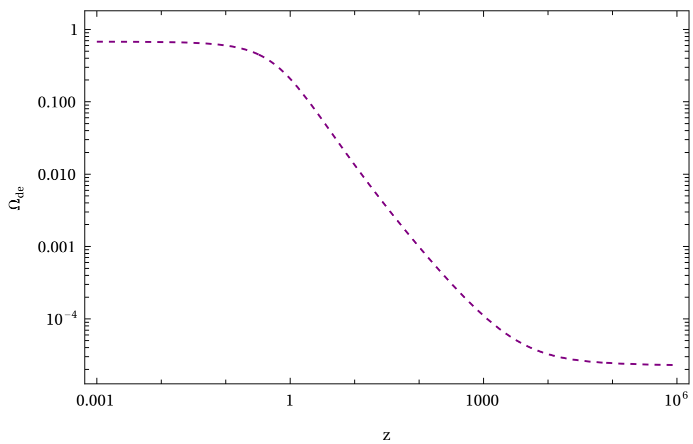
In[∘]:=
```
p55 = LogLogPlot[{Ωde[z] /. sol1}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.01], Purple, Thickness[Medium]]},
  FrameLabel → {"z", "Ωde"}]
```

Out[∘]=



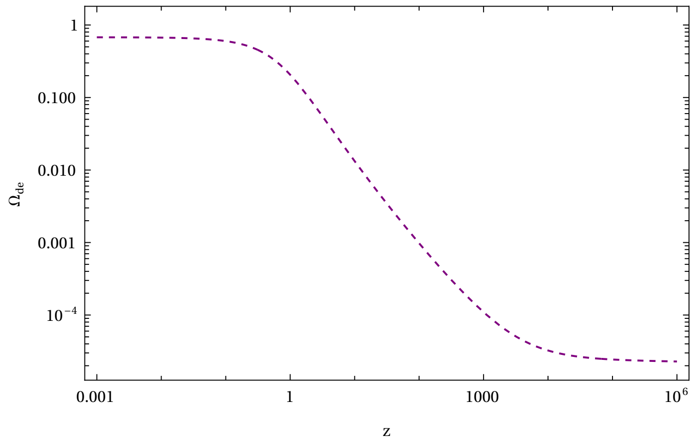
In[∘]:=
```
p56 = LogLogPlot[{Ωde[z] /. sol2}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.01], Purple, Thickness[Medium]]},
  FrameLabel → {"z", "Ωde"}]
```

Out[∘]=



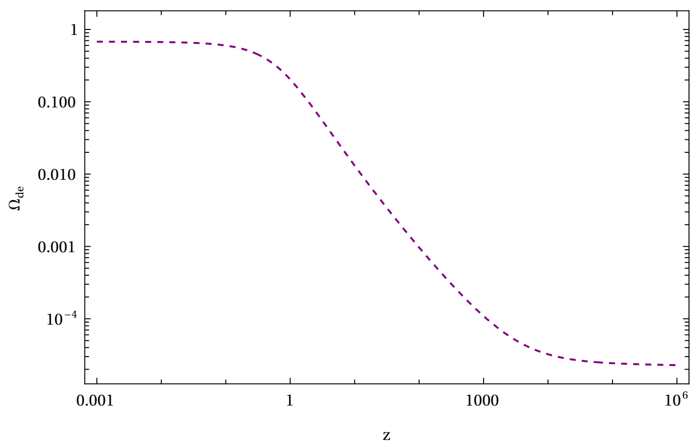
```
In[ ]:= p57 = LogLogPlot[{Ωde[z] /. sol3}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
    FrameStyle → Black, PlotStyle → {Directive[Dashing[0.01], Purple, Thickness[Medium]]},
    FrameLabel → {"z", "Ω_de"}]
```

Out[ ]=




```
In[ ]:= p58 = LogLogPlot[{Ωde[z] /. sol4}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
    FrameStyle → Black, PlotStyle → {Directive[Dashing[0.01], Purple, Thickness[Medium]]},
    FrameLabel → {"z", "Ω_de"}]
```

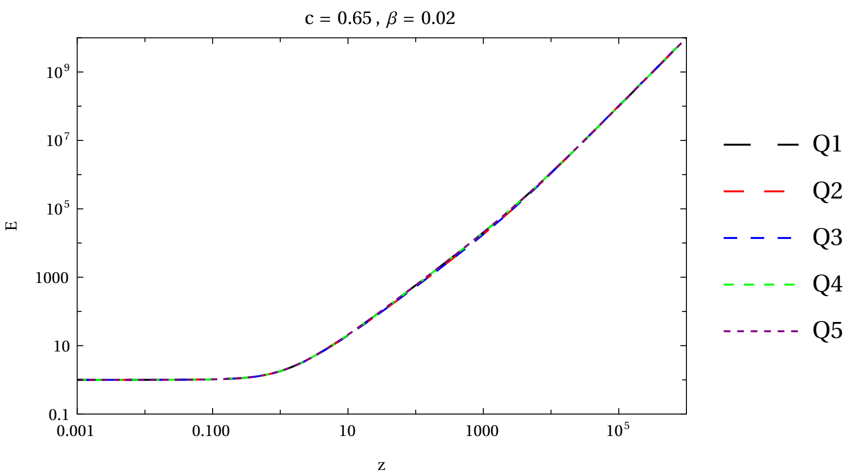
Out[ ]=




```
In[ ]:= (*****************************************************************************)

In[ ]:= (* Comparison of models *)
```

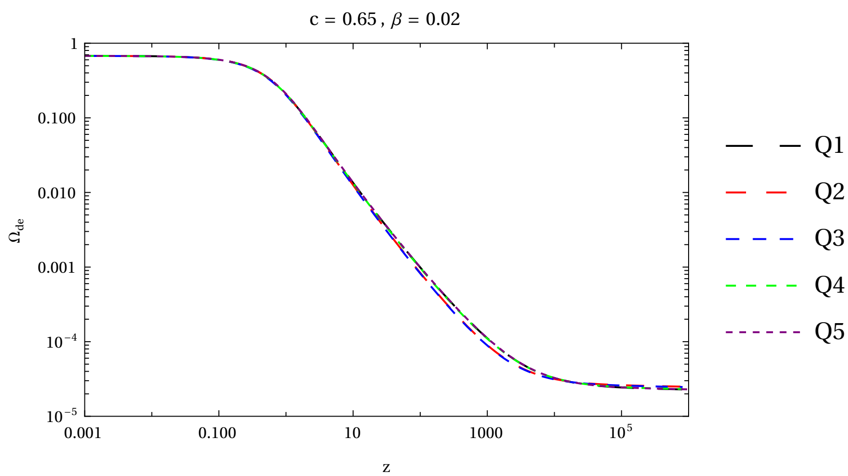
In[◦]:=
```
Show[LogLogPlot[{null, null, null, null, null}, {z, 10^-3, 10^6},
  PlotRange → {{10^-3, 10^6}, {10^-1, 10^10}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.015], Green, Thickness[Medium]]},
    {Directive[Dashing[0.01], Purple, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLabel → "c = 0.65, β = 0.02",
  PlotLegends → {"Q1", "Q2", "Q3", "Q4", "Q5"}], p11, p21, p31, p41, p51]
```

Out[◦]=



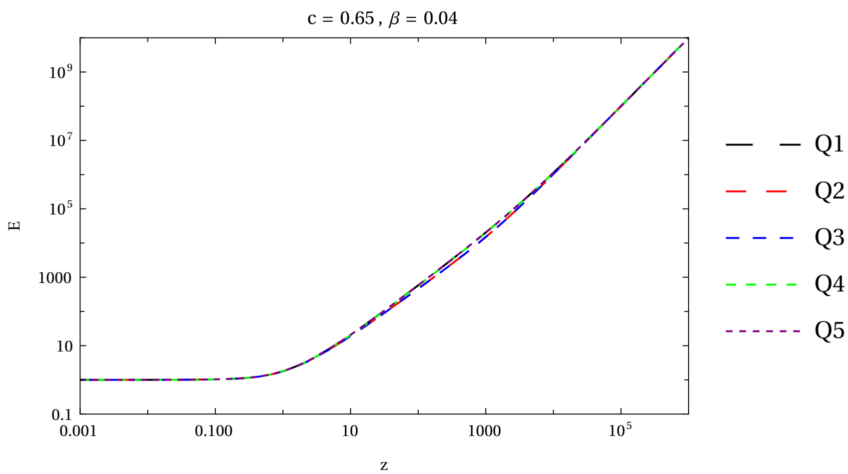
In[ ]:=
```
Show[LogLogPlot[{null, null, null, null, null}, {z, 10^-3, 10^6},
  PlotRange → {{10^-3, 10^6}, {10^-5, 1}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.015], Green, Thickness[Medium]]},
    {Directive[Dashing[0.01], Purple, Thickness[Medium]]}},
  FrameLabel → {"z", "Ω_de"}, PlotLabel → "c = 0.65, β = 0.02",
  PlotLegends → {"Q1", "Q2", "Q3", "Q4", "Q5"}], p15, p25, p35, p45, p55]
```

Out[ ]=



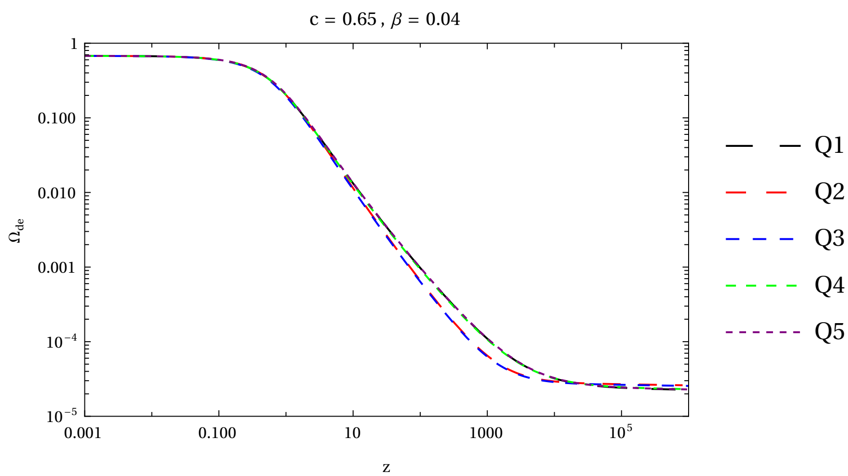
In[ ]:=
```
Show[LogLogPlot[{null, null, null, null, null}, {z, 10^-3, 10^6},
  PlotRange → {{10^-3, 10^6}, {10^-1, 10^10}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.015], Green, Thickness[Medium]]},
    {Directive[Dashing[0.01], Purple, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLabel → "c = 0.65, β = 0.04",
  PlotLegends → {"Q1", "Q2", "Q3", "Q4", "Q5"}], p12, p22, p32, p42, p52]
```

Out[ ]=



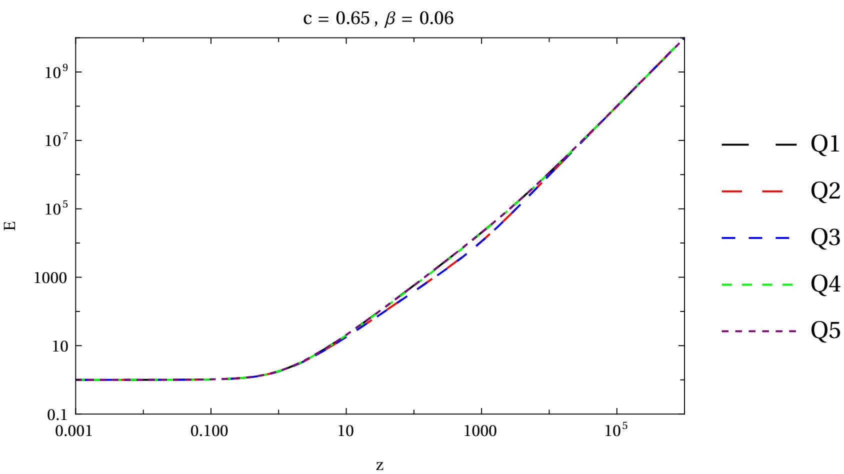
```
In[ ]:= Show[LogLogPlot[{null, null, null, null, null}, {z, 10^-3, 10^6},
   PlotRange → {{10^-3, 10^6}, {10^-5, 1}}, PlotTheme → "Scientific", FrameStyle → Black,
   PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
     {Directive[Dashing[0.03], Red, Thickness[Medium]]},
     {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
     {Directive[Dashing[0.015], Green, Thickness[Medium]]},
     {Directive[Dashing[0.01], Purple, Thickness[Medium]]}},
   FrameLabel → {"z", "Ωde"}, PlotLabel → "c = 0.65, β = 0.04",
   PlotLegends → {"Q1", "Q2", "Q3", "Q4", "Q5"}], p16, p26, p36, p46, p56]
```

Out[ ]=



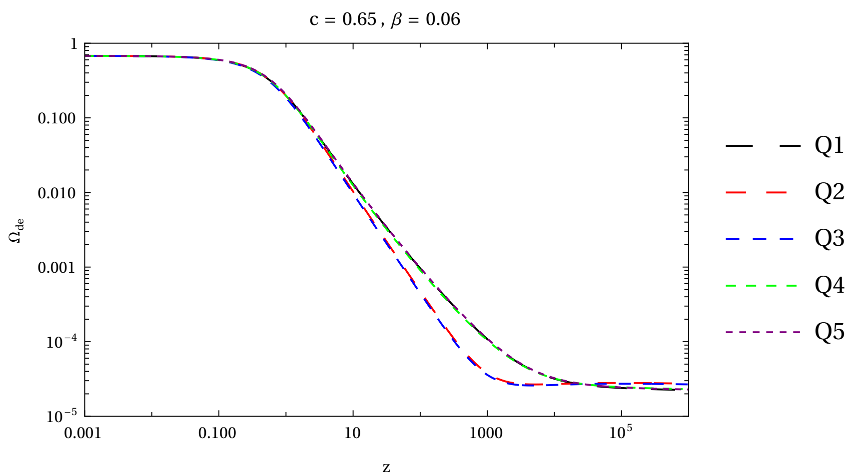
In[ ]:=
```
Show[LogLogPlot[{null, null, null, null, null}, {z, 10^-3, 10^6},
  PlotRange → {{10^-3, 10^6}, {10^-1, 10^10}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.015], Green, Thickness[Medium]]},
    {Directive[Dashing[0.01], Purple, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLabel → "c = 0.65, β = 0.06",
  PlotLegends → {"Q1", "Q2", "Q3", "Q4", "Q5"}], p13, p23, p33, p43, p53]
```

Out[ ]=



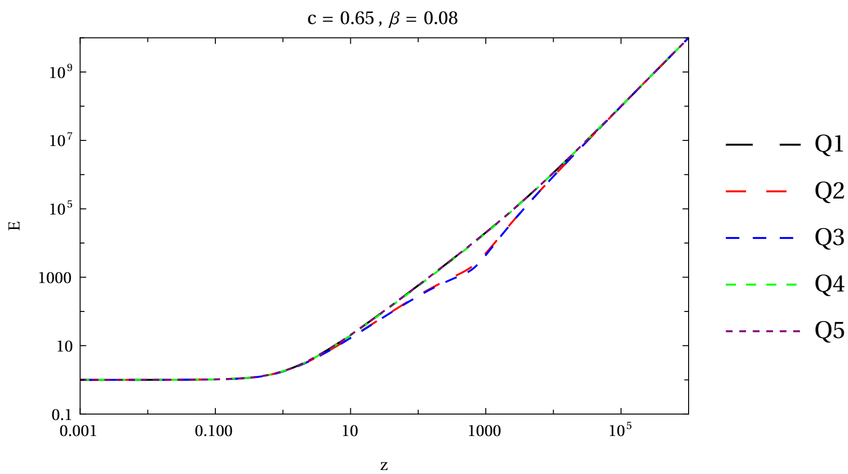
In[ ]:=
```
Show[LogLogPlot[{null, null, null, null, null}, {z, 10^-3, 10^6},
  PlotRange → {{10^-3, 10^6}, {10^-5, 1}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.015], Green, Thickness[Medium]]},
    {Directive[Dashing[0.01], Purple, Thickness[Medium]]}},
  FrameLabel → {"z", "Ωde"}, PlotLabel → "c = 0.65, β = 0.06",
  PlotLegends → {"Q1", "Q2", "Q3", "Q4", "Q5"}], p17, p27, p37, p47, p57]
```

Out[ ]=



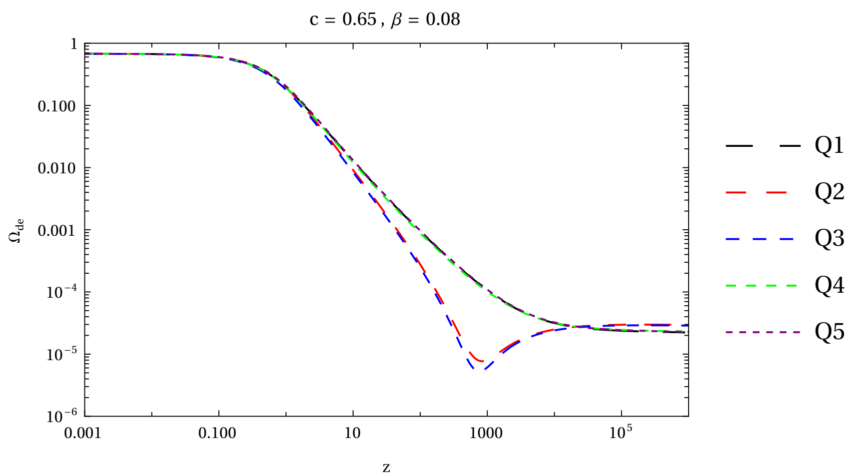
In[ ]:=
```
Show[LogLogPlot[{null, null, null, null, null}, {z, 10^-3, 10^6},
  PlotRange → {{10^-3, 10^6}, {10^-1, 10^10}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.015], Green, Thickness[Medium]]},
    {Directive[Dashing[0.01], Purple, Thickness[Medium]]}},
  FrameLabel → {"z", "E"}, PlotLabel → "c = 0.65, β = 0.08",
  PlotLegends → {"Q1", "Q2", "Q3", "Q4", "Q5"}], p14, p24, p34, p44, p54]
```

Out[ ]=



In[ ]:=
```
Show[LogLogPlot[{null, null, null, null, null}, {z, 10^-3, 10^6},
  PlotRange → {{10^-3, 10^6}, {10^-6, 1}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.04], Black, Thickness[Medium]]},
    {Directive[Dashing[0.03], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.015], Green, Thickness[Medium]]},
    {Directive[Dashing[0.01], Purple, Thickness[Medium]]}},
  FrameLabel → {"z", "Ωde"}, PlotLabel → "c = 0.65, β = 0.08",
  PlotLegends → {"Q1", "Q2", "Q3", "Q4", "Q5"}], p18, p28, p38, p48, p58]
```

Out[ ]=

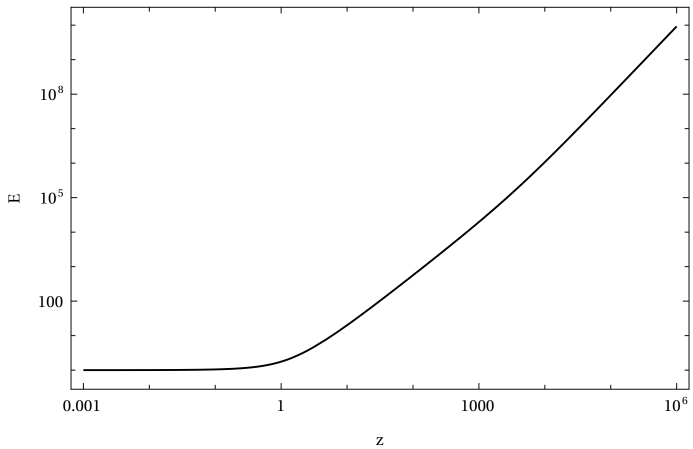


In[ ]:=
```
(*****************************************************************************)
```

```
In[ ]:= (***************************************************************************)

In[ ]:= (* ΛCDM *)

In[ ]:= Clear["Global`*"]

In[ ]:= Ωr[z_] := Ωr0 (1 + z)^4 / e[z]^2

In[ ]:= Ωr0 := Ωm0 / (1 + zeq)

In[ ]:= zeq := 2.5 × 10^4 Ωm0 h^2 (Tcmb / (2.7 Kelvin))^-4

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= 24077.4 h^2 Ωm0

In[ ]:= zeq := 24077.4405856556` h^2 Ωm0

In[ ]:= e[z_] := Sqrt[a[z]^4 ΩΛ0 + a[z] (Ωm0) + Ωr0] / a[z]^2

In[ ]:= a[z_] := 1 / (z + 1)

In[ ]:=

In[ ]:= Ωm0 = 0.2995;
        h = 0.6997;
        ΩΛ0 = 1 - (Ωm0 + Ωr0);

In[ ]:=
```

In[ ]:=
```
pELCDM = LogLogPlot[{e[z]}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Black, Thickness[Medium]}, FrameLabel → {"z", "E"}]
```

Out[ ]=



In[ ]:=

In[ ]:=

$$eLCDM[z\_] := \frac{\sqrt{a[z]^4\,\Omega\Lambda 0LCDM + a[z]\,(\Omega m0LCDM) + \Omega r0LCDM}}{a[z]^2}$$

$$\Omega m0LCDM = 0.2995;$$

$$hLCDM = 0.6997;$$

$$\Omega r0LCDM := \frac{\Omega m0LCDM}{(1 + zeqLCDM)}$$

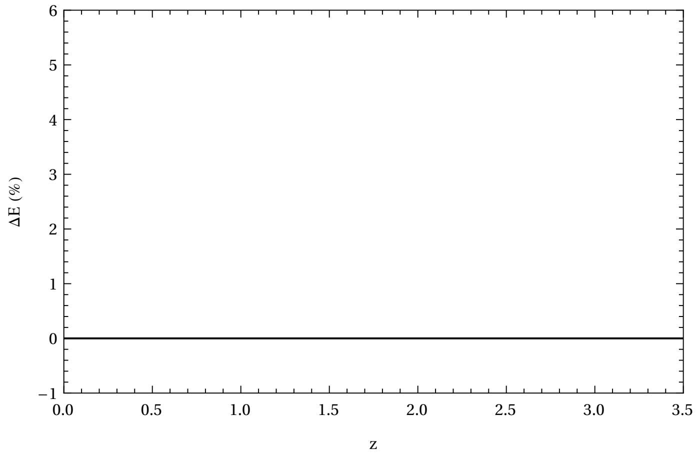
$$\Omega\Lambda 0LCDM := 1 - (\Omega m0LCDM + \Omega r0LCDM);$$

$$zeqLCDM := 2.5 \times 10^4\,\Omega m0LCDM\,hLCDM^2 \left(\frac{Tcmb}{2.7\,Kelvin}\right)^{-4}$$

In[ ]:=

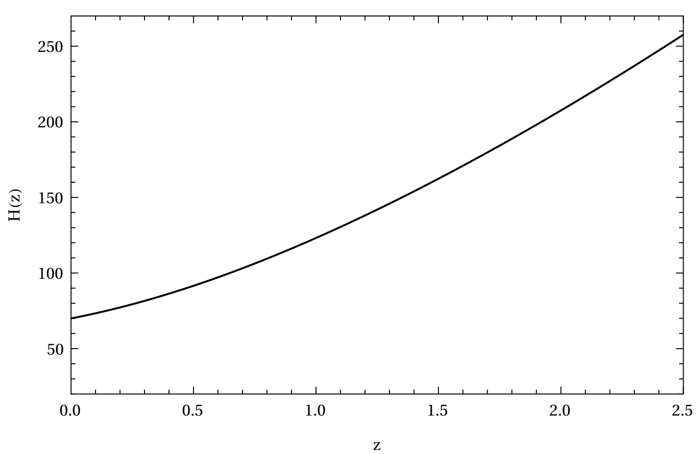
$$\Delta E[z\_] := 100\left(\frac{e[z]}{eLCDM[z]} - 1\right)$$

In[ ]:=
```
pΔELCDM = Plot[{ΔE[z]}, {z, 0, 3.5}, PlotRange → {{0, 3.5}, {-1, 6}},
  PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Black, Thickness[Medium]}, FrameLabel → {"z", "ΔE (%)"}]
```

Out[ ]=


In[ ]:=

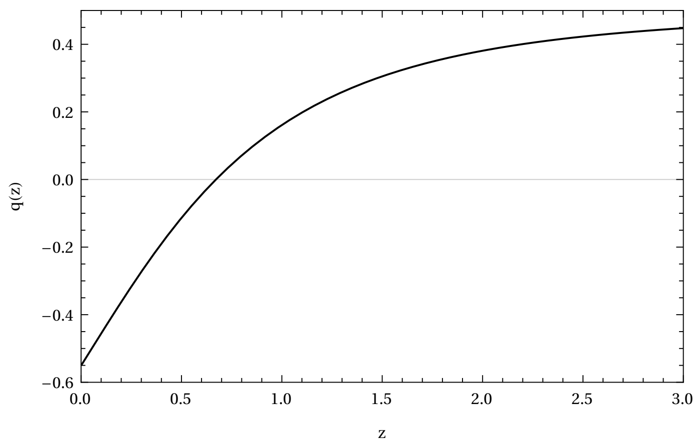
In[ ]:=
```
pHLCDM = Plot[{100 h e[z]}, {z, 0, 2.5}, PlotRange → {{0, 2.5}, {20, 270}},
  PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Black, Thickness[Medium]}, FrameLabel → {"z", "H(z)"}]
```

Out[ ]=


In[ ]:=

In[ ]:= $q[z\_] := \frac{1}{e[z]} e'[z] (1 + z) - 1$

In[ ]:= pqLCDM = Plot[{q[z], 1}, {z, 10^{-3}, 3}, PlotRange → {{0, 3}, {-0.6, 0.5}},
PlotTheme → "Scientific", FrameStyle → Black, PlotStyle →
{{Directive[Black, Thickness[Medium]]}, {Black, Thin}}, FrameLabel → {"z", "q(z)"}]

Out[ ]= 

In[ ]:=

In[ ]:= $\text{tUmodel} = \frac{977.8}{100\, h}\, \text{NIntegrate}\left[\frac{1}{(1+z)\, e[z]}, \{z, 0, \text{Infinity}\}\right]$

Out[ ]= 13.4739

In[ ]:= tULCDM = 13.4738805274518` ;

In[ ]:= $\Delta T = 100 \left(\frac{\text{tUmodel}}{\text{tULCDM}} - 1\right)$

Out[ ]= 0.

In[ ]:= (***)

In[ ]:= (* Q1 *)

In[ ]:= Clear["Global`*"]

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\, (1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+\text{zeq})}$

In[ ]:= $\text{zeq} := 2.5 \times 10^4\, \Omega m0\, h^2 \left(\frac{\text{Tcmb}}{2.7\ \text{Kelvin}}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\ h^2\ \Omega m0$

In[ ]:= zeq := 24077.4405856556` h^2 Ωm0

In[ ]:= Q1[z_] := 3 β H[z] × ρde[z]

In[ ]:= Q[z_] := Q1[z]

In[ ]:= $\Omega I[z\_] := \dfrac{Q[z]}{3\ MP^2\ H[z]^3}$

In[ ]:= $\rho de[z\_] := 3\ MP^2\ H[z]^2\ \Omega de[z]$

In[ ]:= Simplify[ΩI[z]]

Out[ ]= 3 β Ωde[z]

In[ ]:= ΩI[z_] := 3 β Ωde[z]

In[ ]:=

In[ ]:= Ωm0 = 0.3213;
h = 0.6558;
c = 0.8294;
β = 0.0782;

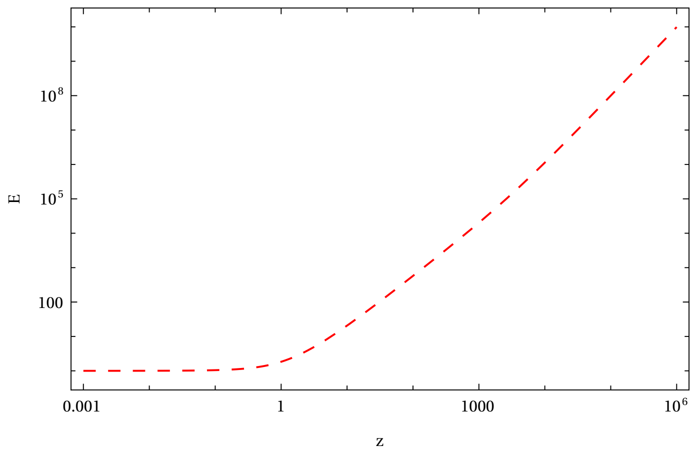
In[ ]:=

In[ ]:= sol1 = NDSolve$\left[\left\{\dfrac{1}{e[z]}\ e'[z] == -\dfrac{\Omega de[z]}{1+z}\left(\dfrac{1}{c}\ \sqrt{\Omega de[z]} + \dfrac{1}{2} + \dfrac{\Omega I[z] - 3 - \Omega r[z]}{2\ \Omega de[z]}\right),\right.\right.$
$\Omega de'[z] == -\dfrac{2\ (1 - \Omega de[z])\ \Omega de[z]}{1+z}\left(\dfrac{1}{c}\ \sqrt{\Omega de[z]} + \dfrac{1}{2} + \dfrac{\Omega I[z] - \Omega r[z]}{2\ (1 - \Omega de[z])}\right),$
$\left.\left. e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\right\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^6\}\right]$

Out[ ]= {{e → InterpolatingFunction[Domain: {{0., $1. \times 10^6$}} Output: scalar],
Ωde → InterpolatingFunction[Domain: {{0., $1. \times 10^6$}} Output: scalar]}}

In[ ]:=

In[ ]:=
```
pEM1 = LogLogPlot[{e[z] /. sol1}, {z, 10^-3, 10^6},
  PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {Directive[Dashing[0.02], Red, Thickness[Medium]]}, FrameLabel → {"z", "E"}]
```

Out[ ]=


In[ ]:=

In[ ]:=

$$\text{eLCDM[z\_]} := \frac{\sqrt{\text{a[z]}^4\,\Omega\Lambda 0\text{LCDM} + \text{a[z]}\,(\Omega\text{m}0\text{LCDM}) + \Omega\text{r}0\text{LCDM}}}{\text{a[z]}^2}$$

$$\text{a[z\_]} := \frac{1}{z+1}$$

Ωm0LCDM = 0.2995;

hLCDM = 0.6997;

$$\Omega\text{r}0\text{LCDM} := \frac{\Omega\text{m}0\text{LCDM}}{(1 + \text{zeqLCDM})}$$

ΩΛ0LCDM := 1 - (Ωm0LCDM + Ωr0LCDM);

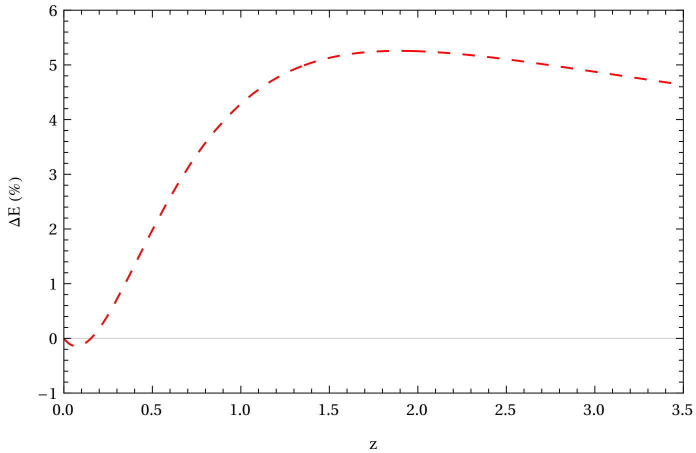
$$\text{zeqLCDM} := 2.5 \times 10^4\,\Omega\text{m}0\text{LCDM}\;\text{hLCDM}^2 \left(\frac{\text{Tcmb}}{2.7\ \text{Kelvin}}\right)^{-4}$$

In[ ]:=

$$\Delta\text{E[z\_]} := 100\left(\frac{\text{e[z]}}{\text{eLCDM[z]}} - 1\right)$$

In[ ]:=
```
pΔEM1 = Plot[{ΔE[z] /. sol1}, {z, 0, 3.5}, PlotRange → {{0, 3.5}, {-1, 6}},
   PlotTheme → "Scientific", FrameStyle → Black, PlotStyle →
    {Directive[Dashing[0.02], Red, Thickness[Medium]]}, FrameLabel → {"z", "ΔE (%)"}]
```

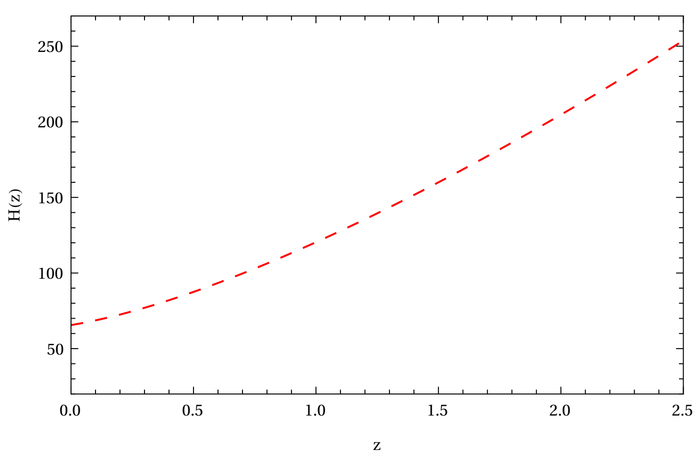
Out[ ]=



In[ ]:=

In[ ]:=
```
pHM1 = Plot[{100 h e[z] /. sol1}, {z, 0, 2.5}, PlotRange → {{0, 2.5}, {20, 270}},
   PlotTheme → "Scientific", FrameStyle → Black, PlotStyle →
    {Directive[Dashing[0.02], Red, Thickness[Medium]]}, FrameLabel → {"z", "H(z)"}]
```

Out[ ]=

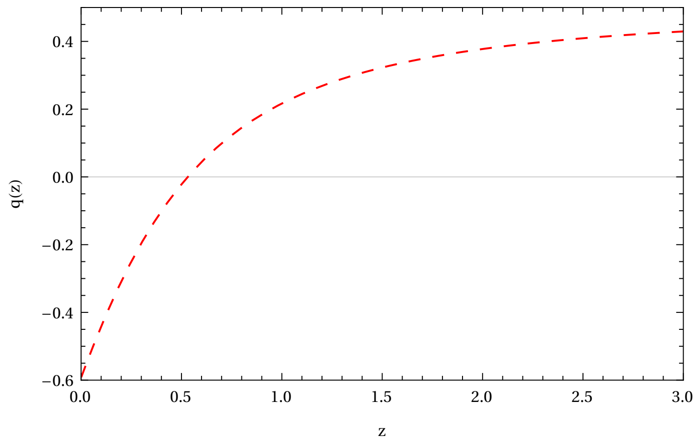


In[ ]:=

In[ ]:= $q[z\_] := \frac{1}{e[z]} e'[z](1+z) - 1$

In[ ]:=
```
pqM1 = Plot[{q[z] /. sol1, 1}, {z, 10^-3, 3},
  PlotRange → {{0, 3}, {-0.6, 0.5}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.02], Red, Thickness[Medium]]}, {Black, Thin}},
  FrameLabel → {"z", "q(z)"}]
```

Out[ ]=



In[ ]:=

In[ ]:= $\text{tUmodel} = \frac{977.8}{100\,h}\,\text{NIntegrate}\left[\frac{1}{(1+z)\,(e[z]\,/.\,\text{sol1})}, \{z, 0, \text{Infinity}\}\right][[1]]$

Out[ ]= 13.9958

In[ ]:= tULCDM = 13.4738805274518` ;

In[ ]:= $\Delta T = 100\left(\frac{\text{tUmodel}}{\text{tULCDM}} - 1\right)$

Out[ ]= 3.87372

In[ ]:= (**)

In[ ]:= (* Q2 *)

In[ ]:= Clear["Global`*"]

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+\text{zeq})}$

In[ ]:= $\text{zeq} := 2.5\times 10^4\,\Omega m0\,h^2\left(\frac{\text{Tcmb}}{2.7\,\text{Kelvin}}\right)^{-4}$

In[ ]:=
```
Tcmb := 2.7255 Kelvin
```

In[ ]:=
```
Simplify[zeq]
```

Out[ ]= $24\,077.4\ h^2\ \Omega m0$

In[ ]:=
```
zeq := 24077.4405856556` h^2 Ωm0
```

In[ ]:=
```
Q2[z_] := 3 β H[z] × ρc[z]
```

In[ ]:=
```
Q[z_] := Q2[z]
```

In[ ]:= $\Omega I[z\_] := \dfrac{Q[z]}{3\ MP^2\ H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\ MP^2\ H0^2\ \Omega c0\ (1 + z)^3$

In[ ]:= $\rho de[z\_] := 3\ MP^2\ H[z]^2\ \Omega de[z]$

In[ ]:=
```
H[z_] := H0 e[z]
```

In[ ]:=
```
Simplify[ΩI[z]]
```

Out[ ]= $\dfrac{3\ (1 + z)^3\ \beta\ \Omega c0}{e[z]^2}$

In[ ]:= $\Omega I[z\_] := \dfrac{3\ (1 + z)^3\ \beta\ \Omega c0}{e[z]^2}$

In[ ]:=

In[ ]:=
```
Ωm0 = 0.3225;
Ωb0 = 0.0518;
Ωc0 = Ωm0 - Ωb0;
h = 0.6399;
c = 0.7538;
β = 0.0092;
```

In[ ]:=

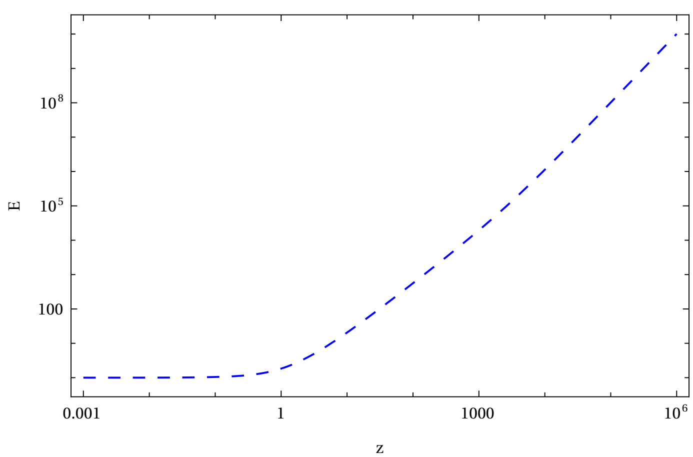
In[ ]:=
$$\text{sol1} = \text{NDSolve}\Big[\Big\{\frac{1}{\text{e[z]}}\,\text{e}'\text{[z]} == -\frac{\Omega\text{de[z]}}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de[z]}}+\frac{1}{2}+\frac{\Omega\text{I[z]}-3-\Omega\text{r[z]}}{2\,\Omega\text{de[z]}}\right),$$
$$\Omega\text{de}'\text{[z]} == -\frac{2\,(1-\Omega\text{de[z]})\,\Omega\text{de[z]}}{1+\text{z}}\left(\frac{1}{\text{c}}\sqrt{\Omega\text{de[z]}}+\frac{1}{2}+\frac{\Omega\text{I[z]}-\Omega\text{r[z]}}{2\,(1-\Omega\text{de[z]})}\right),$$
$$\text{e[0]} == 1,\ \Omega\text{de[0]} == 1-\Omega\text{m0}-\Omega\text{r0}\Big\},\ \{\text{e},\ \Omega\text{de}\},\ \{\text{z},\ 0,\ 10^6\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain : {{0., $1.\times10^6$}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., $1.\times10^6$}} Output : scalar]}}

In[ ]:=

In[ ]:= pEM2 = LogLogPlot[{e[z] /. sol1}, {z, 10^{-3}, 10^6}, PlotTheme → "Scientific",
FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
FrameLabel → {"z", "E"}]

Out[ ]= 

In[ ]:=

In[ ]:=
$$\text{eLCDM[z\_]} := \frac{\sqrt{a[z]^4\ \Omega\Lambda0LCDM + a[z]\ (\Omega m0LCDM) + \Omega r0LCDM}}{a[z]^2}$$

$$\text{a[z\_]} := \frac{1}{z+1}$$

Ωm0LCDM = 0.2995;

hLCDM = 0.6997;

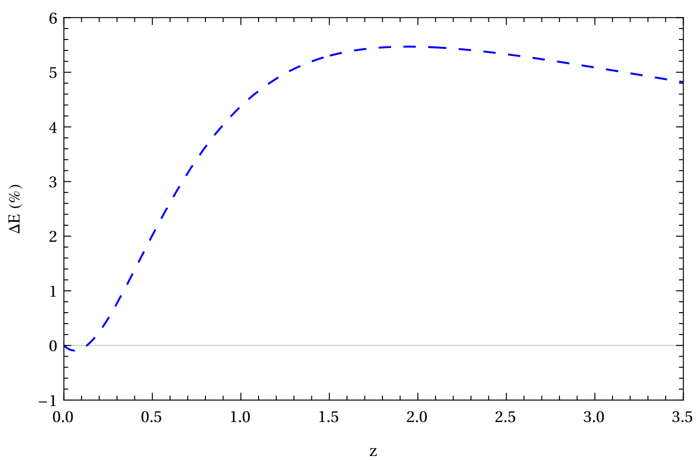
$$\Omega r0LCDM := \frac{\Omega m0LCDM}{(1 + zeqLCDM)}$$

ΩΛ0LCDM := 1 - (Ωm0LCDM + Ωr0LCDM);

$$zeqLCDM := 2.5 \times 10^4\ \Omega m0LCDM\ hLCDM^2 \left(\frac{Tcmb}{2.7\ Kelvin}\right)^{-4}$$

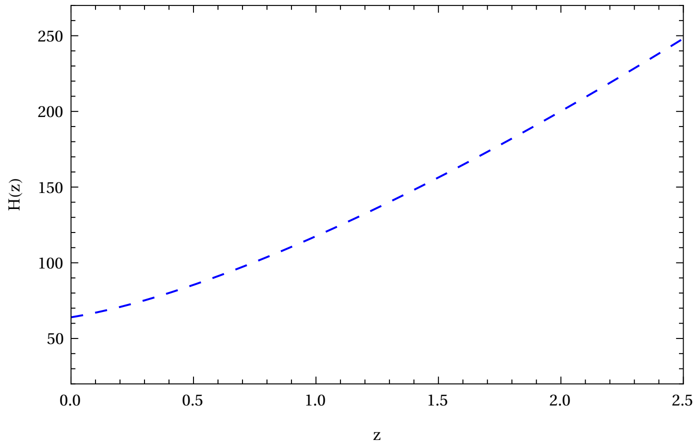
In[ ]:=
$$\Delta E[z\_] := 100 \left(\frac{e[z]}{eLCDM[z]} - 1\right)$$

In[ ]:=
```
pΔEM2 = Plot[{ΔE[z] /. sol1}, {z, 0, 3.5}, PlotRange → {{0, 3.5}, {-1, 6}},
  PlotTheme → "Scientific", FrameStyle → Black, PlotStyle →
   {Directive[Dashing[0.02], Blue, Thickness[Medium]]}, FrameLabel → {"z", "ΔE (%)"}]
```

Out[ ]=


In[ ]:=

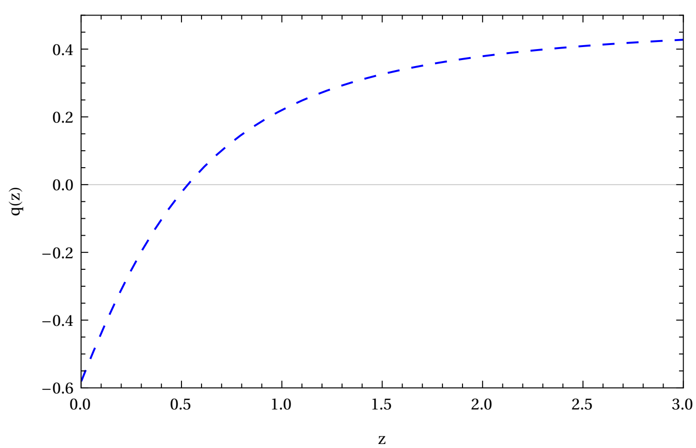
In[ ]:=
```
pHM2 = Plot[{100 h e[z] /. sol1}, {z, 0, 2.5}, PlotRange → {{0, 2.5}, {20, 270}},
  PlotTheme → "Scientific", FrameStyle → Black, PlotStyle →
   {Directive[Dashing[0.02], Blue, Thickness[Medium]]}, FrameLabel → {"z", "H(z)"}]
```

Out[ ]=


In[ ]:=

In[ ]:= $q[z\_] := \frac{1}{e[z]} e'[z](1+z) - 1$

In[ ]:=
```
pqM2 = Plot[{q[z] /. sol1, 1}, {z, 10^-3, 3},
  PlotRange → {{0, 3}, {-0.6, 0.5}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.02], Blue, Thickness[Medium]]}, {Black, Thin}},
  FrameLabel → {"z", "q(z)"}]
```

Out[ ]=


In[ ]:=

In[ ]:= $tUmodel = \frac{977.8}{100\,h}\,\text{NIntegrate}\left[\frac{1}{(1+z)\,(e[z]\,/.\,sol1)},\{z,0,Infinity\}\right][[1]]$

Out[ ]= 14.3376

In[ ]:= tULCDM = 13.4738805274518\` ;

In[ ]:= $\Delta T = 100\left(\frac{tUmodel}{tULCDM}-1\right)$

Out[ ]= 6.41058

In[ ]:= (***)

In[ ]:= (* Q3 *)

In[ ]:= Clear["Global\`*"]

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\,\Omega m0\,h^2\left(\frac{Tcmb}{2.7\,Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\,h^2\,\Omega m0$

In[ ]:= $zeq := 24077.4405856556\text{\`}\ h^2\,\Omega m0$

In[ ]:= $Q3[z\_] := 3\,\beta\,H[z]\,(\rho de[z]+\rho c[z])$

In[ ]:= Q[z_] := Q3[z]

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3\,MP^2\,H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\,MP^2\,H0^2\,\Omega c0\,(1+z)^3$

In[ ]:= $\rho de[z\_] := 3\,MP^2\,H[z]^2\,\Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= `Simplify[ΩI[z]]`

Out[ ]= $$\frac{3\,\beta\,\left((1+z)^3\,\Omega c0+e[z]^2\,\Omega de[z]\right)}{e[z]^2}$$

In[ ]:= $$\Omega I[z\_] := \frac{3\,\beta\,\left((1+z)^3\,\Omega c0+e[z]^2\,\Omega de[z]\right)}{e[z]^2}$$

In[ ]:=

In[ ]:=
```
Ωm0 = 0.3235;
Ωb0 = 0.0547;
Ωc0 = Ωm0 - Ωb0;
h = 0.6394;
c = 0.7538;
β = 0.0092;
```

In[ ]:=

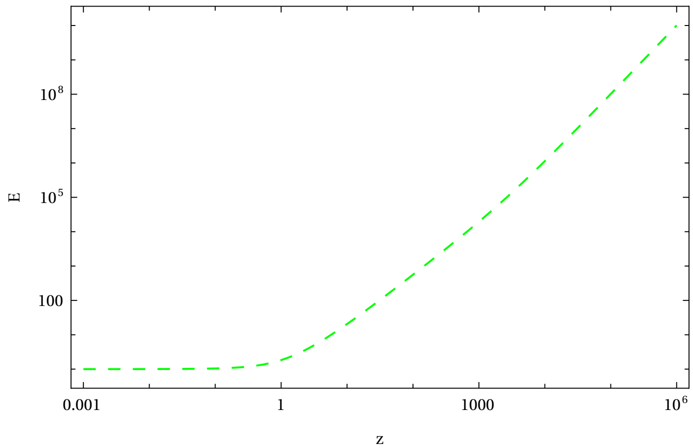
In[ ]:= $$\text{sol1} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\,e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\,\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-3-\Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\,\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-\Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1-\Omega m0-\Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^6\}\Big]$$

Out[ ]= {{e → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar],

Ωde → InterpolatingFunction[Domain : {{0., 1. × 10^6}} Output : scalar]}}

In[ ]:=

In[ ]:=
```
pEM3 = LogLogPlot[{e[z] /. sol1}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Green, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=


In[ ]:=

In[ ]:=
$$eLCDM[z\_] := \frac{\sqrt{a[z]^4\ \Omega\Lambda0LCDM + a[z]\ (\Omega m0LCDM) + \Omega r0LCDM}}{a[z]^2}$$

$$a[z\_] := \frac{1}{z+1}$$

$\Omega m0LCDM = 0.2995;$

$hLCDM = 0.6997;$

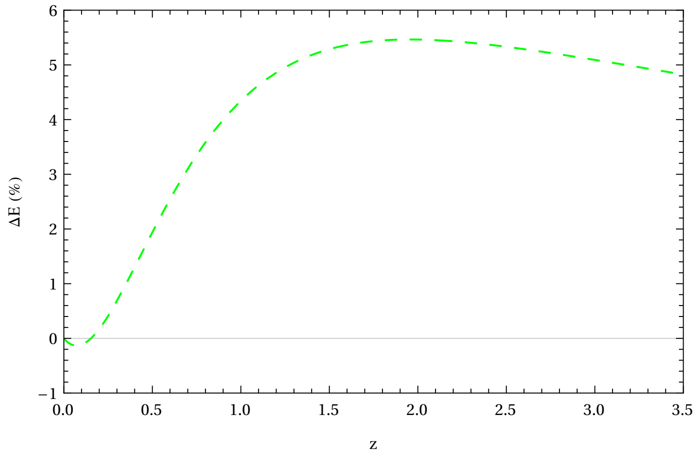
$$\Omega r0LCDM := \frac{\Omega m0LCDM}{(1 + zeqLCDM)}$$

$\Omega\Lambda0LCDM := 1 - (\Omega m0LCDM + \Omega r0LCDM);$

$$zeqLCDM := 2.5 \times 10^4\ \Omega m0LCDM\ hLCDM^2 \left(\frac{Tcmb}{2.7\ Kelvin}\right)^{-4}$$

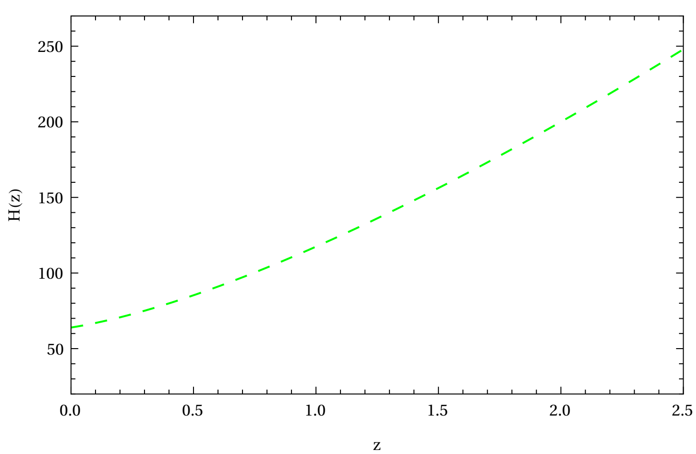
In[ ]:=
$$\Delta E[z\_] := 100 \left(\frac{e[z]}{eLCDM[z]} - 1\right)$$

In[ ]:=
```
pΔEM3 = Plot[{ΔE[z] /. sol1}, {z, 0, 3.5}, PlotRange → {{0, 3.5}, {-1, 6}},
  PlotTheme → "Scientific", FrameStyle → Black, PlotStyle →
   {Directive[Dashing[0.02], Green, Thickness[Medium]]}, FrameLabel → {"z", "ΔE (%)"}]
```

Out[ ]=


In[ ]:=

In[ ]:=
```
pHM3 = Plot[{100 h e[z] /. sol1}, {z, 0, 2.5}, PlotRange → {{0, 2.5}, {20, 270}},
  PlotTheme → "Scientific", FrameStyle → Black, PlotStyle →
   {Directive[Dashing[0.02], Green, Thickness[Medium]]}, FrameLabel → {"z", "H(z)"}]
```

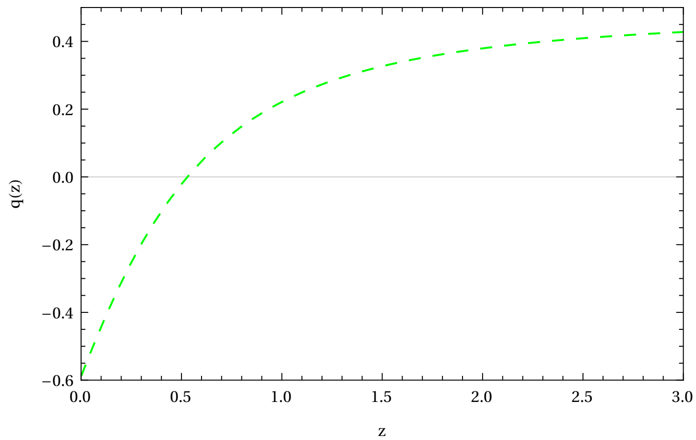
Out[ ]=


In[ ]:=

In[ ]:= $q[z\_] := \frac{1}{e[z]} e'[z] (1 + z) - 1$

In[ ]:=
```
pqM3 = Plot[{q[z] /. sol1, 1}, {z, 10^-3, 3},
  PlotRange → {{0, 3}, {-0.6, 0.5}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.02], Green, Thickness[Medium]]}, {Black, Thin}},
  FrameLabel → {"z", "q(z)"}]
```

Out[ ]=



In[ ]:=

In[ ]:= $tUmodel = \frac{977.8}{100\ h}\ \text{NIntegrate}\left[\frac{1}{(1+z)\,(e[z]\ /.\ sol1)}, \{z, 0, \text{Infinity}\}\right][[1]]$

Out[ ]= 14.3542

In[ ]:=
```
tULCDM = 13.4738805274518` ;
```

In[ ]:= $\Delta T = 100\left(\frac{tUmodel}{tULCDM} - 1\right)$

Out[ ]= 6.53388

In[ ]:=
```
(*****************************************************************************)
```

In[ ]:=
```
(* Q4 *)
```

In[ ]:=
```
Clear["Global`*"]
```

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\,(1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\ \Omega m0\ h^2\left(\frac{Tcmb}{2.7\ Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\ h^2\ \Omega m0$

In[ ]:= zeq := 24077.4405856556` h^2 Ωm0

In[ ]:= $Q4[z\_] := 3\ \beta\ H[z]\ \sqrt{\rho de[z] \times \rho c[z]}$

In[ ]:= Q[z_] := Q4[z]

In[ ]:= $\Omega I[z\_] := \dfrac{Q[z]}{3\ MP^2\ H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\ MP^2\ H0^2\ \Omega c0\ (1 + z)^3$

In[ ]:= $\rho de[z\_] := 3\ MP^2\ H[z]^2\ \Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= Simplify[PowerExpand[Simplify[ΩI[z]]]]

Out[ ]= $\dfrac{3\ (1 + z)^{3/2}\ \beta\ \sqrt{\Omega c0}\ \sqrt{\Omega de[z]}}{e[z]}$

In[ ]:= $\Omega I[z\_] := \dfrac{3\ (1 + z)^{3/2}\ \beta\ \sqrt{\Omega c0}\ \sqrt{\Omega de[z]}}{e[z]}$

In[ ]:=

In[ ]:=
```
Ωm0 = 0.3233;
Ωb0 = 0.0526;
Ωc0 = Ωm0 - Ωb0;
h = 0.6512;
c = 0.7868;
β = 0.0346;
```

In[ ]:=

In[ ]:=

$$\text{sol1} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\, e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - 3 - \Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]} + \frac{1}{2} + \frac{\Omega I[z] - \Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1 - \Omega m0 - \Omega r0\Big\},\ \{e, \Omega de\},\ \{z, 0, 10^6\}\Big]$$

Out[ ]=

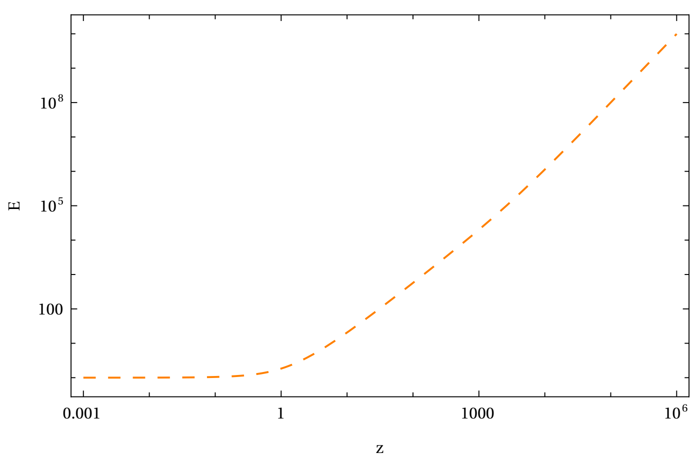
```
{{e → InterpolatingFunction[ Domain: {{0., 1.×10^6}} Output: scalar ],
  Ωde → InterpolatingFunction[ Domain: {{0., 1.×10^6}} Output: scalar ]}}
```

In[ ]:=

In[ ]:=

```
pEM4 = LogLogPlot[{e[z] /. sol1}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Orange, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=



In[ ]:=

In[ ]:= $\text{eLCDM[z\_]} := \dfrac{\sqrt{\text{a[z]}^4\ \Omega\Lambda 0\text{LCDM} + \text{a[z]}\,(\Omega\text{m}0\text{LCDM}) + \Omega\text{r}0\text{LCDM}}}{\text{a[z]}^2}$

$\text{a[z\_]} := \dfrac{1}{z+1}$

```
Ωm0LCDM = 0.2995;
hLCDM = 0.6997;
```

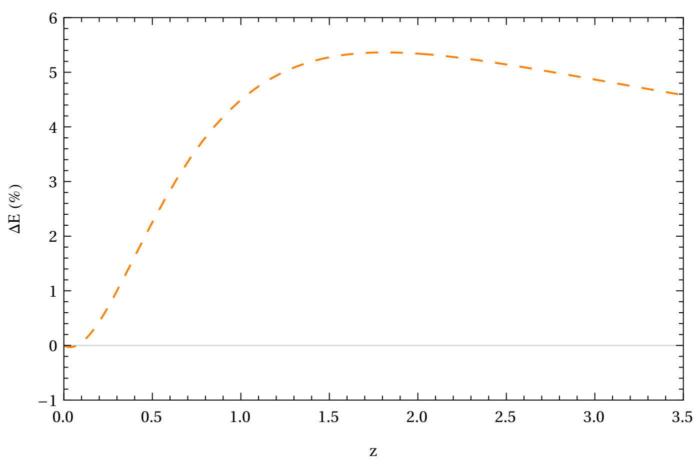
$\Omega\text{r}0\text{LCDM} := \dfrac{\Omega\text{m}0\text{LCDM}}{(1+\text{zeqLCDM})}$

```
ΩΛ0LCDM := 1 - (Ωm0LCDM + Ωr0LCDM);
```

$\text{zeqLCDM} := 2.5 \times 10^4\ \Omega\text{m}0\text{LCDM}\ \text{hLCDM}^2 \left(\dfrac{\text{Tcmb}}{2.7\ \text{Kelvin}}\right)^{-4}$

In[ ]:= $\Delta\text{E[z\_]} := 100\left(\dfrac{\text{e[z]}}{\text{eLCDM[z]}} - 1\right)$

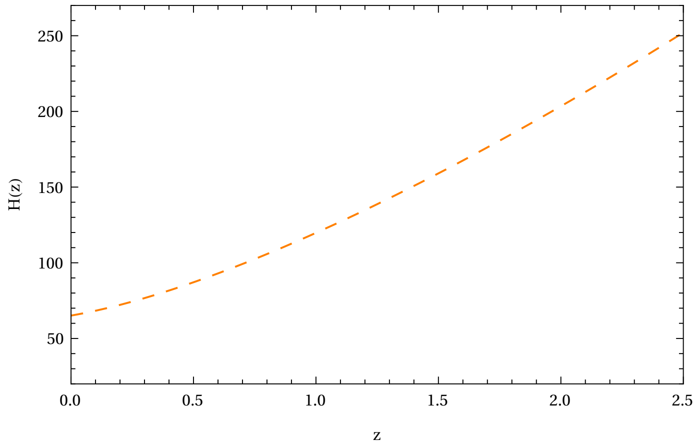
In[ ]:=
```
pΔEM4 = Plot[{ΔE[z] /. sol1}, {z, 0, 3.5}, PlotRange → {{0, 3.5}, {-1, 6}},
  PlotTheme → "Scientific", FrameStyle → Black, PlotStyle →
   {Directive[Dashing[0.02], Orange, Thickness[Medium]]}, FrameLabel → {"z", "ΔE (%)"}]
```

Out[ ]=


In[ ]:=

In[ ]:=
```
pHM4 = Plot[{100 h e[z] /. sol1}, {z, 0, 2.5}, PlotRange → {{0, 2.5}, {20, 270}},
  PlotTheme → "Scientific", FrameStyle → Black, PlotStyle →
   {Directive[Dashing[0.02], Orange, Thickness[Medium]]}, FrameLabel → {"z", "H(z)"}]
```

Out[ ]=

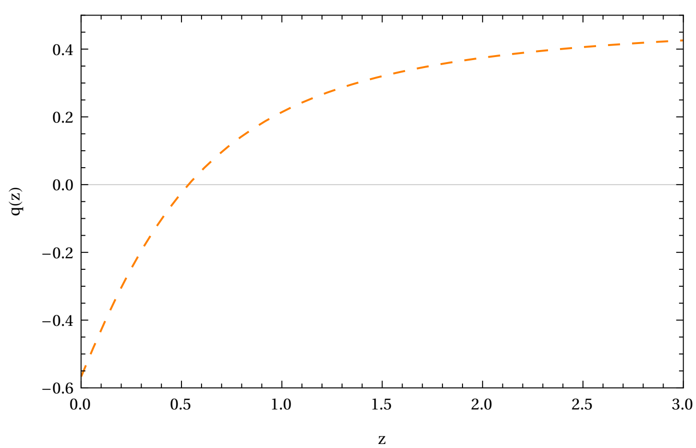


In[ ]:=

In[ ]:= $q[z\_] := \frac{1}{e[z]} e'[z] (1 + z) - 1$

In[ ]:=
```
pqM4 = Plot[{q[z] /. sol1, 1}, {z, 10^-3, 3},
  PlotRange → {{0, 3}, {-0.6, 0.5}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.02], Orange, Thickness[Medium]]}, {Black, Thin}},
  FrameLabel → {"z", "q(z)"}]
```

Out[ ]=



In[ ]:=

In[ ]:= $tUmodel = \frac{977.8}{100\ h}\ NIntegrate\left[\frac{1}{(1+z)\ (e[z]\ /.\ sol1)},\ \{z,\ 0,\ Infinity\}\right][[1]]$

Out[ ]= 14.0777

In[ ]:= tULCDM = 13.4738805274518` ;

In[ ]:= $\Delta T = 100\left(\frac{tUmodel}{tULCDM} - 1\right)$

Out[ ]= 4.48138

In[ ]:= (***)

In[ ]:= (* Q5 *)

In[ ]:= Clear["Global`*"]

In[ ]:= $\Omega r[z\_] := \frac{\Omega r0\ (1+z)^4}{e[z]^2}$

In[ ]:= $\Omega r0 := \frac{\Omega m0}{(1+zeq)}$

In[ ]:= $zeq := 2.5\times 10^4\ \Omega m0\ h^2\left(\frac{Tcmb}{2.7\ Kelvin}\right)^{-4}$

In[ ]:= Tcmb := 2.7255 Kelvin

In[ ]:= Simplify[zeq]

Out[ ]= $24\,077.4\ h^2\ \Omega m0$

In[ ]:= $zeq := 24077.4405856556\text{`}\ \ h^2\ \Omega m0$

In[ ]:= $Q5[z\_] := 3\ \beta\ H[z]\ \frac{\rho de[z]\times\rho c[z]}{\rho de[z]+\rho c[z]}$

In[ ]:= Q[z_] := Q5[z]

In[ ]:= $\Omega I[z\_] := \frac{Q[z]}{3\ MP^2\ H[z]^3}$

In[ ]:= $\rho c[z\_] := 3\ MP^2\ H0^2\ \Omega c0\ (1+z)^3$

In[ ]:= $\rho de[z\_] := 3\ MP^2\ H[z]^2\ \Omega de[z]$

In[ ]:= H[z_] := H0 e[z]

In[ ]:= `Simplify[PowerExpand[Simplify[ΩI[z]]]]`

Out[ ]=
$$\frac{3\,(1+z)^3\,\beta\,\Omega c0\,\Omega de[z]}{(1+z)^3\,\Omega c0+e[z]^2\,\Omega de[z]}$$

In[ ]:=
$$\Omega I[z\_] := \frac{3\,(1+z)^3\,\beta\,\Omega c0\,\Omega de[z]}{(1+z)^3\,\Omega c0+e[z]^2\,\Omega de[z]}$$

In[ ]:=

In[ ]:=
```
Ωm0 = 0.3224;
Ωb0 = 0.0521;
Ωc0 = Ωm0 - Ωb0;
c = 0.7983;
β = 0.0958;
h = 0.6545;
```

In[ ]:=

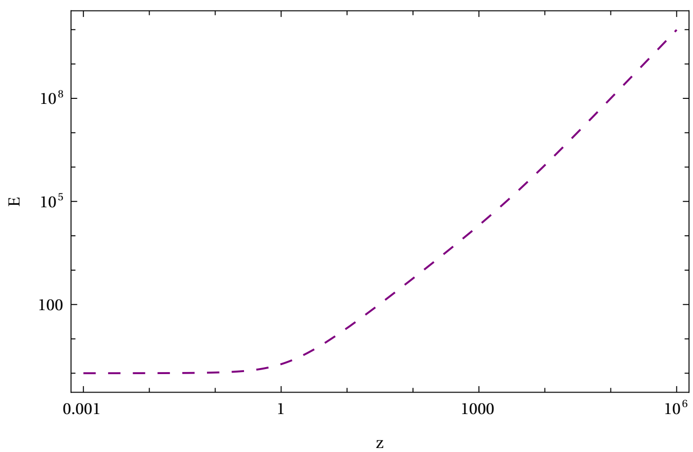
In[ ]:=
$$\text{sol1} = \text{NDSolve}\Big[\Big\{\frac{1}{e[z]}\,e'[z] == -\frac{\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-3-\Omega r[z]}{2\,\Omega de[z]}\right),$$
$$\Omega de'[z] == -\frac{2\,(1-\Omega de[z])\,\Omega de[z]}{1+z}\left(\frac{1}{c}\sqrt{\Omega de[z]}+\frac{1}{2}+\frac{\Omega I[z]-\Omega r[z]}{2\,(1-\Omega de[z])}\right),$$
$$e[0] == 1,\ \Omega de[0] == 1-\Omega m0-\Omega r0\Big\},\ \{e,\ \Omega de\},\ \{z,\ 0,\ 10^6\}\Big]$$

Out[ ]=
```
{{e → InterpolatingFunction[ Domain : {{0., 1. × 10^6}}  Output : scalar ],
  Ωde → InterpolatingFunction[ Domain : {{0., 1. × 10^6}}  Output : scalar ]}}
```

In[ ]:=

In[ ]:=
```
pEM5 = LogLogPlot[{e[z] /. sol1}, {z, 10^-3, 10^6}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {Directive[Dashing[0.02], Purple, Thickness[Medium]]},
  FrameLabel → {"z", "E"}]
```

Out[ ]=



In[ ]:=

In[ ]:=

$$\texttt{eLCDM[z\_] :=} \frac{\sqrt{\texttt{a[z]}^4\ \Omega\Lambda 0\texttt{LCDM} + \texttt{a[z]}\,(\Omega \texttt{m0LCDM}) + \Omega \texttt{r0LCDM}}}{\texttt{a[z]}^2}$$

$$\texttt{a[z\_] :=} \frac{1}{z+1}$$

```
Ωm0LCDM = 0.2995;
hLCDM = 0.6997;
```

$$\Omega\texttt{r0LCDM :=} \frac{\Omega\texttt{m0LCDM}}{(1+\texttt{zeqLCDM})}$$

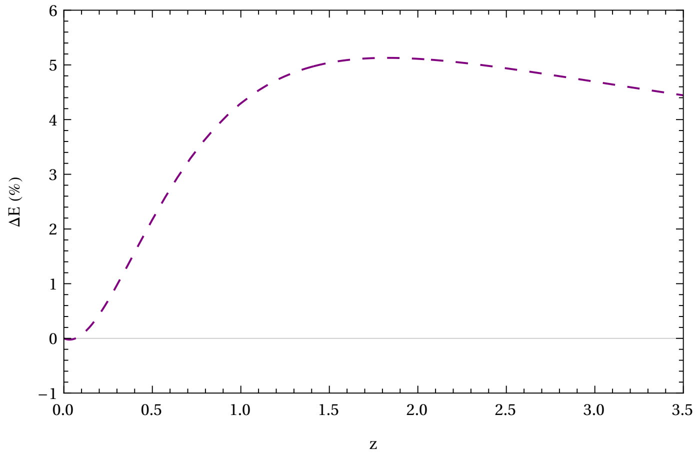
```
ΩΛ0LCDM := 1 - (Ωm0LCDM + Ωr0LCDM);
```

$$\texttt{zeqLCDM := } 2.5\times 10^4\ \Omega\texttt{m0LCDM hLCDM}^2 \left(\frac{\texttt{Tcmb}}{2.7\ \texttt{Kelvin}}\right)^{-4}$$

In[ ]:=

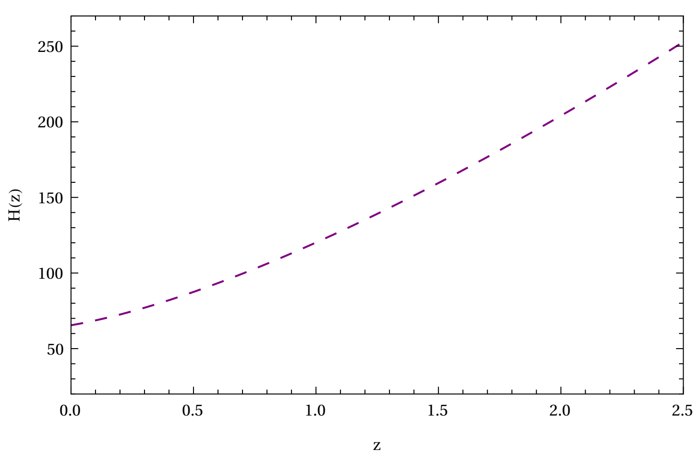
$$\Delta\texttt{E[z\_] := } 100\left(\frac{\texttt{e[z]}}{\texttt{eLCDM[z]}} - 1\right)$$

In[ ]:=
```
pΔEM5 = Plot[{ΔE[z] /. sol1}, {z, 0, 3.5}, PlotRange → {{0, 3.5}, {-1, 6}},
  PlotTheme → "Scientific", FrameStyle → Black, PlotStyle →
   {Directive[Dashing[0.02], Purple, Thickness[Medium]]}, FrameLabel → {"z", "ΔE (%)"}]
```

Out[ ]=


In[ ]:=

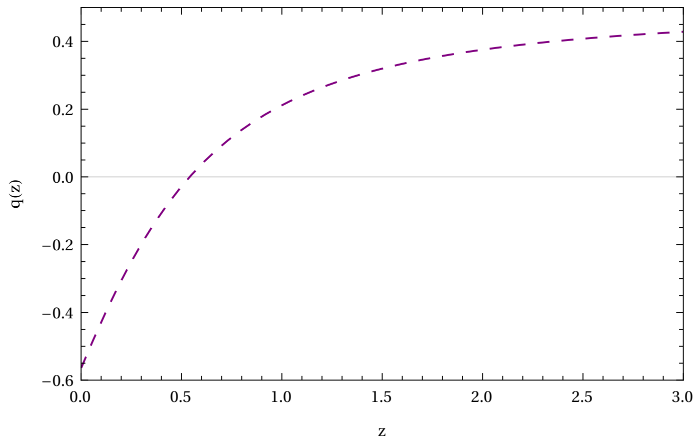
In[ ]:=
```
pHM5 = Plot[{100 h e[z] /. sol1}, {z, 0, 2.5}, PlotRange → {{0, 2.5}, {20, 270}},
  PlotTheme → "Scientific", FrameStyle → Black, PlotStyle →
   {Directive[Dashing[0.02], Purple, Thickness[Medium]]}, FrameLabel → {"z", "H(z)"}]
```

Out[ ]=


In[ ]:=

In[ ]:= $q[z\_] := \frac{1}{e[z]} e'[z](1+z) - 1$

In[ ]:=
```
pqM5 = Plot[{q[z] /. sol1, 1}, {z, 10^-3, 3},
  PlotRange → {{0, 3}, {-0.6, 0.5}}, PlotTheme → "Scientific", FrameStyle → Black,
  PlotStyle → {{Directive[Dashing[0.02], Purple, Thickness[Medium]]}, {Black, Thin}},
  FrameLabel → {"z", "q(z)"}]
```

Out[ ]= 

In[ ]:=

In[ ]:=
$$\text{tUmodel} = \frac{977.8}{100\,h}\,\text{NIntegrate}\left[\frac{1}{(1+z)\,(e[z]\,/.\,\text{sol1})}, \{z, 0, \text{Infinity}\}\right][[1]]$$

Out[ ]= 14.0178

In[ ]:= `tULCDM = 13.4738805274518` ;`

In[ ]:=
$$\Delta T = 100\left(\frac{\text{tUmodel}}{\text{tULCDM}} - 1\right)$$

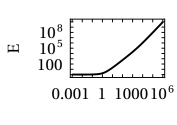
Out[ ]= 4.03704

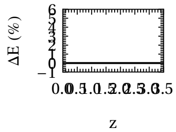
In[ ]:= `(*****************************************************************************)`

In[ ]:= `(* Comparison of models *)`

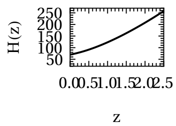
In[ ]:= `Clear["Global`*"]`

In[ ]:=

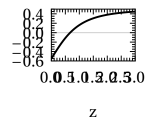
In[ ]:= pELCDM =  ;

In[∘]:= **pΔELCDM =** 

;

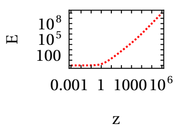
In[∘]:= **pHLCDM =** 

;

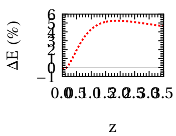
In[∘]:= **pqLCDM =** 

;

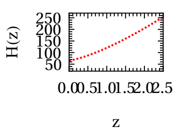
In[∘]:=

In[∘]:= **pEM1 =** 

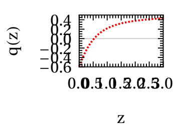
;

In[∘]:= **pΔEM1 =** 

;

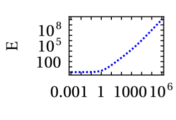
In[∘]:= **pHM1 =** 

;

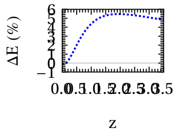
In[∘]:= **pqM1 =** 

;

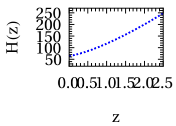
In[∘]:=

In[∘]:= **pEM2 =** 

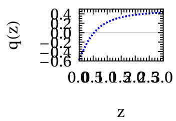
;

In[◦]:= **pΔEM2 =** 

;

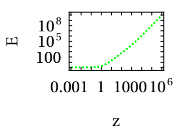
In[◦]:= **pHM2 =** 

;

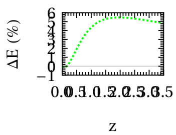
In[◦]:= **pqM2 =** 

;

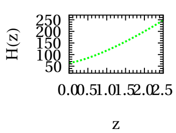
In[◦]:=

In[◦]:= **pEM3 =** 

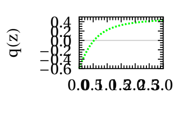
;

In[◦]:= **pΔEM3 =** 

;

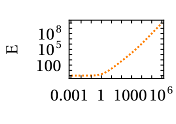
In[◦]:= **pHM3 =** 

;

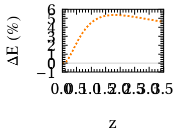
In[◦]:= **pqM3 =** 

;

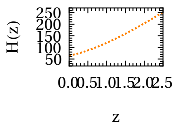
In[◦]:=

In[◦]:= **pEM4 =** 

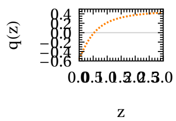
;

In[ ]:= **pΔEM4 =** 

;

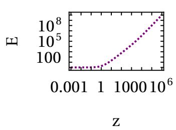
In[ ]:= **pHM4 =** 

;

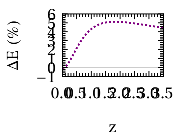
In[ ]:= **pqM4 =** 

;

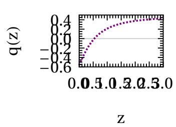
In[ ]:=

In[ ]:= **pEM5 =** 

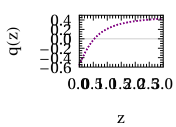
;

In[ ]:= **pΔEM5 =** 

;

In[ ]:= **pHM5 =** 

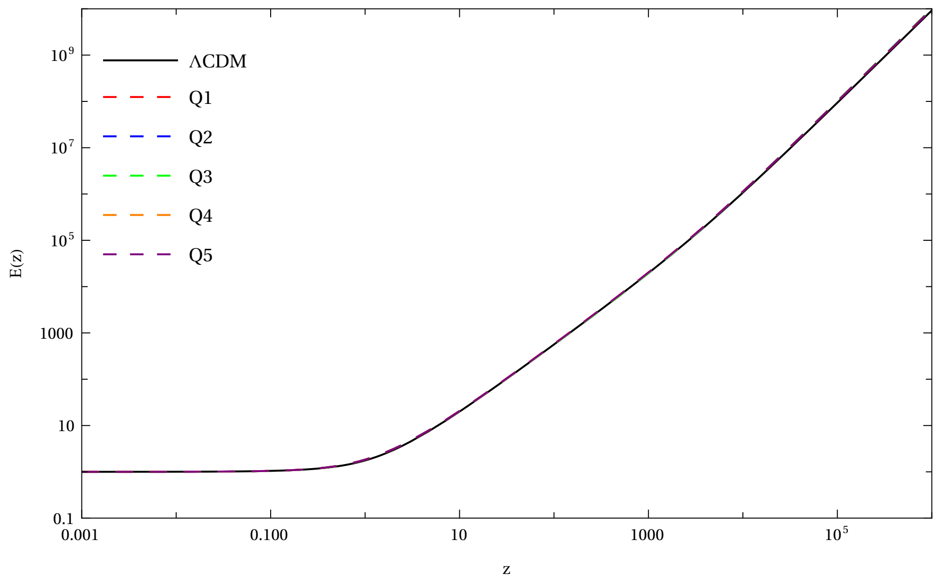
;

In[ ]:= **pqM5 =** 

;

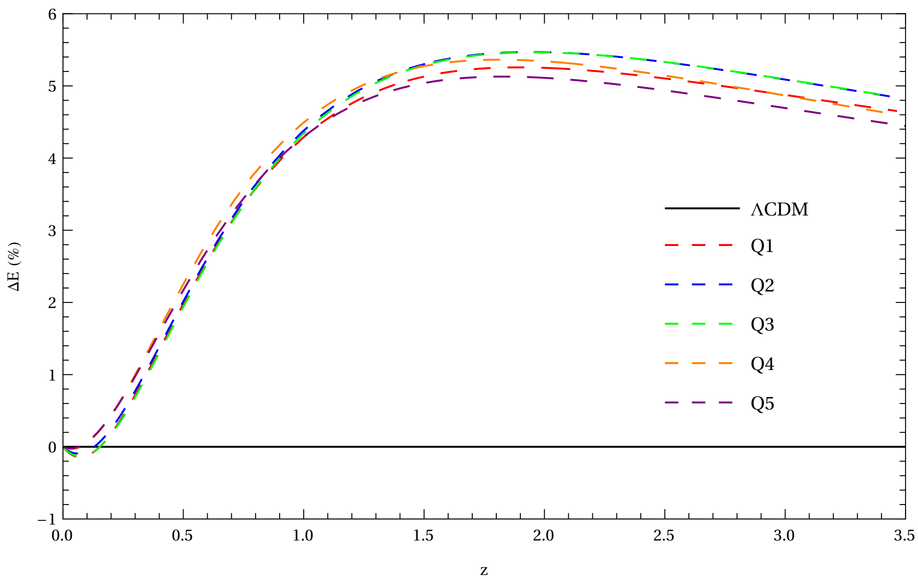
In[ ]:=
```
Show[LogLogPlot[{null, null, null, null, null, null}, {z, 10^-3, 10^6},
  PlotRange → {{10^-3, 10^6}, {10^-1, 10^10}}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {{Directive[Black, Thickness[Medium]]},
    {Directive[Dashing[0.02], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.02], Green, Thickness[Medium]]},
    {Directive[Dashing[0.02], Orange, Thickness[Medium]]},
    {Directive[Dashing[0.02], Purple, Thickness[Medium]]}},
  FrameLabel → {"z", "E(z)"}, PlotLegends →
   Placed[LineLegend[{"ΛCDM", "Q1", "Q2", "Q3", "Q4", "Q5"}], {{0.0, 0}, {-0.6, -1.2}}],
  ImageSize → 500, AspectRatio → 0.6], pELCDM, pEM1, pEM2, pEM3, pEM4, pEM5]
```

Out[ ]=



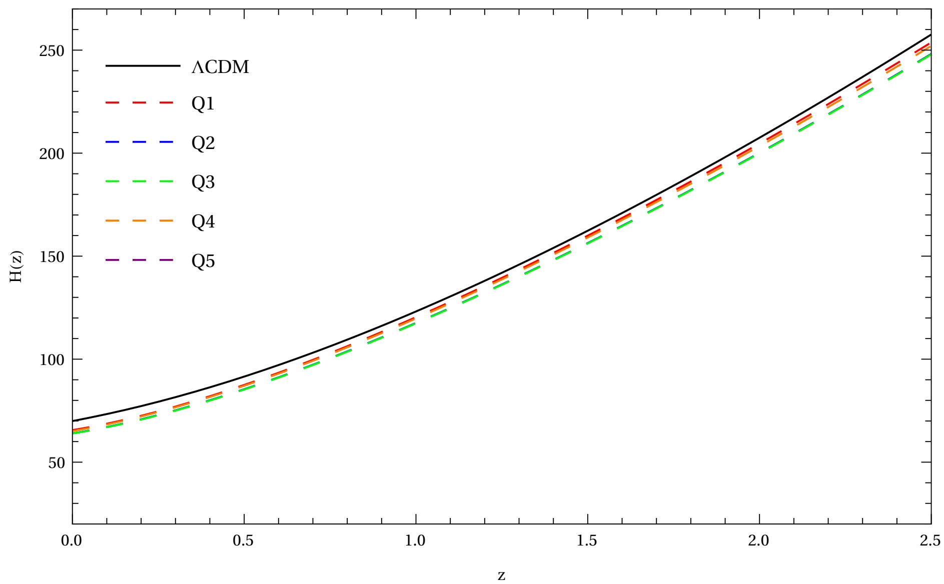
In[∘]:=
```
Show[Plot[{null, null, null, null, null, null},
  {z, 0, 3.5}, PlotRange → {{0, 3.5}, {-1, 6}}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {{Directive[Black, Thickness[Medium]]},
    {Directive[Dashing[0.02], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.02], Green, Thickness[Medium]]},
    {Directive[Dashing[0.02], Orange, Thickness[Medium]]},
    {Directive[Dashing[0.02], Purple, Thickness[Medium]]}},
  FrameLabel → {"z", "ΔE (%)"}, PlotLegends →
   Placed[LineLegend[{"ΛCDM", "Q1", "Q2", "Q3", "Q4", "Q5"}], {{0.0, 0}, {-4, -0.6}}],
  ImageSize → 500, AspectRatio → 0.6], pΔELCDM,
 pΔEM1, pΔEM2, pΔEM3, pΔEM4, pΔEM5]
```

Out[∘]=



```
In[ ]:= Show[Plot[{null, null, null, null, null, null}, {z, 0, 2.5},
   PlotRange → {{0, 2.5}, {20, 270}}, PlotTheme → "Scientific",
   FrameStyle → Black, PlotStyle → {{Directive[Black, Thickness[Medium]]},
     {Directive[Dashing[0.02], Red, Thickness[Medium]]},
     {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
     {Directive[Dashing[0.02], Green, Thickness[Medium]]},
     {Directive[Dashing[0.02], Orange, Thickness[Medium]]},
     {Directive[Dashing[0.02], Purple, Thickness[Medium]]}},
   FrameLabel → {"z", "H(z)"}, PlotLegends →
    Placed[LineLegend[{"ΛCDM", "Q1", "Q2", "Q3", "Q4", "Q5"}], {{0.0, 0}, {-0.6, -1.2}}],
   ImageSize → 500, AspectRatio → 0.6], pHLCDM, pHM1, pHM2, pHM3, pHM4, pHM5]
```

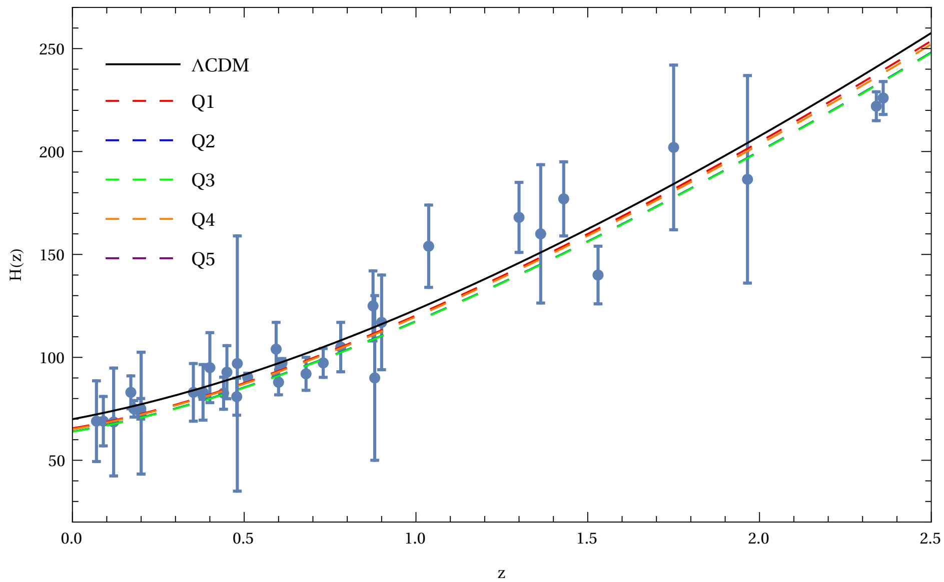
Out[ ]=




```
In[ ]:= dataH = {{0.200, 72.9, 29.6}, {0.070, 69, 19.6}, {0.090, 69, 12}, {0.120, 68.6, 26.2},
    {0.170, 83, 8}, {0.179, 75, 4}, {0.199, 75, 5}, {0.200, 72.9, 29.6},
    {0.270, 77, 14}{0.280, 88.8, 36.6}, {0.352, 83, 14}, {0.380, 81.5, 1.9},
    {0.3802, 83, 13.5}, {0.400, 95, 17}, {0.4004, 77, 10.2}{0.4247, 87.1, 11.2},
    {0.440, 82.6, 7.8}, {0.4497, 92.8, 12.9}, {0.4783, 80.9, 9},
    {0.480, 97, 62}, {0.510, 90.4, 1.9}, {0.593, 104, 13}, {0.600, 87.9, 6.1},
    {0.610, 97.3, 2.1}, {0.680, 92, 8}, {0.730, 97.3, 7}, {0.781, 105, 12},
    {0.875, 125, 17}, {0.880, 90, 40}, {0.900, 117, 23}, {1.037, 154, 20},
    {1.300, 168, 17}, {1.363, 160, 33.6}, {1.430, 177, 18}, {1.530, 140, 14},
    {1.750, 202, 40}, {1.965, 186.5, 50.4}, {2.340, 222, 7}, {2.360, 226, 8}};
```

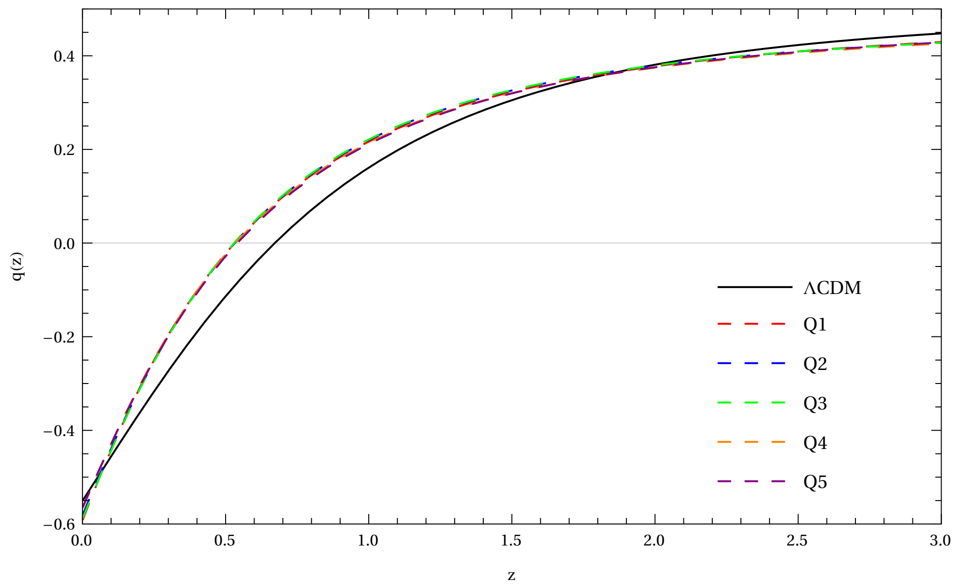
In[ ]:=
```
Show[Plot[{null, null, null, null, null, null}, {z, 0, 2.5},
  PlotRange → {{0, 2.5}, {20, 270}}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {{Directive[Black, Thickness[Medium]]},
    {Directive[Dashing[0.02], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.02], Green, Thickness[Medium]]},
    {Directive[Dashing[0.02], Orange, Thickness[Medium]]},
    {Directive[Dashing[0.02], Purple, Thickness[Medium]]}}, FrameLabel → {"z", "H(z)"},
  PlotLegends → Placed[LineLegend[{"ΛCDM", "Q1", "Q2", "Q3", "Q4", "Q5"}],
    {{0.0, 0}, {-0.6, -1.2}}], ImageSize → 500, AspectRatio → 0.6],
 ErrorListPlot[dataH, ImageSize → Large, Frame → True,
  FrameLabel → {{HoldForm["H(z)"], None}, {HoldForm[z], None}}],
 pHLCDM, pHM1, pHM2, pHM3, pHM4, pHM5]
```

Out[ ]=



In[ ]:=
```
Show[Plot[{null, null, null, null, null, null}, {z, 0, 3},
  PlotRange → {{0, 3}, {-0.6, 0.5}}, PlotTheme → "Scientific",
  FrameStyle → Black, PlotStyle → {{Directive[Black, Thickness[Medium]]},
    {Directive[Dashing[0.02], Red, Thickness[Medium]]},
    {Directive[Dashing[0.02], Blue, Thickness[Medium]]},
    {Directive[Dashing[0.02], Green, Thickness[Medium]]},
    {Directive[Dashing[0.02], Orange, Thickness[Medium]]},
    {Directive[Dashing[0.02], Purple, Thickness[Medium]]}},
  FrameLabel → {"z", "q(z)"}, PlotLegends →
   Placed[LineLegend[{"ΛCDM", "Q1", "Q2", "Q3", "Q4", "Q5"}], {{0.0, 0}, {-4.2, -0.3}}],
  ImageSize → 500, AspectRatio → 0.6], pqLCDM, pqM1, pqM2, pqM3, pqM4, pqM5]
```

Out[ ]=



In[ ]:=
```
(* Age of the Universe: *)
```

In[ ]:=
```
dataT = {{"Model", "tU", "ΔT (%)"}, {"ΛCDM", 13.4738805274518`, 0.0},
   {"Q1", 13.995820961787116`, 3.8737202194416787`},
   {"Q2", 14.337634864484775`, 6.410583315423901`},
   {"Q3", 14.354247738950846`, 6.533880196617292`}, {"Q4", 14.077696558940106`,
    4.481381813190977`}, {"Q5", 14.017827112609494`, 4.03704474037343`}};
```

In[ ]:=
```
Grid[dataT]
```

Out[ ]=

Model	t_U	ΔT (%)
ΛCDM	13.4739	0.
Q1	13.9958	3.87372
Q2	14.3376	6.41058
Q3	14.3542	6.53388
Q4	14.0777	4.48138
Q5	14.0178	4.03704

In[ ]:=
```
Insert[%, {Background → {None, {GrayLevel[0.7], {White}}},
  Dividers → {Black, {2 → Black}}, Frame → True, Spacings → {2, {2, {0.7}, 2}}}, 2]
```

Out[ ]=

Model	t_U	ΔT (%)
ΛCDM	13.4739	0.
Q1	13.9958	3.87372
Q2	14.3376	6.41058
Q3	14.3542	6.53388
Q4	14.0777	4.48138
Q5	14.0178	4.03704

In[ ]:=
```
(*******************************************************************************)
```